A Universal Magnetic Helicity Integral

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A magnetic helicity integral is presented which can be applied to domains which are not magnetically closed, i.e. have a non-vanishing normal component of the magnetic field on the boundary. Contrary to the relative helicity integral, which was previously used for magnetically open domains, it does not rely on a reference field and thus avoids all problems related to the choice of a particular reference field. Instead it uses a gauge condition on the vector potential which corresponds to a particular topologically unique closure of the magnetic field in the external space. For magnetically closed domains the integral reduces to the classical helicity integral, and has additional elegant properties which go beyond that of relative helicity.

INTRODUCTION

Magnetic helicity is an important quantity in describing the structure and evolution of magnetic fields in many fields of physics, in particular in plasma physics and astrophysics. It is defined as an integral over a magnetically closed volume; that is, the normal component of the magnetic field vanishes on the boundary ∂V of the volume:

$$H(\mathbf{B}) := \int_V \mathbf{A}\cdot\mathbf{B}\; dV \;; \quad \mathbf{B}\cdot\mathbf{n}|_{\partial V} = 0. \tag{1}$$

The integral measures - roughly speaking - the Gaussian linkage of magnetic flux within V. More precisely, it is the asymptotic linking number of pairs of field lines averaged over the volume [1]. It is an important property of this integral that we can derive an equation of continuity for the helicity density, which assumes nothing but the homogenous Maxwell's equations (here τ is used to denote the time, while t will later be used for tangential components):

$$\partial_\tau(\mathbf{A}\cdot\mathbf{B}) + \nabla\cdot(\Phi\mathbf{B} + \mathbf{E}\times\mathbf{A}) = -2\;\mathbf{E}\cdot\mathbf{B}\;. \tag{2}$$

This shows that the integral is a topological invariant, i.e. it does not change under a deformation of the field within V, as given for instance by the motion of a magnetic field embedded in an ideal plasma, satisfying

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0\;. \tag{3}$$

Under such a condition, (2) becomes

$$\partial_\tau(\mathbf{A}\cdot\mathbf{B}) + \nabla\cdot((\Phi - \mathbf{v}\cdot\mathbf{A})\mathbf{B} + \mathbf{v}\mathbf{A}\cdot\mathbf{B}) = 0\;, \tag{4}$$

so that integrating over a volume with $\mathbf{v}\cdot\mathbf{n} = 0$ on the boundary (or more generally a comoving volume) results in

$$\begin{aligned}\frac{d}{d\tau}\int_V \mathbf{A}\cdot\mathbf{B}\; dV &= \int_V \partial_\tau(\mathbf{A}\cdot\mathbf{B}) + \nabla\cdot(\mathbf{v}\;\mathbf{A}\cdot\mathbf{B})\; dV \\ &= -\int_{\partial V}(\Phi - \mathbf{v}\cdot\mathbf{A})\mathbf{B}\cdot\mathbf{n}\, da = 0. \end{aligned} \tag{5}$$

Moreover, the total helicity is often an approximate invariant even for non-ideal plasmas, and is therefore a valuable tool in determining the evolution of many technical and natural plasmas. One of the most prominent results is the prediction of the relaxed state of a Reversed-Field Pinch [2], but there are many more applications, see [3] for an overview.

However, the boundary condition $\mathbf{B}\cdot\mathbf{n}|_{\partial V}$ on the integral, which is necessary to ensure the gauge invariance of the quantity, restricts its application in many cases where the magnetic field crosses the boundary. Typical examples are the vacuum vessels of technical plasmas where an external magnetic field crosses the boundaries, or the atmospheres of stars or planets, where the studied volume is usually bounded by the surface of the body, through which the magnetic field emerges.

