Linking in Four Dimensions

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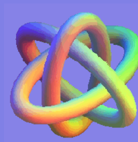


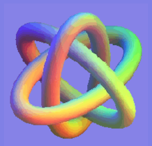
1. Generalization of 3D-Linking

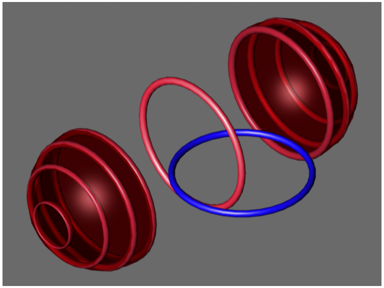
The different but equivalent linking concepts in three dimensions can be carried over to four dimensions. Several methods can be used to derive linking integrals over corresponding fields. These are:

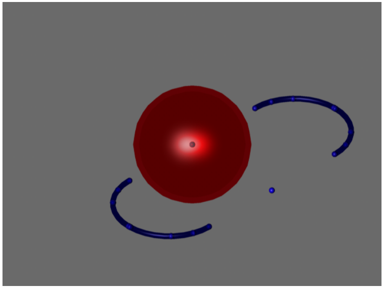
- Homology and intersection pairings
- Alexander duality, de Rham cohomology and the Massey double product
- The generalization of the Gauss-linking formula

Analogy:

$$\begin{array}{lll} I\!R^3: & lk(S^1,S^1) & \sim & H(B_1,B_2) = \int A_1 \wedge dA_2 \\ & & & B = dA \\ I\!R^4: & lk(S^2,S^1) & \sim & N(F_1,H(F_2)) = \int A_1 \wedge H(F_2) \\ & & & H(F_2) = A_2 \wedge F_2 \\ & & & F_2 \wedge F_2 = 0; \quad F_1 \wedge F_2 = 0 \end{array}$$







$$N(F_1, H(F_2)) = \int A_1 \wedge H(F_2)$$

- $N(F_1, H(F_2))$ is the Novikov-Invariant
- $N(F_1, H(F_2))$ describes a cross-linking between the electromagnetic field F_1 and the helicity (4-vector) field H of another electromagnetic field F_2.
- $N(F_1, H(F_1)) = 0$, there is no self-linking for this invariant.
- The restrictions $F \wedge F = 0 \Rightarrow \mathbf{E} \cdot \mathbf{B} = 0$ correspond to the constraint of ideal MHD.
- It is a 4D-topological invariant. It distinguishes processes instead of 3D configurations.
- It can be related to the measuring of helicity separation

2. Further generalization: triple linking

$$N(F_1|F_2, F_3) = \int F_1 \wedge A_2 \wedge A_3$$
$$F_1 \wedge F_3 = 0; \quad F_1 \wedge F_2 = 0$$

$N(F_1|F_2, F_3)$ describes a relative linking between the cross-helicity of the electromagnetic fields F_1 and $F_2$and the electromagnetic field F_3.