Kinetic theory and transport processes for a test-particle in magnetized plasma [1]

Yannis KOURAKIS

U.L.B. Université Libre de Bruxelles

Physique des Plasmas / Association Euratom - Etat Belge

C.P. 231, boulevard du Triomphe, B-1050 Brussels, Belgium

email: ikouraki@ulb.ac.be

Thesis work with: A. Grecos, D. Carati, B. Weyssow & L. Brenig.

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1 Introduction - Motivation of the problem

- *Plasma*: long - range interactions;

LANDAU description for *homogeneous, electrostatic* plasma [2]:

$$\frac{\partial f(\mathbf{v};t)}{\partial t} + \frac{1}{m}\,\mathbf{F}_{mf}\,\frac{\partial f(\mathbf{v};t)}{\partial \mathbf{v}} = \mathcal{C}_{Landau}\{f\}$$

$\mathbf{F}_{mf}$ *should* be the mean-field (VLASOV) force [3].

1st Question:

What if an external force field is present ?

(e.g. Lorentz forces in plasma)

- *Anti-paradigm 1*:

Extrapolation: **F** may represent an external force ("by the hand") - NOT rigorously considering the *influence of the field on the collision term*:

$$\frac{\partial f(\mathbf{v};t)}{\partial t} + \frac{1}{m}\,\mathbf{F}_{external}\,\frac{\partial f(\mathbf{v};t)}{\partial \mathbf{v}} = \mathcal{C}_{Landau}\{f\}$$

2nd Question:

What if the d.f. is NOT uniform i.e. $f = f(\mathbf{x}, \mathbf{v};\, t)$ *?*

- *Anti-paradigm 2*:

Phenomenological generalizations of the Landau equation:

$$\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial t} + \mathbf{v}\,\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial \mathbf{x}} + \frac{1}{m}\,\mathbf{F}\,\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial \mathbf{v}} = \mathcal{C}_{Landau}\{f\}$$

(*same* rhs, *no field*)

---

[2] up to λ^2 (in the interaction).

[3] The Vlasov term cancels in *our* model; see below.

- thus, we're after a *plasma kinetic equation* in the form:

$$\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial t} + \mathbf{v}\,\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial \mathbf{x}} + \frac{1}{m}\,\mathbf{F}\frac{\partial f(\mathbf{x},\mathbf{v};t)}{\partial \mathbf{v}} = \mathcal{C}\{f;\, interactions;\, field\}$$

2 The model

Two subsystems, weakly coupled to each other:

- a 'tagged' test-particle Σ

and

- a heat bath (the "reservoir" R): N particles (species α_j)

AND

+ an external magnetic field:

$$\mathbf{B} = B\,\hat{z}$$

(uniform & stationary)

2.1 Equations of motion for the test-particle

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \frac{1}{m}[\mathbf{F_{ext}}(\mathbf{x},\mathbf{v};t) + \lambda\,\mathbf{F_{int}}(\mathbf{x},\mathbf{v};\mathbf{X_R};t)] \end{aligned} \tag{1}$$

* $\mathbf{F_{ext}}(\mathbf{x},\mathbf{v};t)$ is due to the external field; e.g. for a uniform magnetic field along the z-axis:

$$\mathbf{F_{ext}}(\mathbf{x},\mathbf{v};t) = \frac{e}{mc}(\mathbf{v}\times\mathbf{B})$$

* $\mathbf{F_{int}}(\mathbf{x},\mathbf{v};\mathbf{X_R};t)$ denotes the "*stochastic*" force due to interactions of the test-particle with the reservoir particles surrounding it i.e.

$$\begin{aligned} \mathbf{F_{int}}(\mathbf{x},\mathbf{v};\mathbf{X_R};t) &= \sum_{j=1}^{N} \mathbf{F_{intj}}(|\mathbf{x}(t)-\mathbf{x_j}(t)|) \\ &= -\sum_{j=1}^{N} \frac{\partial V(|\mathbf{x}(t)-\mathbf{x_j}(t)|)}{\partial \mathbf{x}(t)} \end{aligned}$$

