Structure formation by modulational interactions between lower-hybrid and dispersive Alfvén waves

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The nonlinear interaction between finite amplitude lower-hybrid (LH) and dispersive Alfvén (DA) waves is considered. For this purpose, a set of equations is derived, which governs the three-dimensional dynamics of nonlinearly coupled LH and DA waves propagating obliquely to an external magnetic field. The interaction between the two wave modes is mediated by the Reynolds stress of the LH waves and the density perturbation associated with the DA wave. Additional terms due to self-interaction nonlinearity of the DA wave are included in order to describe large-amplitude DA waves. Three-wave decay and modulational interactions of the system are analyzed by means of a newly derived nonlinear dispersion relation, which predicts the instability of a finite-amplitude LH pump wave. The fully nonlinear dynamics and the LH wave collapse are studied by solving the governing equations numerically. © 2005 American Institute of Physics. [DOI: 10.1063/1.1896373]

I. INTRODUCTION

Lower-hybrid (LH) waves are frequently observed in space plasmas, both in the ionosphere and in the magnetosphere. The LH waves can be excited by beam instabilities and nonlinear processes, e.g., by the parametric decay of Langmuir waves or decay of an extraordinary polarized electromagnetic wave. Dispersive Alfvén waves, where the nonideality arises from the finite electron mass and parallel electron kinetics, are very important in understanding many plasma phenomena in both space and laboratory plasmas, see Ref. 4 for a comprehensive review.

One of the most interesting manifestations of the LH wave turbulence is the localized bursts of LH waves associated with density cavities, which have been observed in the ionosphere and in the magnetosphere. This phenomenon was first observed by the Marie sounding rocket in the upper ionosphere. The waves appeared as spikes in the measured spectrogram, and were therefore initially termed “spikelets.” It was later confirmed that the wave activity coincides with density depletions that are elongated along the geomagnetic field lines. The relative depth of the cavities is typically a few percent. The perpendicular width is observed to be typically a few ion gyroradii, while the parallel dimension is estimated to be several order of magnitudes larger than the perpendicular one. These elongated density structures are associated with localized waves in the LH frequency range, and are therefore termed LH cavities or LH solitary structures (LHSS). They are frequently observed by spacecrafts in the Earth’s ionosphere and magnetosphere. The LHSS are also found to be a source of transverse ion acceleration. The LHSS are always observed to be immersed in nonlocalized wave activity near the lower-hybrid frequency \( \omega_{LH} \). The nonlocalized waves are collectively referred to as the “hiss” and are believed to be generated by precipitating electrons. The electric field associated with the hiss is strongest above \( \omega_{LH} \), but the wave activity below \( \omega_{LH} \) is also observed. The wave activity above \( \omega_{LH} \) is mainly attributed to electrostatic LH waves. Both the observational and theoretical literature on LHSS is extensive, see a recent review by Schuck et al. and many references therein. The structure of the electric field inside the cavities is well described by electrostatic and electromagnetic theories for linear waves in an inhomogeneous plasma. However, the low-frequency dynamics and the formation of LHSS are not yet well understood. It has been suggested by many authors that the initial formation of LHSS can be described by the parametric interaction between the LH and ion acoustic waves. Although this interpretation is not widely accepted, it was argued in Ref. 19 that the observational data do not provide evidence to rule out the possibility that LHSS are generated by the modulational instability.

The theory for modulational interactions between the LH and ion acoustic waves is well developed. Recently, there has been a great deal of interest in understanding the nonlinear interaction between the LH and dispersive Alfvén (DA) waves, and efforts to deduce a set of equations describing the modulational interaction between the two modes have been made. The treatment in Refs. 19, 20, and 22 is limited to extremely low-\( \beta \) plasma and describes excitation of inertial Alfvén waves. The dispersion of the Alfvén wave was considered to be exclusively due to the finite electron mass and the effects due to the parallel electron kinetics were neglected. The theory in Ref. 20 is valid for medium and extremely low-\( \beta \) plasmas, and it describes the excitation of inertial Alfvén waves as well as kinetic Alfvén waves by the LH waves. It has been shown that nonlinear structures can be formed as a result of the modulational interaction. No detailed observations of correlation between the LH and DA waves have, to our knowledge, been reported in the literature. However, in Ref. 19, a limited data set from the Freja satellite was examined and evidence for correlation between the two modes has been found. Even though the data analysis in Ref. 19 is limited to a single event, one can expect that future analysis of wave data from the Cluster and FAST sat-
ellites will provide detailed observations of the correlation between LH and DA waves.

In previous descriptions of the LH turbulence, the interaction between the LH waves and ion acoustic waves has been investigated, where the low-frequency ion dynamics was assumed to be unmagnetized with wave frequencies much larger than the ion gyrofrequency. In this model, the ponderomotive force created by the high-frequency electron motion along the magnetic field direction couples the LH waves with the ion acoustic waves. On the other hand, the high-frequency $\mathbf{E} \times \mathbf{B}$ drift in the density perturbation associated with the ion acoustic waves causes a significant nonlinear frequency shift and couples the two modes. The resulting interaction is of vector type. Similar to the case of the Langmuir turbulence, it was shown by Musher and Sturman that the LH waves can self-focus and subsequently collapse into small-scale structures.

