A third-order topological invariant for three magnetic fields

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If you notice a third-order link
You can colour it green, brown and pink;
Invariants follows,
As hills follow hollow;
So a field triply-linked cannot shrink. – HKM 2001

Abstract The topology of divergence-free fields is important in many parts of physics, e.g. in magnetohydrodynamics, plasma physics, hydrodynamics, superfluids etc. With the focus on applications in magnetohydrodynamics, our principal aim is the characterisation of magnetic fields by the means of invariants.

In this report an introduction to the problem of finding higher order invariants is given. Then a third-order link integral of three magnetic fields is presented, which can be shown to be a topological invariant and therefore an invariant in ideal magnetohydrodynamics. This integral generalises the known third-order link invariant derived from the Massey triple product, which could only be applied to isolated flux tubes. As an example three magnetic fields not confined to flux tubes are given that possess a third-order linkage.

1. Introduction

In recent years topological considerations have become increasingly important in the study of physical problems. In plasma physics for example the topology of magnetic fields plays an important role when dealing with problems such as the stability and time evolution of a plasma or when estimating its energy content.

The structure of magnetic fields is studied with the aid of topological measures of complexity, i.e. measures which are invariant under diffeo-

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morphic isotopies of the magnetic field. The simplest of such measures is magnetic helicity. For a magnetic field $\mathbf{B}$ with vector potential $\mathbf{A}$ it is defined as the volume integral

$$H^{(2)}(\mathbf{B}) = \int_V \mathbf{A} \cdot \mathbf{B}\, d^3x,$$

which is gauge invariant if $\mathbf{B} \cdot \mathbf{n}|_{\partial V} = 0$. Magnetic helicity measures the total mutual linkage of magnetic flux (see Moffatt 1969). In ideal magnetohydrodynamics, i.e. in the regime of large magnetic Reynolds numbers, the magnetic field is frozen into the plasma flow and magnetic helicity is a conserved quantity. Since it is quadratic in magnetic flux it is often referred to as a second-order topological invariant. As examples the Hopf link and the double twisted torus are shown in the following figure, which both have the same magnetic helicity of $H^{(2)} = 2\phi_0^2$.

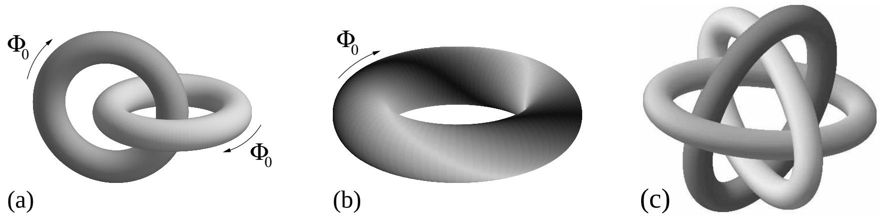

Figure 1. (a) Hopf link, (b) double twisted torus and (c) Borromean rings

It has been known for a long time that magnetic field configurations exist which are non-trivially linked, but for which the magnetic helicity vanishes. Helicity e.g. fails to detect the interlocking of three flux tubes in the form of the Borromean rings (Figure 1c). This raises the question whether higher order invariants exist which detect these linkages.

2. Higher order invariants

In knot theory different invariants are known which distinguish for example the Borromean rings from unlinked rings. To use these invariants e.g. to measure the complexity of magnetic fields, they have to be expressed in terms of physical quantities, i.e. in terms of the magnetic field $\mathbf{B}$. For most knot invariants a translation into a physical setting is unknown and still no higher order invariant $H^{(n)}(\mathbf{B})$ with $n > 2$ has been found which is applicable to an arbitrary magnetic field $\mathbf{B}$. We will see that matters simplify, if the field splits into m isolated flux tubes U_i, each with magnetic field B^i. In this case $H^{(n)}$ is to be considered as an invariant $H^{(n)}(\mathbf{B}^1, \ldots, \mathbf{B}^m)$ which measures the n-th order *cross-linkage* of m distinct magnetic flux tubes.

It was recognised by Monastyrsky & Retakh (1986) and independently by Berger (1990) and Evans & Berger (1992) that the link invariants based on so-called Massey higher products (see Massey 1958, 1969) can be written as invariants applicable to magnetic flux tubes. Massey higher products in fact yield a hierarchy of higher order invariants, which are expressible as cross-linkage integrals of the above mentioned type. Unfortunately, their usage is restricted to magnetic fields confined to flux tubes which must not possess a linkage lower than the linkage which is measured. The situation is summarised in the following table

Total-linkage	*Cross-linkage*
$H^{(2)}(\mathbf{B}) = \int_V \mathbf{A}\cdot\mathbf{B}\; d^3x$	$H^{(2)}(\mathbf{B}_1,\mathbf{B}_2) = \int_V \mathbf{A}_1\cdot\mathbf{B}_2\; d^3x$ cross-helicity for arbitrary fields
$H^{(3)}(\mathbf{B}) = \boxed{?}$	$H^{(3)}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3)$ fields restricted to flux tubes with no $2nd$ order linkage
$H^{(4)}(\mathbf{B}) = \boxed{?}$	$H^{(4)}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3,\mathbf{B}_4)$ fields restricted to flux tubes with no $2nd$ or $3rd$ order linkage

where $H^{(3)}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3)$ and $H^{(4)}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3,\mathbf{B}_4)$ are constructed respectively from the Massey triple and quadruple product.