The "random" force $\mathbf{F_{int}}(t)$ is thus described by a *Gaussian* process, determined by a vanishing mean-value:

$$\langle \mathbf{F_{int}}(t)\rangle_R \equiv I\!\!E\{\mathbf{F_{int}}(\mathbf{x},\mathbf{v};\ t)\} = 0$$

and the two-time correlation function:

$$\underline{\underline{\mathbf{C}}} = \underline{\underline{\mathbf{C}}}(\mathbf{x},\mathbf{v};t_1,t_2) = \langle \mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2)\rangle_R \equiv I\!\!E\,\{\mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2)\}$$

$$\equiv \int_{\Gamma_R} d\mathbf{X}_R\ \sigma_R(\mathbf{X}_R)\,\mathbf{F_{int}}(t_1)\,\mathbf{F_{int}}(t_2)$$

* The interaction potential V is assumed to be a long-range electrostatic potential $V(r)$.

2.2 Solution of the dynamical problem for the "free" particle

* Simple case: $\mathbf{B} = B\,\hat{z}$

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \frac{e}{mc}\,(\mathbf{v}\times\mathbf{B})\end{pmatrix} \equiv \begin{pmatrix}\mathbf{v}\\ \Omega\,(\mathbf{v}\,\times\hat{b})\end{pmatrix}$$

(known) exact solution (helicoidal motion):

$$\begin{aligned}
x(t) &= x + \Omega^{-1}\,v_x sin\Omega t + s\,\Omega^{-1} v_y(1 - cos\Omega t)\\
y(t) &= y - s\,\Omega^{-1}\,v_x\,(1 - cos\Omega t) + \Omega^{-1}\,v_y\,sin\Omega t\\
z(t) &= z + v_z\,t\\
v_x(t) &= v_x\,cos\Omega t + s\,v_x\,sin\Omega t\\
v_y(t) &= -\,s\,v_x\,sin\Omega t + v_y\,cos\Omega t\\
v_z(t) &= v_z = const.
\end{aligned}$$

i.e.

$$\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{pmatrix} = \begin{pmatrix}\underline{\underline{\mathbf{I}}} & \underline{\underline{\mathbf{N}}}(t)\\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{R}}}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} \tag{2}$$

that is:

$$\underline{\underline{\mathbf{M}}}(t) = \underline{\underline{\mathbf{I}}}, \qquad \underline{\underline{\mathbf{M}}}'(t) = \underline{\underline{\mathbf{0}}}$$

$$\underline{\underline{\mathbf{N}}}'(t) = \underline{\underline{\mathbf{R}}}^{\alpha}(t) = \begin{pmatrix}\cos\Omega t & s\ \sin\Omega t & 0\\ -s\ \sin\Omega t & \cos\Omega t & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\underline{\underline{\mathbf{N}}}(t) = \int_0^t dt'\ \underline{\underline{\mathbf{R}}}^{\alpha}(t) = \Omega^{-1}\begin{pmatrix}\sin\Omega t & s\ (1-\cos\Omega t) & 0\\ s\ (\cos\Omega t - 1) & \sin\Omega t & 0\\ 0 & 0 & \Omega t\end{pmatrix} \tag{3}$$

* Definitions: $\Omega = \Omega_j \equiv \frac{|e_\alpha| B}{m_\alpha c}$, $\qquad s^j = sgn(e_j) = \pm 1.$

3 Statistical description - the Generalized Master Equation

* <u>*weak-coupling approximation*</u>: the mutual interaction between particles is taken to be very small), as compared to their average kinetic energy (i.e. λ ($\lambda \ll 1$).

* <u>*(Non-markovian) "Generalized Master Equation" (GME)*</u> to 2nd order in the interaction:

$$\begin{aligned} \partial_t f(\mathbf{X};t) &= L_0\, f(\mathbf{X};t) \\ &+ \lambda^2 \sum_{\alpha'} n_{\alpha'} \int_0^t d\tau \int_\Gamma d\mathbf{X_1}\; L'_{\Sigma 1}\; U^0(\tau)\; L'_{\Sigma 1}\; \phi_{eq}^{\alpha'}(\mathbf{X_1^{\alpha'}})\; f(\mathbf{X};t-\tau) \end{aligned} \tag{4}$$