In the present paper, we consider the modulational interaction between the LH waves and DA waves. We derive and analyze a set of equations governing the three-dimensional dynamics of nonlinearly coupled LH and DA waves that are propagating obliquely to an external magnetic field. The LH waves are considered to be electrostatic, and the present treatment describes waves on the resonance cone as well as pressure driven oscillations. The DA wave has frequencies much smaller than the ion gyrofrequency, and therefore the ions are assumed to be strongly magnetized and the ion motion is described by the drift approximation. The ion dynamics along the magnetic field is neglected, which is appropriate for the description of the DA waves where the dispersion arises from the finite magnetic field-aligned wave electric field, the ion polarization drift, the Hall effect, the parallel electron kinetics, and the parallel electron inertia. However, for waves with parallel phase velocities comparable to the ion sound speed, the parallel ion dynamics must be included. Here, the DA waves are linearly coupled to the ion acoustic waves. This limit is not covered in the present treatment. The DA waves accompany a quasineutral density perturbation and a sheared magnetic field which modulate the LH waves. On the other hand, the ponderomotive force associated with LH waves couples back to the low-frequency perturbations and modifies the DA perturbations. In addition to the coupling mechanisms presented in Ref. 21, the set of equations derived in the present paper includes a scalar nonlinearity considered in Refs. 19 and 22. This scalar interaction is important when the scale length of the LH waves is extremely small in comparison to the length scale of the DA waves. We derive a nonlinear dispersion relation which describes the nonlinear generation of DA waves by a large-amplitude LH pump wave. The dispersion relation generalizes previous results to include effects originating from the parallel electron kinetics and allows us to describe the excitation of kinetic Alfvén waves and modified convective cells. Furthermore, we present numerical solutions of the three-dimensional equations, where the fully nonlinear dynamics and the wave collapse are studied.

The paper is organized in the following fashion. The basic equations for low- and medium-$\beta$ plasma are presented in Sec. II. In addition, some basic properties of the LH and DA waves are briefly reviewed. In Sec. III we derive a nonlinear dispersion relation describing the excitation of DA waves by a LH pump wave in an extremely low-$\beta$ plasma. Analytical results as well as numerical calculations of the growth rates are presented. In Sec. IV we present numerical solutions of the three-dimensional equations in an extremely low-$\beta$ plasma. The paper is concluded by a summary in Sec. V.

II. BASIC EQUATIONS

Before considering the equation governing the nonlinear evolution we shall review some basic properties of the LH and DA waves. The LH waves are electrostatic oscillations in a magnetized plasma with wave frequencies $\omega$ much smaller than the electron gyrofrequency $\omega_{ce}=eB_0/(mc)$, but much larger than the ion gyrofrequency $\omega_{ci}=eB_0/(mi)$, i.e., $\omega_{ci}<<\omega<<\omega_{ce}$, where $e$ is the magnitude of the electron charge, $B_0$ is the strength of the external magnetic field, $m_e$ ($m_i$) is the electron (ion) mass, and $c$ is the speed of light in vacuum. In the linear approximation, the LH waves are characterized by the approximate dispersion relation

$$\omega^2 = \omega_{LH}^2 \left(1 + k_L^2 \rho_L^2 + \frac{k_P^2}{k_I^2} \frac{m_i}{m_e} \right),$$

where $\omega_{LH}^2 = \omega_{ce}^2/(1+\omega_{pe}^2/\omega_{ce}^2)^{1/2}$ is the LH frequency, $\omega_{pe}^2 = (4\pi n_e e^2/m_i)^{1/2}$ and $\omega_{ce}^2 = (4\pi m_e e^2/m_i)^{1/2}$ are the electron and ion plasma frequencies, respectively, $n_0$ is the background plasma number density, and $k_L$ and $k_P$ are the perpendicular (to the external magnetic field direction) and parallel wave numbers, respectively. In the electrostatic cold plasma approximation, the LH waves propagate in the form of a resonance cone. However, for large $k_L$ the finite Larmor radius effects must be included, which gives rise to an additional perpendicular dispersion. The strength of the finite Larmor radius effects is described by the thermal dispersion length

$$\rho_T = \left(\frac{3T_i}{\omega_{LH}^2 m_i} + \frac{2T_e}{\omega_{ce}^2 m_e} \frac{\omega_{pe}^2}{\omega_{ce}^2 + \omega_{pe}^2} \right)^{1/2},$$

where $T_e$ and $T_i$ are the electron and ion temperatures, respectively. For sufficiently large wavelengths, viz., $k\lambda_s^{-1}$, where $\lambda_s = c/\omega_{pe}$ is the electron skin depth, the LH waves connect to the electromagnetic $R$ whistler/fast magnetosonic wave mode. This transition is not described by Eq. (1), but it is straightforward to include lowest order electromagnetic effects due to the finite electron current parallel to the magnetic field. It is important to note that for wave propagation almost transverse to the ambient magnetic field direction, the parallel current is small, but for exact perpendicular propagation the parallel current density is zero. However, the present treatment of the LH waves is restricted to the electrostatic limit $kh_s^{-1} \gg 1$, in which case Eq. (1) provides an appropriate description. The group velocity of the LH waves is
\[ v_s = \omega_{\text{ LH}} \left( k_1 \rho_T^2 + k_2^2 \frac{m_i}{m_e} \right) + \frac{k_3 \omega_{\text{ LH}}^2 m_i}{m_e} \frac{\hat{z}}{z}. \] (3)

Notice that the LH wave is a backward wave in the perpendicular direction, i.e., it has a negative group velocity, when \( k_\perp \) is sufficiently small, i.e., when \( k_4^2 \rho_T^2 < k_2^2 m_i / m_e \).