In such cases it was previously necessary to resort to the calculation of the relative helicity, i.e. the helicity was calculated with respect to a reference field $\mathbf{B}_{\rm ref}$ satisfying the same boundary conditions. One can prove [4, 5] that for an arbitrary closure of the magnetic field outside V, denoted $\mathbf{B}_{\rm ext}$, the relative helicity

$$\begin{aligned} H(\mathbf{B}|\mathbf{B}_{\rm ref}) &= H(\mathbf{B}+\mathbf{B}_{\rm ext}) - H(\mathbf{B}_{\rm ref}+\mathbf{B}_{\rm ext}) \quad (6)\\ &= \int (\mathbf{A}+\mathbf{A}_{\rm ref})(\mathbf{B}-\mathbf{B}_{\rm ref})\; dV\;, \quad (7)\end{aligned}$$

is actually independent of the external closure of the field. For convenience the reference field is often chosen to be a potential field (see e.g. [6]). The introduction of a reference field, however, not only complicates the calculation of magnetic helicity, but also complicates its already difficult interpretation. For instance, the question arises as to whether a change of relative helicity in a volume has a physical meaning, or whether it is only due to our particular choice of reference field.

In this contribution it is proposed to replace the reference field by a more general boundary condition on the vector potential and it is shown that this leads to a well defined quantity.

DEFINITION

The volume $V \subset I\!\!R^3$ we refer to is assumed to be simply connected and without cavities. In other words: it has vanishing first and second Betty numbers. The first condition ensures that the vector potential is unique up to a gradient of a function, the second that a vector potential always exist (alternatively we can require $\int \mathbf{B}\cdot\mathbf{n}da = 0$ over the boundary of any cavity). More general volumes such as solid tori can be considered as well, if additional constraints are included. For a solid torus, for example, we have to impose $\int \mathbf{A}\cdot d\mathbf{l} = 0$ around the hole of the torus.

Definition: The universal magnetic helicity of a magnetic field in a simply connected volume $V \subset I\!\!R^3$ is defined as

$$H_V(\mathbf{B}) = \int_V \mathbf{A}\cdot\mathbf{B}\ dV \ \ \textit{with}\ \nabla_t\cdot\mathbf{A}_t = 0 \ \textit{on}\ \partial V. \tag{8}$$

Here the boundary condition $\nabla_t\cdot\mathbf{A}_t = 0$ is understood to be the divergence of the tangential component of $\mathbf{A}$ on the boundary ∂V, i.e. the divergence is taken with respect to the boundary coordinates only.

In order for the quantity $H_V(\mathbf{B})$ to be well defined, we have to prove firstly that it is not gauge dependent, and secondly that the boundary condition can be satisfied for any field.

Gauge invariance: First note that the boundary condition $\nabla_t\cdot\mathbf{A}_t = 0$, together with $\nabla\times\mathbf{A}_t|_{\partial V} = \mathbf{B}\cdot\mathbf{n}$, uniquely determines $\mathbf{A}_t$, since a gauge $\mathbf{A}\to\mathbf{A}+\nabla\phi$ consistent with the boundary condition requires $\triangle_t\phi|_{\partial V} = 0$ on the boundary, which is a closed manifold, and therefore $\phi|_{\partial V} = constant$. Thus

$$\begin{aligned} H_V(\mathbf{B}) \ &\to\ H_V(\mathbf{B}) + \int_{\partial V}\phi\mathbf{B}\cdot\mathbf{n}\, da \\ &=\ H_V(\mathbf{B}) + \phi\int_{\partial V}\mathbf{B}\cdot\mathbf{n}\, da = H_V(\mathbf{B})\ . \end{aligned} \tag{9}$$

Existence: Starting with an arbitrary vector potential for $\mathbf{B}$ in V, with $\nabla_t\cdot\mathbf{A}_t = \rho$, we note that it is possible to find a function $\phi(\mathbf{y})$, $\mathbf{y}\in\partial V$ defined on the boundary with $\triangle\phi|_{\partial V} = \rho$. This function can be extended to a function on all of V, for instance, by letting it smoothly fall off to zero within an ϵ-neighbourhood of the boundary $\phi(\mathbf{x}) := \phi(\mathbf{y})\exp(-z^2/(\epsilon^2 - z^2))$ where z is a coordinate locally perpendicular to the boundary. Thus $\hat{\mathbf{A}} = \mathbf{A}-\nabla\phi$ has the desired property $\nabla_t\cdot\hat{\mathbf{A}}_t = 0$.