* $U^0(\tau) \;=\; e^{L_0 \tau}$

* Rem.: the mean-field *(Vlasov)* term, in order λ^1, disappears once we take the reservoir state to be homogeneous);

* L_0 is the *"free"* Liouville operator defined previously;

the *binary interaction* Liouville operator $L'_{\Sigma 1}$ is given by:

$$\begin{aligned} L'_{\Sigma 1} &= -\mathbf{F_{int}}(|\mathbf{x_\Sigma} - \mathbf{x_1}|)\Big(\frac{1}{m_\Sigma}\frac{\partial}{\partial \mathbf{v_\Sigma}} - \frac{1}{m_1}\frac{\partial}{\partial \mathbf{v_1}}\Big) \\ &\equiv \frac{\partial V(|\mathbf{x}-\mathbf{x_1}|)}{\partial \mathbf{x}}\Big(\frac{1}{m}\frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_1}\frac{\partial}{\partial \mathbf{v_1}}\Big) \end{aligned} \tag{5}$$

* <u>Problem:</u> (*'Which'*) Markovian approximation (?)

Two methods to be tested in the following...

(see Poster for formal details)

4 Quasi-markovian approximation: the Θ- operator

4.1 Quasi-markovian master equation

"WILD" Markovian assumption:

$$f(t-\tau) \approx U^0(-\tau)\, f(t)$$

so (4) becomes the *'quasi-markovian' Master Equation*:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} =$$

$$n_{\alpha'} \int_0^t d\tau \int_\Gamma d\mathbf{X_1}\; L'_{\Sigma 1}\; U(\tau)\; L'_{\Sigma 1}\; U(-\tau)\; \phi_{eq}^{\alpha'}(\mathbf{v_1^{\alpha'}})\; f(\mathbf{x},\mathbf{v};t)$$

$$= L_0 f(\mathbf{X}) + \int_0^t d\tau\, \mathcal{K}(\tau)\, f(\mathbf{X}) \quad \equiv \quad \Theta_2(t)\, f \quad (6)$$

* Rem.: the interaction Liouville operator L_I is the binary interaction operator $L_{\Sigma,1}$ (cf. definition (5)).

* Important ! The propagator has to be evaluated *by taking into account the external field*; $U(t)$ does *not* commute with the velocity gradient $\frac{\partial}{\partial \mathbf{v}}$. Indeed, if we define:

$$\underline{\mathbf{D}}_{\mathbf{V_i}}(t) \equiv U(t)\frac{\partial}{\partial \mathbf{v_i}}U(-t) \qquad i = \Sigma,\ 1^R$$

(note that $\underline{\mathbf{D}}_{\mathbf{V_i}}(0) = \frac{\partial}{\partial \mathbf{v_i}}$)

we find:

$$\underline{\mathbf{D}}_{\mathbf{V_i}}(t) = \underline{\underline{\mathbf{N_i}}}^{\mathbf{T}}(t)\ \frac{\partial}{\partial \mathbf{x_i}} + \underline{\underline{\mathbf{N}'_\mathbf{i}}}^{\mathbf{T}}(t)\ \frac{\partial}{\partial \mathbf{v_i}} \qquad (7)$$

(obviously: $\underline{\mathbf{D}}_{\mathbf{V_1}}(t)\ \phi(\mathbf{v_1}) = \underline{\underline{\mathbf{N}'}}^{\mathbf{T}}(t)\ \frac{\partial \phi(\mathbf{v_1})}{\partial \mathbf{v_1}}$);

(cf. [Misguich et al.1975]).

4.2 "Quasi-markovian" Fokker-Planck equation

The following PDE is obtained:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}[\mathbf{A}(\mathbf{x},\mathbf{v})\frac{\partial}{\partial \mathbf{v}} + \mathbf{B}(\mathbf{x},\mathbf{v})\frac{\partial}{\partial \mathbf{x}} + \mu\,\mathbf{a}(\mathbf{x},\mathbf{v})]\,f \tag{8}$$

$(f = f(\mathbf{x},\mathbf{v};t); \quad \mu \equiv m/m_1^{\alpha'})$.