In a low-\( \beta (\beta \ll 1) \) plasma, the DA wave is characterized by the dispersion relation
\[ \omega^2 = \frac{k_2^2 \rho_T^2}{1 + k_4^2 \lambda_{\parallel}^2 + k_5^2 \lambda_{\perp}^2} (1 + k_2^2 \rho_T^2), \] (4)
where \( v_A = B_0 / \sqrt{4 \pi n m_i} \) is the Alfvén speed, \( \lambda_i = c / \omega_{pi} \) is the ion skin depth, and \( \rho_T = (T_e / m_e)^{1/2} / \omega \) is the ion gyroradius at the electron temperature \( T_e \). The dispersion relation (4) is a generalization of \( \omega = k \nu_A \), predicted by the ideal magnetohydrodynamics. The term \( k_2^2 \rho_T^2 \) originates from the ion polarization contribution to the ion density fluctuation, the \( k_4^2 \lambda_{\parallel}^2 \) term arises from the finite parallel electron inertia, and the \( k_5^2 \lambda_{\perp}^2 \) term is a correction due to finite \( \omega / \omega_c \). The dispersive properties described by Eq. (4) depend on the plasma \( \beta \). In a medium-\( \beta \) plasma \( (m_e / m_i \ll \beta \ll 1) \) with \( \omega / \omega_c = \sqrt{k_2^2 \rho_T^2} \), \( k_2 \lambda_{\parallel} \ll 1 \), and \( k_2 \rho_T \sim 1 \), Eq. (4) describes the kinetic Alfvén wave with
\[ \omega = k_2 \nu_A (1 + k_2^2 \rho_T^2)^{1/2}. \] (5)

For an extremely low-\( \beta \) plasma \( (\beta \ll m_e / m_i) \) with \( k_2 \lambda_{\parallel} \ll 1 \) and \( k_2 \rho_T \sim 1 \), Eq. (4) describes the dispersive inertial Alfvén wave (DIAW) and the modified convective cells. The DIAW is characterized by \( k_4 \rho_T \ll 1 \) and \( k_4 \lambda_{\parallel} \sim 1 \) for which the dispersion relation (4) becomes
\[ \omega = \frac{k_2 \nu_A}{(1 + k_4^2 \lambda_{\parallel}^2)^{1/2}}. \] (6)

For sufficiently large \( k_\perp \), viz., \( k_\perp \lambda_{\parallel} \gg 1 \) and significant \( k_2 \rho_T \), the DA wave is known as the modified convective cell, and is described by the approximate dispersion relation
\[ \omega = \frac{k_2 \nu_A}{(1 + k_2^2 \rho_T^2)^{1/2}}. \] (7)

Both the DIAW and the convective cell are backward waves in the perpendicular direction.

**A. LH wave equation**

Here, we derive the governing equation for the evolution of the LH wave. The latter is coupled to the DA wave through the fluctuations in the plasma number density and in the ambient magnetic field. In addition, the low-frequency fluid motion associated with the DA wave convects the LH perturbations and gives rise to additional nonlinear frequency shifts. The high-frequency electric field \( \mathbf{E}_L \), which is associated with the LH waves, is assumed to be electrostatic, i.e., \( \mathbf{E}_L = - \nabla \phi_L \), where \( \phi_L \) is the electrostatic potential. Since the LH frequency regime is characterized by \( \omega \ll \omega \ll \omega_c \), the electron motion perpendicular to \( \mathbf{B}_0 = B_0 \hat{z} \) can be evaluated in the drift approximation, while the ions can be considered as unmagnetized. The perpendicular electron velocity \( v_{e \perp} \) is given by
\[ \begin{align*}
\mathbf{v}_{e \perp} &= \frac{c}{B_0} \hat{z} \times \nabla \phi_L + \frac{c}{B_0 \omega_{ce}} \frac{\partial}{\partial t} \nabla \phi_L - \frac{1}{\omega_{ce}} \hat{z} \times \left[ (\mathbf{v}_{e \perp} \cdot \nabla) \mathbf{u}_{e \perp} + (\mathbf{u}_{e \perp} \cdot \nabla) \mathbf{v}_{e \perp} \right].
\end{align*} \] (8)

where the first two terms are linear and represents the \( \mathbf{E} \times \mathbf{B} \) and polarization drifts, respectively. The DA wave accompanies a sheared magnetic field \( \mathbf{B} = \mathbf{B}_0 \hat{z} \), where \( \hat{z} \) is the \( z \) component of the vector potential associated with the DA wave. The field fluctuation gives rise to the third term in Eq. (8). The last term in (8) is due to advection, and \( \mathbf{u}_{e \perp} \) is the low-frequency electron fluid velocity associated with the DA wave. For our purposes, in the right-hand side of Eq. (8), \( \mathbf{u}_{e \perp} \) can be evaluated in the lowest order drift approximation, yielding \( \mathbf{u}_{e \perp} = c / B_0 \hat{z} \times \nabla \phi_L \), where \( \phi_L \) is the electrostatic potential associated with the DA wave. The parallel component of the electron fluid velocity is governed by
\[ \partial_t \mathbf{v}_e = -(e/m_e) \nabla \phi_L. \] (9)