PROPERTIES

Showing that the universal helicity is well defined is not enough to justify its name. It must also reduce to the total helicity (1) for the case of a vanishing normal component of $\mathbf{B}$ on the boundary. Since (8) includes the case of vanishing $\mathbf{B}\cdot\mathbf{n}$ this is obviously the case.

In addition, we can prove that it is a topological invariant for any deformation of the magnetic field inside V, i.e. for any deformation which leaves the boundary unaffected: $\mathbf{v} = 0|_{\partial V}$. Evaluating (3) on the boundary gives $\mathbf{E}_t = -\partial_t\mathbf{A}_t - (\nabla\Phi)_t = 0$. Since $\mathbf{A}_t$ is determined solely by $\mathbf{B}_n$, which is constant in time, we get $\partial_t\mathbf{A}_t = 0$ and hence $\Phi = const.$ on the boundary. Thus (5) vanishes, now due to $\Phi = const.$ and $\mathbf{v} = 0|_{\partial V}$ instead of $\mathbf{B}\cdot\mathbf{n} = 0$.

Another important property of the universal helicity integral is its additivity with respect to magnetic fields. The rule is the same as for the total helicity:

$$H_V(\mathbf{B}^a+\mathbf{B}^b) = H_V(\mathbf{B}^a)+H_V(\mathbf{B}^b)+2H_V(\mathbf{B}^a,\mathbf{B}^b), \tag{10}$$

where

$$H_V(\mathbf{B}^a,\mathbf{B}^b) = \int_V \mathbf{A}^a\cdot\mathbf{B}^b\ dV = \int_V \mathbf{A}^b\cdot\mathbf{B}^a\ dV \tag{11}$$

is the mutual or cross helicity integral. The equivalence of the two integrals in (11) can be shown by using the condition $\nabla_t\cdot\mathbf{A}_t = 0$, which implies a representation

$$\mathbf{A}_t = \nabla\Psi\times\mathbf{n}. \tag{12}$$

This can be used to prove that the difference between the two integrals vanishes:

$$\begin{aligned} \int_V \nabla\cdot\left(\mathbf{A}^a\times\mathbf{A}^b\right) dV \ &=\ \int_{\partial V}\left(\mathbf{A}^a\times\mathbf{A}^b\right)\cdot\mathbf{n}\ da \\ &=\ \int_V\left(\nabla\Psi^b\times\nabla\Psi^a\right)\cdot\mathbf{n}\ da \\ &=\ \int_{\partial V}\left(\nabla\times\Psi^b\nabla\Psi^a\right)\cdot\mathbf{n}\ da = 0. \end{aligned} \tag{13}$$

Furthermore, the universal helicity is additive with respect to complementary volumes. Complementary here describes that the volumes V^a and V^b are adjacent to one another, and that they satisfy $\mathbf{B}^a\cdot\mathbf{n} = \mathbf{B}^b\cdot\mathbf{n}$ on their common boundary, and have $\mathbf{B}^a\cdot\mathbf{n} = \mathbf{B}^b\cdot\mathbf{n} = 0$ on all other boundaries. Note that this still assumes that the volumes are simply connected and have no holes. Thus, the total volume $V^a\cup V^b$ has vanishing normal magnetic field on its boundary, and we can calculate its (classical) total helicity

$$H_{V^a\cup V^b}(\mathbf{B}^a+\mathbf{B}^b) = H_{V^a}(\mathbf{B}^a) + H_{V^b}(\mathbf{B}^b). \tag{14}$$

The proof relies on the fact that the total helicity on the left hand side is gauge invariant, so that we can choose a gauge for the vector potential such that $\nabla_t\cdot\mathbf{A}_t = 0$ holds both on its boundary and on the interface between V^a and V^b. Then the total vector potential can be split in two parts with $\mathbf{A}^a = \mathbf{A}|_{V^a}$ ($\mathbf{A}^a = 0$ on V^b) and $\mathbf{A}^b$ analogously defined. This implies (14).