* After an algebraic manipulation, the QMFPE (8) takes the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{q}}(\underline{\mathcal{F}}^{\mathcal{O}}\,f) + \frac{\partial}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{q}} : (\underline{\underline{\mathbf{D}}}^{\mathcal{O}}\,f) \tag{9}$$

where $q \equiv (\mathbf{x},\mathbf{v})$.

The 6x6 *diffusion matrix* is:

$$\underline{\underline{\mathbf{D}}}^{\mathcal{O}}(\mathbf{x},\mathbf{v}) = \begin{pmatrix} \underline{\underline{\mathbf{0}}} & \frac{1}{2}\underline{\underline{\mathbf{B}}}^{\mathbf{T}} \\ \frac{1}{2}\underline{\underline{\mathbf{B}}} & \underline{\underline{\mathbf{A}}} \end{pmatrix} \tag{10}$$

and the 6d *dynamical friction vector* $\mathcal{F}^{\mathcal{O}}$ reads: $\mathcal{F}^{\mathcal{O}} = (\mathbf{0},\mathbf{F})^T$ where $\mathbf{F}$ is the 3d vector defined by:

$$F_i = -\mu\,a_i + \frac{\partial A_{ij}}{\partial v_j} + \frac{\partial B_{ij}}{\partial x_j} \qquad i,j = 1,2,3$$

* In the *homogeneous case*:

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}(A_{ij}\,f) - \frac{\partial}{\partial v_i}(F_i\,f) \tag{9-HOM}$$

$(f = f(\mathbf{v};t))$

4.3 Coefficients

$$
\begin{aligned}
\left\{ \begin{array}{c} \underline{\underline{\mathbf{A}}}(\mathbf{x},\mathbf{v}) \\ \underline{\underline{\mathbf{B}}}(\mathbf{x},\mathbf{v}) \end{array} \right\} &= \frac{n}{m^2}\int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \\
&\qquad \mathbf{F_{int}}(|\mathbf{x}-\mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau)-\mathbf{x_1}(-\tau)|) \left\{ \begin{array}{c} \underline{\underline{\mathbf{R}}}^{\mathbf{T}}(\tau) \\ \underline{\underline{\mathbf{N}}}^{\mathbf{T}}(\tau) \end{array} \right\} \\
&= \frac{n}{m^2}\int_0^\infty d\tau\, \underline{\underline{\mathbf{C}}}(\mathbf{x},\mathbf{v};t,t-\tau) \left\{ \begin{array}{c} \underline{\underline{\mathbf{R}}}^{\mathbf{T}}(\tau) \\ \underline{\underline{\mathbf{N}}}^{\mathbf{T}}(\tau) \end{array} \right\} \\
\underline{\mathbf{a}}(\mathbf{x},\mathbf{v}) &= -\frac{n}{m^2}\int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \\
&\qquad \mathbf{F_{int}}(|\mathbf{x}-\mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau)-\mathbf{x_1}(-\tau)|)\, \underline{\underline{\mathbf{N}}}'^{\mathbf{T}}_{\mathbf{1}}(\tau) \frac{\partial \phi(\mathbf{v_1})}{\partial \mathbf{v_1}} \\
&= -\frac{n}{m^2}\int_0^\infty d\tau\, \underline{\mathbf{d}}(\mathbf{x},\mathbf{v};t,t-\tau) \qquad (11)
\end{aligned}
$$

4.4 Problem: The positivity issue !

* Properties: The K.E. should preserve (i) the reality, (ii) the normalization and (iii) the positivity of the (probability) distribution function. Furthermore, (iv) an H-theorem should be satisfied.

* The diffusion matrice $\underline{\underline{\mathbf{D}}}$ should be positive definite, i.e. one should have, for any $\mathbf{a} \in \Re^6$:

$$(\mathbf{a}, \underline{\underline{\mathbf{D}}}\mathbf{a}) = \mathbf{a}^T \underline{\underline{\mathbf{D}}}\mathbf{a} = \mathbf{a}^T \underline{\underline{\mathbf{D}}}^{SYM}\mathbf{a} \ \geq \ 0$$

This criterion is definitely *not* satisfied here, as $det\, \underline{\underline{\mathbf{D}}}^{\mathcal{O}} = -(det \underline{\underline{\mathbf{C}}})^2 \leq 0$.