In the LH frequency domain, the ions can be regarded as unmagnetized since \( \omega \ll \omega_i \) and the ion fluid velocity \( \mathbf{v}_i \) associated with the LH wave is determined by
\[ \partial_t \mathbf{v}_i = -(e/m_i) \nabla \phi_i. \] (10)

Equations (8)–(10) can be used to calculate the high-frequency current density \( \mathbf{J} = e n_0 (1 + \eta)(\mathbf{v}_e - \mathbf{v}_i) + e n_e \mathbf{u}_e \), where \( n_e (n_i) \) is the high-frequency electron (ion) density fluctuation. \( \eta = (c / B_0 \omega_{ce}) \nabla \cdot \phi_L \) is the quasi-neutral density perturbation associated with the DA wave, which arises from the ion polarization drift. By substituting Eqs. (8)–(10) into the charge density conservation equation \( \partial_t \rho + \nabla \cdot \mathbf{J} = 0 \) and using Poisson’s equation \( \nabla \phi_L = -4 \pi \rho \), we obtain the LH wave equation
\[ \mathcal{L}_L \phi_L = -\omega^2_{\text{ LH}} \nabla^2 \cdot \left[ (\eta \nabla \phi_L) - \omega^2_{pe} \partial_t (\eta \partial_t \phi_L) \right] - \frac{\omega_{ce}^2}{\omega_{ce}} \partial_t (\nabla \phi_L \times \nabla \phi_L) - \frac{c}{B_0} \partial_t (\hat{z} \times \nabla \phi_L \cdot \nabla \phi_L) - \frac{c}{B_0 \omega_{ce}} \partial_t \cdot (\hat{z} \times \nabla \phi_L \cdot \nabla \phi_L) + \hat{z} \times \nabla \phi_L \cdot \nabla \phi_L) - \frac{e}{m_e} \frac{\omega_{ci}^2}{\omega_{ci}} \partial_t \cdot [(\partial_t \nabla \phi_L) \partial_t^2 \phi_L] + \frac{\omega_{ci}^2}{B_0} \partial_t [\hat{z} \times \nabla A_L \cdot \nabla \partial_t \phi_L]. \] (11)

The operator \( \mathcal{L}_L \) describes the linear dispersion of the LH waves and is given by
\[ \mathcal{L}_L = \partial_t^2 \nabla^2 + (\omega_{pe}^2 / \omega_{ce}^2) + \omega_{pe}^2 \partial_z^2 + \omega_{pi}^2 \partial_z^2 \]
\[ + \partial_t^2 [\alpha_1 \nabla^2 + \alpha_2 \partial_z^2 + \alpha_3 \partial_z^2]. \] (12)

In addition to the cold plasma effects described by Eqs.
(8)–(10), we have also included corrections due to the finite pressure in the linear part of Eq. (11). The thermal dispersion is described by the coefficients $a_1$, $a_2$, and $a_3$. In the linear approximation, we recover the dispersion relation given in Eq. (1).

B. DA wave equation

In this section, we derive a set of equations governing the DA waves. The DA waves accompany a sheared magnetic field $\mathbf{B}_1$, as well as a parallel electric field $E_z$. The electric and magnetic fields of the DA waves are given by $\mathbf{E}_A=-\nabla \phi_A - \hat{z} e^{-1} \partial_z \mathbf{A}$, and $\mathbf{B}_L=\nabla \times \hat{z} \times \mathbf{A}$, respectively. As the frequency of the considered DA waves is much smaller than $\omega_{ci}$, both the electron and ion velocity can be calculated in the drift approximation. The parallel phase velocity of the DA waves is assumed to be much larger than the ion sound speed. This assumption allows us to neglect the parallel ion dynamics and excludes the ion acoustic branch. The perpendicular electron velocity associated with the DA wave is

$$u_{e\perp} \approx \frac{c}{B_0} \hat{z} \times \nabla \times \phi_A - \frac{e}{B_0} \mathbf{E}_A - \frac{1}{\omega_{ce}} \mathbf{v}_{e\perp} \cdot \nabla \phi_A - u_{ez} \frac{\hat{z} \times \nabla A_z}{B_0},$$

(13)

where $u_{ez}$ is the parallel electron velocity and the angular bracket denotes the ensemble average over the LH wave period. The first two terms are linear and arise due to the $\mathbf{E} \times \mathbf{B}$ and diamagnetic drifts, respectively. The third term is a self-interaction nonlinearity due to field line bending associated with the electromagnetic DA wave. The last term arises from the parallel Reynolds stress of the high frequency LH wave. The electron polarization drift has been neglected in Eq. (13), as the contribution of the ion polarization drift to the current is larger by a factor $m_i/m_e \gg 1$. The perpendicular ion velocity is

$$u_\perp \approx \frac{c}{B_0} \hat{z} \times \nabla \times \phi_A - \frac{e}{B_0} \mathbf{E}_A - \frac{1}{\omega_{ci}} \mathbf{v}_{i\perp} \cdot \nabla \phi_A - \frac{1}{\omega_{ci}} \left( \frac{\partial}{\partial t} - \omega_c \hat{z} \right) \langle \mathbf{v}_i \cdot \nabla \mathbf{v}_i \rangle,$$