3. Generalisation of the third-order invariant

In this section we generalise the third-order invariant known from the Massey triple product to an invariant of three magnetic fields not confined to flux tubes.

Theorem: *Let $\mathbf{B}^1$, $\mathbf{B}^2$ and $\mathbf{B}^3$ be three magnetic fields with potentials $\mathbf{A}^i$, and let $V \subset \mathbb{R}^3$. In an ideal dynamics, i.e. with $\mathbf{E}^i + \mathbf{v}\times\mathbf{B}^i = 0$ in all three components, the integral*

$$H^{(3)}(\mathbf{B}^1,\mathbf{B}^2,\mathbf{B}^3) = \int_V \mathbf{A}^1\cdot\mathbf{A}^2\times\mathbf{A}^3\; d^3x \qquad (1)$$

is a gauge invariant, conserved quantity, if the three potentials obey (i) $\nabla\cdot(\mathbf{A}^i\times\mathbf{A}^j) = 0$ for all $i,j = 1,2,3$, and (ii) the boundary condition $\mathbf{A}^i\times\mathbf{n}|_{\partial V} = 0$ for $\mathbf{n}$ being the normal vector to the boundary of the integration volume V.

A complete proof of this theorem is given in Hornig & Mayer (2001) and only the main ideas shall be presented here: For an arbitrary gauge field $F = dA - iqA \wedge A$ we have the following general identity

$$tr(F \wedge F) = d\, tr(A \wedge dA - iq\frac{2}{3}A \wedge A \wedge A), \qquad (2)$$

where the term on the left is known as the second Chern class and the three-form on the right as the Chern-Simons three-form.

For the gauge group $U(1)$, which yields electrodynamics on $\mathbb{M}^4$, identity (2), together with an ideal dynamics, i.e. with $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ leads to the continuity equation for helicity density

$$\partial_t\left(\mathbf{A} \cdot \mathbf{B}\right) + \nabla \cdot \left(\mathbf{v}\left(\mathbf{A} \cdot \mathbf{B}\right)\right) = \nabla \cdot \left((-A_0 + \mathbf{A} \cdot \mathbf{v})\mathbf{B}\right).$$

This proves the well known result that the integral over helicity density, for the boundary condition $\mathbf{B} \cdot \mathbf{n}|_{\partial V} = 0$, is a conserved quantity.

Using instead of the group $U(1)$ the gauge group $SU(2)$, the potential A naturally has three components $\sigma_i A^i = A$. A summation over repeated indices is understood. Here the Pauli matrices were chosen as a basis for the Lie algebra $su(2)$. Interpreted as three independent potentials of electromagnetic fields $F^i_{EM} = dA^i$, identity (2) together with $\mathbf{E}^i + \mathbf{v} \times \mathbf{B}^i = 0$, leads (i) to the helicity conservation of $\mathbf{B} = \mathbf{B}^1 + \mathbf{B}^2 + \mathbf{B}^3$ and (ii) to a new continuity equation, which with $\mathbf{G}^1 = \mathbf{A}^2 \times \mathbf{A}^3$ and cyclic permutations of it, reads

$$\partial_t(\mathbf{A}^1 \cdot \mathbf{A}^2 \times \mathbf{A}^3) + \nabla \cdot (\mathbf{v}(\mathbf{A}^1 \cdot \mathbf{A}^2 \times \mathbf{A}^3)) = \nabla \cdot (\tfrac{1}{3}\mathbf{v}(\mathbf{A}^i \cdot \mathbf{G}^i) - A^i_0 \mathbf{G}^i).$$

Analogous to the case of helicity this continuity equation yields a conservation law for the density $\mathbf{A}^1 \cdot \mathbf{A}^2 \times \mathbf{A}^3$. Together with the gauge invariance (see Hornig & Mayer 2001) this proves the above theorem.

Let us note that the presented derivation reveals a close relationship between magnetic helicity and the generalised third-order invariant. In fact, both conservation laws result from the same identity (2).

In the special case of three fields confined to flux tubes, condition (i) of the theorem implies that the tubes must be mutually unlinked. It can easily be confirmed that in this case integral (1) coincides with expressions derived from the Massey triple product given by Monastyrsky & Retakh (1986), Berger (1990) and Ruzmaikin & Akhmetiev (1994).