* Consequense: the Quasi-Markovian F.P. equation (8) *does not guarantee the preservation of the positivity* of the probability distribution function f.

5 Towards a "Markovian" kinetic equation...

* Consider the operator Φ:

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt'\, U(-t')\, \mathcal{O}\, U(t') \tag{12}$$

5.1 (i) the homogeneous case

* The "old" $\mathcal{O}$- and the "new" Φ-operators coincide in this case!

5.2 M-FP equation: (ii) the general (inhomogeneous) case

* The Markovian FPE reads:

$$\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\frac{\partial f}{\partial \mathbf{v}} &= \\
&= \frac{\partial^2}{\partial v_i \partial v_j}[A_{ij}^{SYM}(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)] \\
&\quad +2\frac{\partial^2}{\partial v_i \partial x_j}[D_{VXij}(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)] \\
&\quad +\frac{\partial^2}{\partial x_i \partial x_j}[D_{XXij}(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)] \\
&\quad -\frac{\partial}{\partial v_i}\Big[\mathcal{F}_{(V)}^{(\Phi)}(\mathbf{v})\Big] f(\mathbf{x},\mathbf{v};t) \\
&\quad -\Big[\mathcal{F}_{(X)}^{(\Phi)}(\mathbf{v})\Big]\frac{\partial}{\partial x_i} f(\mathbf{x},\mathbf{v};t)
\end{aligned} \tag{13}$$

(see Poster).

*

6 Coefficients

The diffusion coefficients in the FPE are defined by:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} \frac{1}{m_{\alpha'}^2} \int_0^t d\tau \left\{ \begin{array}{c} \left\{ C_\perp^{\alpha,\alpha'} \right\} \\ C_\parallel^{\alpha,\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\Omega^\alpha\tau \\ (-s^\alpha)\,\frac{1}{2}\,\sin\Omega^\alpha\tau \\ (1+\frac{1}{2}\cos\Omega^\alpha\tau) \end{array} \right\} \\ 1 \end{array} \right\} \tag{14}$$

where $C_{\{\perp,\parallel\}}^{\alpha,\alpha'}(v_\perp, v_\perp; \Omega)$ are elements of the (diagonal) force-correlation matrix $\mathbf{C}(\tau) = \langle \mathbf{F_{int}}(t)\; \mathbf{F_{int}}(t-\tau)\rangle_R$; they are given by:

$$C_{\{\perp,\parallel\}} = n_{\alpha'}(2\pi)^3 \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^\alpha(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\, k_{\{\perp,\parallel\}}^2 \tag{15}$$

(a summation over n, m is understood) where v_i ($v_{1,i}$), $i = 1, 2, 3$ denote the velocity coordinates of the test- (R-) particle and $\tilde{V}_k$ stands for the Fourier transform of $V(r)$.

Remember that $V = V(|\mathbf{r}|) = V(r)$ implies $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$. The dynamical friction terms in (3) are given by:

$$\mathcal{F}_x = (1+\mu)\,(\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y}) \qquad \mathcal{F}_y = (1+\mu)\,(-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y})$$
$$\mathcal{F}_z = (1+\mu)\,\frac{\partial D_\parallel}{\partial v_z} \tag{16}$$

($\mu = m_\alpha/m_{\alpha'}$) [1]. Note the explicit dependence on the magnetic field as well as on the form of the reservoir equilibrium d.f. $\phi_{eq} = \phi_{eq}(v_\perp, v_\parallel)$ and the interaction potential $V(r)$.

6.1 Non-dimensional expressions

6.1.1 Correlations

Notice that the above relations imply a set of expressions for the force correlation functions $C^{\alpha}_{\{\perp,\|\}}(\tau)$, readily obtained by comparing (9), (10) to (5). The integration variable $k_\perp$ therein can be re-scaled to the non-dimensional variable: $x = \frac{\tilde{k}_\perp}{k_D} = (1 + \frac{k_\perp^2}{k_D^2})^{1/2}$; relation (6) thus becomes:

$$C^{\alpha}_{\{\perp,\|\}}(\tau) \;=\; 4\,n\,e^4\,k_D \int_1^{x_{max}} dx\; e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\Omega\tau}{2}} \Big(1 - \frac{1}{x^2}\Big)^{\{1,0\}} \\ J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\Omega\tau}{2})\;\tilde{F}_{\{\perp,\|\}} \tag{17}$$