(14)

where we have assumed $\hat{z} \times \nabla \phi_A \cdot \nabla \perp \gg (B_0/c)|\mathbf{v}_e| \partial_t / \partial z$. The electron fluid velocity $u_{ez}$ parallel to $\hat{z}$ is governed by the equation

$$\left( \frac{\partial}{\partial t} + \frac{c}{B_0} \hat{z} \times \nabla \times \phi_A \cdot \nabla \perp \right) u_{ez} = \frac{e}{m_e c} \left[ \frac{\partial \phi_A}{\partial \perp} + \left( \frac{\partial}{\partial t} + \frac{c}{B_0} \hat{z} \times \nabla \phi_A \cdot \nabla \perp \right) A_z \right. \left. - \frac{e}{m_e c} \frac{\partial}{\partial \perp} \left( \frac{1}{B_0} \hat{z} \times \nabla A_z \right) \nabla \perp \phi_A \right] - \langle \mathbf{v}_{e\perp} \cdot \nabla \mathbf{v}_{e\perp} \rangle - \frac{1}{2} \langle \partial_z v_{ez}^2 \rangle,$$

(15)

where the self-interaction nonlinearity on the left-hand side is due to advection and the nonlinearities on the right-hand side are due to the nonlinear $\mathbf{E} \times \mathbf{B}$ drift. Furthermore, a finite parallel electron pressure term has been included in the electron equation of motion. The nonlinear terms in the right-hand side of Eq. (15) can be evaluated using the linearized electron fluid velocity associated with the LH waves. As the time scales of the LH and DA waves are well separated, the LH wave potential can be represented as $\phi_L = \tilde{\phi}_L \exp(-i\omega_L t) + \text{c.c.}$, where $\tilde{\phi}_L = \phi_L(\mathbf{r}, t)$ is a temporally slowly varying envelop function and $\omega_L$ is the frequency of the LH wave. With this representation of $\phi_L$, the parallel Reynolds stress in Eq. (15) can be written as

$$\langle \mathbf{v}_{e\perp} \cdot \nabla \mathbf{v}_{e\perp} \rangle + \frac{1}{2} \langle \partial_z v_{ez}^2 \rangle = \frac{1}{B_0 \omega_L m_e c} \left[ \left( \nabla \times \tilde{\phi}_L \right) \cdot \nabla \phi_L - \left( \nabla \times \phi_L \right) \cdot \nabla \tilde{\phi}_L \right] + e^2 \frac{\partial}{\partial \perp} \left[ \frac{1}{m_e^2 \omega_{ce}^2} \left( \nabla \tilde{\phi}_L \right) \cdot \nabla \tilde{\phi}_L \right] \left[ \nabla \phi_L \right]^2 + e^2 \frac{\partial}{\partial \perp} \left[ \frac{1}{m_e^2 \omega_{ce}^2} \left( \nabla \tilde{\phi}_L \right) \cdot \nabla \tilde{\phi}_L \right] \left[ \nabla \phi_L \right]^2,$$

(16)

where the first term on the right-hand side is due to the $\mathbf{E} \times \mathbf{B}$ drift, the second term is due to the electron polarization current, and the last term is due to the nonlinear parallel electron convection. The electron polarization term provides an interaction of scalar type and was investigated in Refs. 19 and 22. The perpendicular Reynolds stress in Eqs. (13) and (14) can be evaluated in terms of $\tilde{\phi}_L$.

The parallel ion current density is negligibly small in comparison to the electron current density. Substituting Eqs. (15) and (16) into the parallel component of Ampère’s law, i.e., $\nabla \times \mathbf{A}_z = 4\pi n_0 \mathbf{u}_e / c$, we obtain

$$d_i (1 - \kappa^2 \nabla^2) \mathbf{A}_z + c \partial_t \phi_A - \rho_e^2 \partial_z \nabla \perp \phi_A = i \frac{c^2}{B_0 \omega_L m_e c} \left[ \left( \nabla \times \tilde{\phi}_L \right) \cdot \nabla \phi_L - \left( \nabla \times \phi_L \right) \cdot \nabla \tilde{\phi}_L \right] + \frac{e c}{m_e \omega_{ce}^2} \frac{\partial}{\partial \perp} \left( \nabla \tilde{\phi}_L \right) \left[ \nabla \phi_L \right]^2 + \frac{e c}{m_e \omega_{ce}^2} \frac{\partial}{\partial \perp} \left( \nabla \tilde{\phi}_L \right) \left[ \nabla \phi_L \right]^2,$$

(17)

where the differential operators $d_i = \partial_t + (c/B_0) \hat{z} \times \nabla \phi_A \cdot \nabla$ and $d_z = \partial_z - B_0 / c^2 \times \nabla \phi_A \cdot \nabla$ include the self-nonlinearities. The continuity equation in conjunction with the quasineutrality condition and the parallel component of Ampère’s law gives

$$d_i \nabla \perp \phi_A + \frac{v_{ez}^2}{c} d_z \nabla \perp A_z = - \frac{e}{m_e \omega_L} \partial_t \left[ \left| \nabla \tilde{\phi}_L \right|^2 + \frac{e m_e}{B_0} \partial_z \mathbf{A}_z \right] \times \left( \hat{z} \times \nabla \phi_L \right) \cdot \nabla \perp A_z^* + \text{c.c.},$$