Finally we want to remark, that the third-order invariant can be interpreted as the cross-helicity

$$H^{(3)}(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3) = \int_V \mathbf{A}^1 \cdot \mathbf{A}^2 \times \mathbf{A}^3 d^3x = H^{(2)}(\mathbf{B}^i, \mathbf{G}^i)$$

of the pair of divergence-free fields $\mathbf{B}^i$ and $\mathbf{G}^i$ for each $i = 1, 2, 3$.

4. Example magnetic field

We know that the requirement $\nabla \cdot (\mathbf{A}^i \times \mathbf{A}^l) = 0$ of the above theorem can always be satisfied for fields confined to mutually unlinked flux tubes. However, it is not easily satisfied for three arbitrary fields. In the following Yang-Mills theory is used to construct a natural example of three magnetic fields not confined to flux tubes, to which the invariant $H^{(3)}$ can be applied.

In an $SU(2)$ gauge symmetry the field strength $F = dA - iqA \wedge A$ has components $F^i = dA^i + q\,\epsilon_{ijk} A^j \wedge A^k$, for $i = 1, 2, 3$. Taking the exterior derivative we observe that $dF = 0$ is sufficient for all $d(A^i \wedge A^j)$ to vanish. These potential components meet the above requirement and therefore shall be interpreted as three potentials of independent electromagnetic fields. Since $dF = 0$ is trivially fulfilled for $F = 0$, i.e. for a pure gauge Yang-Mills field, we are now going to construct such a field on $\mathbb{R}^3$, a time slice of $\mathbb{R}^4$. Using the mapping $g : \mathbb{R}^3 \to SU(2)$

$$g(\mathbf{x}) = \exp\Big(\frac{i\pi\; x^i \sigma_i}{\sqrt{||\mathbf{x}||^2 + \lambda^2}}\Big),$$

we perform a gauge transformation $\sigma_i A^i \to \sigma_i A'^{\,i} = g^{-1}(\mathbf{x})(\sigma_i A^i) g(\mathbf{x}) + \frac{1}{q} i g^{-1}(\mathbf{x}) dg(\mathbf{x})$, from the classical vacuum $A^i = 0$ to

$$A^i \sigma_i = \tfrac{1}{q} i g^{-1}(\mathbf{x}) dg(\mathbf{x}), \qquad \text{with } q \neq 0.$$

This yields the potential of a vacuum Yang-Mills field (see Hornig & Mayer 2001) with a winding number of the vacuum of 1 (see Itzykson-Zuber 1980). The three independent magnetic fields, calculated from $\mathbf{B}^i = \mathrm{curl}(\mathbf{A}^i)$, are well defined and decay sufficiently with $||\mathbf{B}^i|| \to r^{-4}$

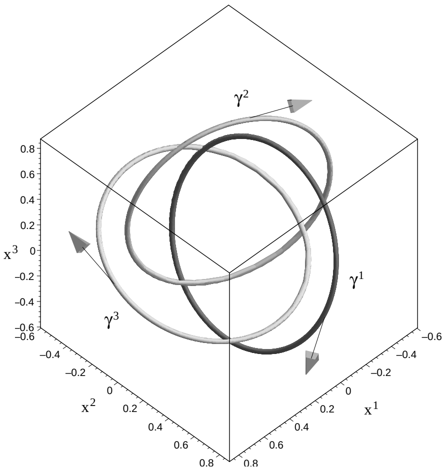

Figure 2. Three numerically integrated field lines are shown. Each field line γ^i belongs to the field $\mathbf{B}^i$. The symmetrically chosen integration starting points are indicated by the arrow positions. All field lines are mutually linked and their total Gauß linking number is -3.

for large radii. As is shown in Hornig & Mayer (2001) the non-vanishing winding number of the Yang-Mills vacuum directly implies a non-trivial third-order linkage. Explicitly calculating integral (1) we find

$$H^{(3)}(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3) = -16\pi^2(2/q)^3.$$

The constructed magnetic fields are in fact highly symmetric. Each field $\mathbf{B}^i$ is invariant with respect to rotations about the $\mathbf{x}^i$-axis and all three fields are similar in that they can be obtained from one field by rotations which map the axis of symmetry on one another. As seen in Figure (2) all field lines are closed and have an elliptical shape. Even though all cross-helicities vanish, individual field lines among different fields are in general linked. To be more precise, every pair of such field lines is either linked exactly once or intersects twice.

5. Conclusions

An integral expression is given which generalises the third-order invariant known from the Massey triple product. It is interesting that in the presented derivation helicity and $H^{(3)}$ are the result of the same identity. An example shows that the new invariant $H^{(3)}$ is a real generalisation.

Still an open question is whether a third-order invariant for the "total-linkage" of a single arbitrary magnetic field exists and whether it can be constructed from the "cross-linkage" $H^{(3)}(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3)$ similar to the case of helicity. There might e.g. exist a subdivision of a single magnetic field into three components, such that $H^{(3)}(\mathbf{B}) = H^{(3)}(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3)$. Unfortunately, the antisymmetry of the "cross-linkage" $H^{(3)}(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3)$ seems to be one of the key problems for a further generalisation analogous to helicity.

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