INTERPRETATION AND EXAMPLE

Equation (14) provides access to an interpretation of the universal helicity integral in terms of the total helicity integral. Imagine that the vector potential of $\mathbf{B}^b$ is given by

$$\mathbf{A}^b = \alpha\ \mathbf{a}\ ; \quad \nabla\times\mathbf{a} = 0 \quad \Rightarrow \quad \mathbf{B}^b = \nabla\alpha\times\mathbf{a} \qquad (15)$$

and correspondingly $\mathbf{A}^b\cdot\mathbf{B}^b \equiv 0$, so that $H_{V^b}(\mathbf{B}^b)$ vanishes. In this case the universal helicity integral of $\mathbf{B}^a$ is just the total helicity of the field $\mathbf{B}^a$ closed by a field $\mathbf{B}^b$ of vanishing helicity density. For this interpretation to hold in general, we have to prove that for any boundary condition on $\mathbf{B}^b$ given by $\mathbf{B}^b\cdot\mathbf{n} = \mathbf{B}^a\cdot\mathbf{n}$, there exists a $\mathbf{B}^b$ satisfying (15).

The existence and topological uniqueness of this field can be deduced from the boundary condition. Eq. (12) ensures the existence of a potential Φ on the boundary. $\mathbf{A}_t$ is tangent to the contour lines of Φ, (see Fig. 1). On the boundary there exists another family of lines which are perpendicular to the contour lines of Φ and parallel to the gradient of Φ. They form a foliation of lines everywhere on the surface except for the points where $\nabla\Phi$ vanishes. This foliation of lines can be extended perpendicular to the boundary to form a foliation of surfaces within a limited distance of the boundary, as shown in Fig. 2 for one line only. Define $\mathbf{A}^b = \mathbf{A}_t \exp(-z^2/(\epsilon^2 - z^2))$, with z being a coordinate locally perpendicular to the boundary. Then $\mathbf{A}^b$ is a vector field which vanishes outside a distance ϵ from the boundary, and is everywhere perpendicular to the constructed foliation. Thus $\mathbf{A}^b\cdot\mathbf{B}^b \equiv 0$ and the foliation is a foliation of flux surfaces for the magnetic field lines (see Fig. 2) . Moreover we can consider the foliation to be level surfaces of a (not single valued) function Ψ, so that $\mathbf{A}^b = \alpha\nabla\Psi$ has indeed a representation in form of (15).

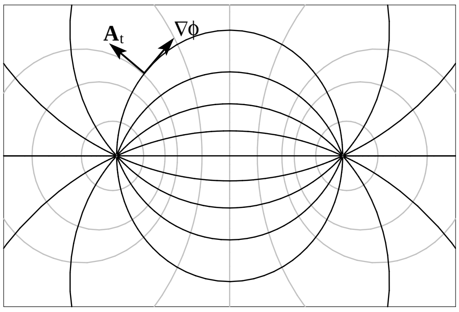

FIG. 1: An example showing contour lines of Φ (grey) and its perpendicular foliation (black) corresponding to two adjacent regions of positive and negative $\mathbf{B}\cdot\mathbf{n}$ on the boundary.