(18)

Equations (11), (17), and (18) are the desired equations for investigating the parametric excitation of the DA waves by large amplitude LH waves. This set of equations will be analyzed analytically as well as numerically in order to investigate the three-wave decay and modulational interactions. In addition, the self-nonlinearities in the DA equations allow studies of interactions between LH waves and nonlinear Alfvénic structures.
III. NONLINEAR DISPERSION RELATION

In this section, we consider the parametric excitation of the DA waves by a LH pump wave in an extremely low-β plasma. This extremely low-β limit is appropriate for describing phenomena in the Earth’s ionospheric plasma. As mentioned above, in an extremely low-β plasma the dispersion of the DA wave with \( q_l \sim 1 \), where \( q \) is the wave number of the DA wave, is dominated by the finite parallel electron inertial effect. For shorter wavelengths, viz., \( q_L \gg 1 \), the parallel electron kinetics is important and the wave mode can be characterized as the modified convective cell. Thus, depending on the length scale of the excited DA wave, one must include or neglect the parallel electron kinetics.

For the purpose of describing the initial state of the LH-DA turbulence, we consider small amplitudes of the DA wave and neglect the self-interaction nonlinearities, i.e., \( d_t = \tilde{\sigma}_t \) and \( d_c = \tilde{\sigma}_c \). Equations (17) and (18) can then be combined to eliminate \( \tilde{\Lambda}_1 \) and to obtain one single equation governing \( \phi_A \). In the resulting equation, the nonlinearities originating from the right-hand side of Eq. (18) are proportional to \( \tilde{\sigma}_t \), while the nonlinearities originating from Eq. (17) do not depend on temporal derivatives. Thus, for sufficiently slow variations the main nonlinearities arise from the parallel Reynolds stress. The total parallel Reynolds stress in Eq. (16) is a sum of three different contributions. An order of magnitude estimate shows that the contribution from the electron polarization drift is much larger than the one from the electron po-

\[ \phi_L = \phi_{0L} e^{i(k \cdot x - \omega_L t)} + \phi_{cl} e^{i(k \cdot q - \omega_cl)}, \]

where \( \omega_L \) is the frequency of the LH pump wave and \( \Theta_L \) is the electrostatic potential of the LH wave pump (sidebands). The amplitudes of the up-shifted and down-shifted satellites can be calculated from Eq. (20). We have

\[ D_A \phi = \frac{c}{B_0} \frac{\omega_L^2}{\omega_{cl}^2} q^2 (kL_z - qL_z), \]

where \( \omega_{kl} \) are the frequencies of the sidebands. The amplitude of the DA wave is given by

\[ D_A a = -i \frac{c}{B_0} q_L^2 \frac{\omega_L^2}{\omega_{cl}^2} (kL_z - qL_z)(\phi^*_L \phi_B + \phi^*_B \phi_L), \]

where \( \Omega_A = (1 + \lambda_1^2 q^2 - \rho^2 q^2 (1 + q^2 \rho^2))^2 \). By combining Eqs. (22), (23), and (25) we obtain a nonlinear dispersion relation

\[ 1 + \frac{\omega_{kl}}{\omega_{cl}^2} \frac{\Omega_A^2}{\Omega^2} \rho^2 \frac{q^2}{k^2} E_{TH}^2 (kL_z - qL_z) \times \left[ \frac{\alpha_s}{\Omega + \omega_{kl} - \alpha_k} + \frac{\alpha_s}{\Omega + \omega_{kl} - \alpha_k} \right] = 0, \]

where \( \Omega_A \) is the frequency of the DA wave, as predicted by the linear theory, \( E_{TH} = \omega_{cl} B_0 / (\omega_{cl} m) \) is a characteristic field strength of the interaction, and \( \alpha_s = q^2 / (k |q|^2)^2 \). Equation (26) generalizes the dispersion relation obtained in Refs. 19 and 20 by including finite electron temperature effects. In the limit \( q \rho = 0 \) we recover the previous result. The finite \( T_e \) effect included in Eq. (26) is important in the short wave limit \( q \lambda_1 \gg 1 \), where Eq. (26) describes the excitation of the modified convective cells. Especially, this is important in the case of the three-wave decay of a LH pump wave. Namely, the wave number for the electrostatic LH waves satisfies \( k \lambda_1 \sim 1 \) and the three-wave decay instability usually excites the DA waves with \( q \sim k \). Thus, the thermal correction cannot be ignored for the three-wave decay case.