$\tilde{F} = \tilde{F}(\phi(x,\tau), \tilde{v}_\|)$ is given by:

$$\tilde{F}^{\alpha'}_{\{\perp,\|\}} = \pm\sqrt{\pi}\,\phi + \frac{\pi}{4} \sum_{s=+1,-1} \Big[(1 \mp 2\,\phi^2 \mp s2\,\phi\,\tilde{v}_\|)\,e^{(\phi+s\,\tilde{v}_\|)^2}\,Erfc(\phi + s\,\tilde{v}_\|)\Big] \tag{18}$$

where

$$\phi = \frac{1}{\sqrt{2}}\,\omega_{p,\alpha}\,\tau\,x\,, \qquad \tilde{v}_\| = v_\|/\sqrt{\sigma}$$

$$\tilde{v}_\perp = v_\perp/\sqrt{\sigma}\,, \qquad \lambda = \sqrt{\sigma}\,\frac{k_D}{\Omega} = \cdots = \sqrt{2}\,\frac{\omega_p}{\Omega}$$

(having set $\sigma_\perp = \sigma_\| = \sigma$ for simplicity). Remember that $\sigma_\alpha = 2\,k_B\,T_\alpha/m_\alpha = 2\,v_{th,\alpha}^2$ is related to the thermal velocity (i.e. the temperature), $\Omega_\alpha = e_\alpha\,B/m_\alpha\,c$ is the cyclotron (gyroscopic) frequency, $k_D = \frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha}$ is the Debye wave-number and $\omega_{p,\alpha} = (\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha})^{1/2}$ is the plasma (Langmuir) frequency (so $\omega_p = \sqrt{\sigma\,k_D/2}$). Notice the interplay of collision and magnetic field scales through $\lambda \approx \frac{T_{gyro}}{T_{coll}} \equiv \frac{v_{thermal}}{v_{Alfven}}$.

6.1.2 Diffusion coefficients

As we saw, the final formulae for the diffusion coefficients can be simplified by rescaling the integration variables $k_\perp$ (Fourier wave-number) and τ (time) therein to the non-dimensional variables $x \equiv \frac{\tilde{k}_\perp}{k_D} = (1 + \frac{k_\perp^2}{k_D^2})^{1/2}$ and $\tau' = \Omega\tau$. The diffusion coefficients $D_*(t)$ are thus given by:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{2\sqrt{2}\, n\, e^4}{m^2 \sqrt{k_B T}} \, \lambda \int_0^t d\tau' \int_1^{x_{max}} dx \; e^{\lambda^2 (1-x^2) \sin^2 \frac{\tau'}{2}} \Big(1 - \frac{1}{x^2}\Big)^{\{1,0\}} e^{-\tilde{v}_\parallel^2}$$

$$J_O(2\lambda \sqrt{x^2-1}\, \tilde{v}_\perp \, \sin \frac{\tau'}{2}) \; \tilde{F}_{\{\perp,\parallel\}} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos\tau' \\ (-s^\alpha)\, \frac{1}{2} \, \sin\tau' \\ (1 + \frac{1}{2} \cos\tau') \end{array} \right\} \\ 1 \end{array} \right\} \quad (19)$$

where all quantities in the *rhs* except $\frac{2\sqrt{2}\, n\, e^4}{m^2 \sqrt{k_B T}} \equiv D_0$ are non-dimensional; J_0 is a Bessel function of the first kind; $\tilde{F} = \tilde{F}(\phi(x,\tau'), \tilde{v}_\parallel)$ is given by:

$$\tilde{F}^{\alpha'}_{\{\perp,\parallel\}} = \pm\sqrt{\pi}\, \phi + \frac{\pi}{4} \sum_{s=+1,-1} \Big[(1 \mp 2\, \phi^2 \mp s 2\, \phi\, \tilde{v}_\parallel)\, e^{(\phi + s\, \tilde{v}_\parallel)^2}\, Erfc(\phi + s\, \tilde{v}_\parallel) \Big] \quad (20)$$

where

$$\phi = \frac{1}{2} \lambda\, \tau'\, x\,, \qquad \lambda = \sqrt{2}\, \frac{\omega_p}{\Omega}\,, \qquad \tilde{v}_* = (\frac{m v_*^2}{2 k_B T})^{1/2}\,, \qquad * \in \{\perp, \parallel\}$$