Topological uniqueness. Note that not only is the foliation of flux surfaces of $\mathbf{B}^b$ unique, but so too is the connectivity of points on the boundary by the field lines on each surface. The latter is due to a unique representation of $\mathbf{A}^b$ in the form of (15). Since α in this representation is determined by $\mathbf{A}^t$ on the boundary, and is constant along field lines, the topology of the field lines connected to the boundary does not depend on the functional z-dependence chosen for $\mathbf{A}^b$. If, however, the functional z dependence is not monotonically decaying we may get closed field lines in the leafs which do not contribute to the helicity but are unnecessary for the closure of $\mathbf{B}$. Hence, choosing a monotonously decaying z-dependence, we obtain a topologically unique external field $\mathbf{B}^b$ with the property $\mathbf{A}^b\cdot\mathbf{B}^b = 0$.

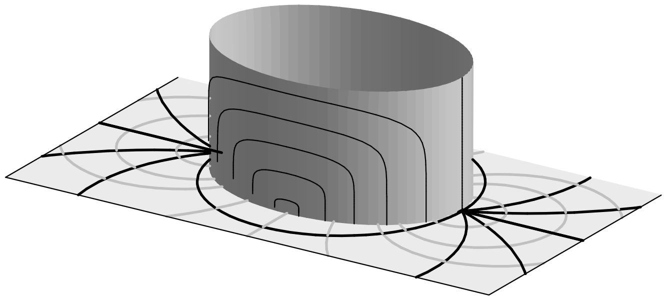

FIG. 2: A leaf of the perpendicular foliation of surfaces constructed from the line field perpendicular to Φ on the boundary. A set of magnetic field lines are also displayed on the leaf, which connect the different polarities of the magnetic field on the boundary.

Gauge freedom. It is also instructive to look at how the boundary condition $\nabla_t\cdot\mathbf{A}_t = 0$ affects the gauge freedom. Consider a simple untwisted closed magnetic flux tube (Fig. 3). It is possible to find a vector potential $\hat{\mathbf{A}}$ such that the helicity density vanishes everywhere, $\hat{\mathbf{A}}\cdot\mathbf{B} \equiv 0$. Now, any gauge $\hat{\mathbf{A}} \to \mathbf{A} = \hat{\mathbf{A}} + \nabla\phi$ with non-vanishing $\mathbf{B}\cdot\nabla\phi$ will generate regions of positive and negative helicity density along the flux tube. The contributions of these regions will cancel for the total helicity integral over the whole flux tube. They will, however, not cancel in general if the integral is taken only over one half of the flux tube (domain V, Fig. 3). This is one of the fundamental problems of defining a helicity integral for a magnetically non-closed domain. The condition $\nabla_t\cdot\mathbf{A}_t = 0$ precludes this situation, in that it requires $\nabla_t\phi = 0$ and hence $\phi = const.$ on the boundary, so that the situation sketched in the Fig. 3 cannot occur.

Example. For a non-trivial example, consider a field consisting of two untwisted flux tubes with the same magnetic flux Φ as shown in Fig. 4(a). There are no closed field lines in the volume under consideration, so there is no way of calculating the helicity of this configuration with the classical helicity integral. The configuration can be considered as the sum of two simple flux tubes as shown in Fig. 4(b), which are cut along a horizontal plane into two mirror symmetric parts. An explicit calculation results in $H_V(\mathbf{B}) = -\Phi^2$ for the universal helicity in the upper half space V. Note that the

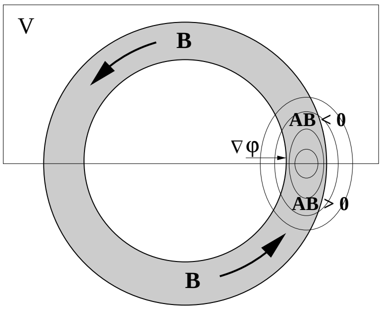

FIG. 3: Choosing a certain gauge potential ϕ, a simple untwisted closed magnetic flux tube can have regions of cancelling helicity density. If these regions are separated by the boundary of the domain V, then the condition $\nabla_t \cdot \mathbf{A}_t = 0$ is violated.