The excitation of long wavelength DA waves via the modulational instability of the LH wave pump is investigated by considering the limits \( \lambda_1^{-1} \sim q \gg k \gg \omega_{cl} / \alpha_{cl} \). For the modulational instability, both the upper and lower sidebands are resonant, whereas the low-frequency perturbations are.
nonresonant. This allows us to analyze the nonlinear dispersion relation in the limiting case \( \Omega \ll \Omega_A \). Here, Eq. (26) takes the form

\[
(\Omega - \mathbf{q} \cdot \mathbf{v}_g)^2 = \delta^2 \left[ 1 - \frac{\omega_{\text{LH}} q_x^2 E_0^2}{4 \delta k^2 E_{\text{TH}}^2 \sin^2 \alpha} \right],
\]

where \( \mathbf{v}_g \) is the group velocity of the LH pump wave,

\[
\delta = \frac{\omega_{\text{LH}}}{2} \left[ k_p^2 + \frac{k^2 m_i}{k_m^2} (4 \cos^2 \alpha - 1) \right] q_x^2 k^2 - 2 \omega_{\text{LH}} \frac{m_i k q_x q_y \cos \alpha}{k^4} + \frac{\omega_{\text{LH}} m_i q_z^2}{2 m_e k^2},
\]

and \( \cos \alpha = k_x q_x / (k, q_x) \). As \( \delta \) is quadratic in the components of \( \mathbf{q} \), it follows from the solution of Eq. (27) that the pump wave is unstable when \( E_0 > E_{\text{mod}} < q \). Thus, the modulational instability cannot, according to this condition, excite long wavelength DA waves, as the threshold scales as \( q \).

For \( q \sim k \gg \lambda^{-1} \), Eq. (26) describes a three-wave decay process, where the LH pump decays into a down-shifted LH sideband and a modified convective cell. For small amplitudes, the up-shifted sideband is not strongly excited and we can omit the first term in the square bracket in Eq. (26). We then obtain

\[
(\Omega^2 - \Omega_A^2) (\Omega + \omega_{\text{LH}} - \omega_{\text{LH}}) + \frac{\omega_{\text{LH}} q_x^2 \lambda_x^2}{8} E_0^2 (k_x q_x)^2 q^2 \Omega_A^2 + \frac{q_x^2 \Omega_A^2 q^2}{k_x^2} = 0,
\]

which predicts a three-wave decay instability near the resonance surface \( \omega_{\text{LH}} = \omega_{\text{LH}} + \Omega_A \). The growth rate of the instability is

\[
\gamma = \frac{q_x^2 \omega_{\text{LH}}}{8} \left[ \frac{q_x^2 \lambda_x^2}{1 + \frac{q_x^2 \lambda_x^2}{E_{\text{TH}}^2}} \right] \left[ \frac{\sin^2 \alpha}{q_x^2 \Omega_A^2} \right]^{1/2}.
\]

The decay spectrum has a fairly complicated structure as the negative perpendicular dispersion of the LH wave allows several intersections between the surfaces \( \omega_{\text{LH}} = \omega_{\text{LH}} + \Omega_A \) and \( \Omega = \Omega_A \). The main features of the resonance surface are illustrated in Fig. 1 for \( k = k_x \hat{x} \) with \( k_x = (2 \pi / 20) \text{ m}^{-1} \). The plasma parameters used in Fig. 1 are \( n_0 = 900 \text{ cm}^{-3} \), \( B_0 = 0.25 \text{ G} \), \( m_i / m_e = 16 \times 1836 \), \( T_e = 3000 \text{ K} \), and \( T_i = 2400 \text{ K} \). This set of plasma parameters is appropriate for the Earth’s upper ionosphere.\(^7\) The plasma there has \( \beta = 1.6 \times 10^{-5} \), \( \lambda_c = 180 \text{ m} \), \( \lambda_x = c / \omega_p \approx 30 \text{ km} \), and \( \rho_i = 8 \text{ m} \). Figure 2 displays the growth rate of the three-wave decay interaction for a moderate amplitude (\( E_0 = 0.5 \text{ mV/m} \)) LH pump wave. The growth rate \( \gamma = \Im \Omega \) has been obtained by solving the nonlinear dispersion relation, Eq. (26), numerically. In this amplitude regime, the instability occurs close to the resonance surfaces, as depicted in Fig. 1. The two local maximas are attributed to the negative perpendicular LH dispersion. Two local maximas \( q_x \) are seen in the panel (a). For larger \( q_x \), the regions of instability are merging. The maximum \( \gamma \) is at \( q_x = k_x, q_x = k_x / 2 \), and \( q_x = 2 \times 10^{-5} \text{ m}^{-1} \).

![Fig. 1. Projections of the resonance surface \( \omega = \omega_{\text{LH}} + \Omega_A \). Panels (a)–(c) show the projection of the resonance surface in the \( q_x - q_y \) plane for \( q_x = 1.7 \times 10^{-5} \text{ m}^{-1} \), \( q_x = 2.0 \times 10^{-5} \text{ m}^{-1} \), and \( q_x = 2.2 \times 10^{-5} \text{ m}^{-1} \), respectively. Panel (d) shows the projection on the \( q_x - q_z \) plane for \( q_x = k_x \).](image1)