configuration in the lower half space of Fig. 4(b) can be obtained from the upper one if we reverse one flux tube. Therefore the sign of the helicity integral changes: $H_{V'}(\mathbf{B}) = -H_V(\mathbf{B})$. This is consistent with the rule (14), which requires $0 = H_{V'}(\mathbf{B}) + H_V(\mathbf{B})$. Another

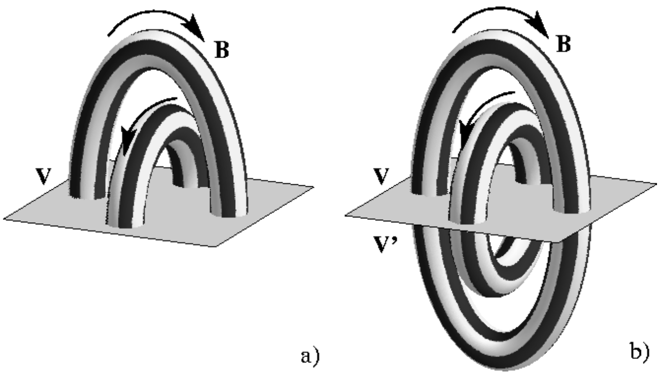

FIG. 4: a) The example under consideration and b) a closure of the field with $H_{V'}(\mathbf{B}) = -H_V(\mathbf{B})$.

way of determining the universal helicity of Fig. 4(a) is to close the magnetic field with an identical copy in V' as in Fig. 5(a). This configuration has a total helicity of $-2\Phi^2$ due to the linkage of the two flux tubes (see e.g. [5]). Hence $-2\Phi^2 = H_{V'}(\mathbf{B}) + H_V(\mathbf{B}) = 2H_V(\mathbf{B})$ and therefore $H_V(\mathbf{B}) = -\phi^2$.

In Fig. 5(b) a closure of the field is shown which is topologically equivalent to the corresponding boundary foliation. The resulting field now has only one flux tube which has a uniform twist of -2π and thus a total helicity of $H_V(\mathbf{B}) = -\Phi^2$, in perfect agreement with our previous result.

SUMMARY

In this letter it was shown how the total helicity integral can naturally be generalized to allow for magnetic

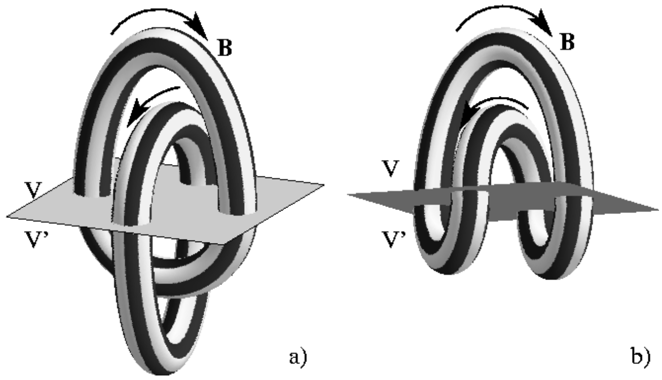

FIG. 5: a) A closure of the field with $H_{V'}(\mathbf{B}) = H_V(\mathbf{B})$ and b) the closure corresponding to the boundary foliation.

fields which are not closed within the domain, i.e. which have a non-vanishing normal component on the boundary. The construction does not require an explicit reference field as the relative helicity integral does, which was previously used in this situation. Instead we have a gauge condition for $\mathbf{A}$ on the boundary which corresponds to closing the domain with a topologically unique field. This field is an external complement with zero helicity density to the field in the given domain. The new integral has all desirable properties, i.e. it is gauge invariant, topologically invariant, and it reduces to the total helicity whenever the latter is well defined. Moreover, it shows the proper additivity with respect to fields and complementary volumes, which was missing for the relative helicity. This facilitates not only many calculations of helicity, but also its interpretation.

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