For larger amplitudes of the pump wave, both the up-shifted and down-shifted sidebands are excited. For \( q_x = q_y \hat{y} \) and \( k = k_x \hat{x} \) the dispersion relation Eq. (26) can be solved analytically, giving

\[
\Omega^2 = \frac{\Omega_A^2 + \delta_x^2}{2} \pm \left[ \left( \frac{\Omega_A^2 - \delta_x^2}{2} \right)^2 + \Omega_A^2 \Gamma \right]^{1/2},
\]

where

\[
\Gamma = \frac{\omega_{\text{LH}}^2 q_x^2 \lambda_x^2}{8} E_0^2 \frac{q_x^2}{k_x^2} \left( \frac{q_x^2 \lambda_x^2}{E_{\text{TH}}^2} + \frac{q_x^2 m_i}{q_x^2 \lambda_x^2} \right)
\]

and \( \delta_x = (\omega_{\text{LH}}/2) [q_x^2 \rho_i^2 + (q_x^2 / q_x^2)(m_i / m_e)] \). As seen from the solution in Eq. (31), there is a purely growing instability for sufficiently large \( E_0 \) and for a limited range of \( q_x \). The insta-
plasma parameters as used in Fig. 1. The wave equation (20) governing the LH can be simplified further. Namely, the time scales of the LH and DA waves are well separated, and we can use the WKB representation $\phi_l = \phi_L(x,t)\exp(-i\omega_{1,2}t) + c.c.$ With this representation, Eq. (20) can be reduced to an equation describing the envelope function $\tilde{\phi}_k$. We have

$$\bar{L}_k \tilde{\phi}_k = i\frac{\omega_{1,2}}{c_i} (\nabla \tilde{\phi}_k \times \nabla \eta)_z,$$

(33)

where $\bar{L}_k = -(2i/\omega_{1,2}) \bar{q}^2 \nabla^2 - \bar{q}^2 \nabla^2 + (m_i/m_e) \bar{q}^2.$ A pseudospectral method was used to approximate the spatial derivatives on a $64 \times 64 \times 64$ spatial grid and the solution was advanced in time using a standard fourth-order Runge–Kutta method. A monochromatic LH wave with the wave vector $k = (2\pi/20)\hat{r}$ m$^{-1}$ and amplitude $E_0$ was given as an initial condition. For the DA wave, low-amplitude noise was given as an initial condition. By considering the two cases $E_0 = 0.5$ mV/m and $E_0 = 6$ mV/m, we have investigated the nonlinear evolution of the three-wave decay instability and the modified decay instability.

A. Low-amplitude LH pump

The three-wave decay instability is investigated by solving the governing equations with a small-amplitude LH pump, we take $E_0 = 0.5$ mV/m. Here, we observed that the numerical solution exhibits the three-wave decay instability, illustrated in Fig. 2. The growth rate of the fastest growing mode is in agreement with the growth rate shown in Fig. 2. Due to depletion of the pump wave, the instability ceased to grow at $\Omega_{Lh} = 30$ ($t = 6$ s). No further instabilities were observed before the end of the simulation at $\Omega_{Lh} = 60$.

B. Large-amplitude LH pump

In order to investigate the nonlinear evolution of the modified decay instability, we solve the governing equations (19) and (20) for a strong LH pump wave with the amplitude $E_0 = 6$ mV/m. The initial stage is dominated by the modified decay instability and the growth rate of the fastest growing mode agrees with the growth rate shown in Fig. 2. Figure 5 displays the result of our simulation. Panels (a)–(c) show the time evolution of $\eta$ in the $x$–$y$ plane for a fix $z$. It is clearly seen in the panel (a) that the DA waves with $q_x \perp k \parallel \hat{x}$ are excited by the purely growing instability. At $\omega_{1,2}t = 4.2 \times 10^3$, the amplitude of $\eta$ has grown to $\sim 0.1\%$ and regions with relatively large $\eta$ and large electric fields have been formed, see panels (b) and (c). The electric field inside the density perturbation is enhanced by roughly one order of magnitude at the time illustrated in the panel (c). Panel (d) shows the structure of the density fluctuation along $B_0$ in the $y$–$z$ plane for $x/\lambda = 6$. For larger $t$ the amplitude of the density fluctuations continue to grow and the spatial size decreases. The spatial size of the structure became compatible with the grid size at $\omega_{1,2}t = 5.0 \times 10^4$, and the simulation was halted. We did not observe any stationary state of the structure formation. The collapsing structures accompany a
sheared magnetic field. However, the collapse is essentially electrostatic, as the perpendicular length scale of the structure is \( \sim \rho_e \) and much smaller than \( \lambda_e \).

V. SUMMARY

In the present paper, we have considered the nonlinear interaction between LH and DA waves. We have derived a set of equations describing the parametric interaction between the two wave modes. The obtained set of equations describes the excitation of dispersive inertial Alfvén waves and kinetic Alfvén waves. In addition to the nonlinear coupling mechanisms presented in Ref. 21, we have included a scalar nonlinearity due to the electron polarization current previously considered in Refs. 19 and 22. The vector and scalar nonlinearities as well as self-interaction nonlinearities are treated on an equal footing, and the derived equations can be used in future large-scale computer simulations to study the LH-DA wave turbulence. In the present treatment, the governing equations have been analyzed both analytically and numerically. By solving the governing equations numerically, we have shown that small scale structures, i.e., smaller than the wavelength of the LH pump wave, can be generated. The scale length of the excited waves is much shorter than \( \lambda_e \), and effects due to the parallel kinetics must be included. The present results thus offer a clue to the generation of current filaments due to the presence of large amplitude LH waves in the Earth's ionospheric as well as in laboratory experiments.

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