Remember that $k_D = (\frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha})^{1/2}$ is the Debye wave-number and $\omega_{p,\alpha} = (\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha})^{1/2}$ is the plasma (Langmuir) frequency. Notice the interplay of collision and gyration scales through $\lambda \approx \frac{T_{gyro}}{T_{coll}} \equiv \frac{v_{thermal}}{v_{Alfven}}$.

7 A numerical parametric study

(cf. Poster)

Typical values:

temperature $T = 10\,KeV$

particle density $n = 10^{14}\,cm^{-3} = 10^{20}\,m^{-3}$

implying a $\omega_{p,e} = 5.64 \cdot 10^{11}\,s^{-1}$

$\Omega_e = 1.76\,10^{11} \times B \quad s^{-1}$,

(B being expressed in Tesla).

* Correlations are found to decrease quite fast in time. The magnetic field seems to *enhance correlation*, since the higher its magnitude B ($\sim \Omega$), the higher the value of $C_\perp(\tau)$; see figure 1.

Physically speaking, this fact reflects particle confinement by the magnetic field, since particles 'stick' to their helicoidal trajectory around the magnetic field lines and thus 'feel' each other for longer periods of time.

* Diffusion coefficient $D_\perp(t)$:

The asymptotic value $D(\infty)$ depends on the field Ω.

* Relaxation times: Remember that the diffusion coefficients $D_{\perp,\parallel}(t)$ are related to the *inverse* of the time needed for relaxation towards equilibrium [8]. We therefore see that the magnetic field *favours* thermalization (i.e. relaxation of the distribution function towards a maxwellian state).

This seems to agree with physical intuition (the more 'confined' the particles, the more they influence each other and the more efficient collisions are towards relaxation).

8 Influence on moment equations

Defining the moments:

Particle density:

$$\int d\mathbf{v}\, f(\mathbf{x},\mathbf{v};\, t) \;=\; n_\alpha(\mathbf{x},t)$$

Mean velocity:

$$\int d\mathbf{v}\, v_r\, f(\mathbf{x},\mathbf{v};\, t) \;=\; n_\alpha(\mathbf{x},t)\, u_r^\alpha(\mathbf{x},t)$$

Pressure:

$$\int d\mathbf{v}\, (v_r - u_r)(v_m - u_m)\, f(\mathbf{x},\mathbf{v};\, t) \;=\; P_{rm}^\alpha(\mathbf{x},t)$$

etc. we obtain a set of *modified* evolution equations:

$$\begin{aligned}
\frac{\partial n_\alpha}{\partial t} &= \frac{\partial}{\partial x_r}(n_\alpha\, u_r^\alpha) \\
\frac{\partial}{\partial t}(m_\alpha\, n_\alpha\, u_r^\alpha) &= \frac{\partial}{\partial x_m}(m_\alpha\, n_\alpha\, u_r^\alpha\, u_m^\alpha + P_{rm}) + e_\alpha\, n_\alpha\, (E_r + \frac{1}{c}\epsilon_{rmn}\, u_m\, B_n) + \mathcal{R}_r^\alpha \\
n_\alpha \frac{\partial T_\alpha}{\partial t} &= -n_\alpha u_r^\alpha \frac{\partial T_\alpha}{\partial x_r} - \frac{2}{3} n_\alpha T_\alpha \frac{\partial u_r^\alpha}{\partial x_r} - \frac{2}{3}\pi_{mn}^\alpha \frac{\partial u_n^\alpha}{\partial x_m} - \frac{2}{3}\frac{\partial q_m^\alpha}{\partial x_m} + \frac{2}{3}\mathcal{Q}^\alpha
\end{aligned} \tag{21}$$

where the *collisional terms* $\mathcal{R}_r^\alpha$, $\mathcal{Q}^\alpha$ now contain *space diffusion terms*:

Friction vector:

$$\mathcal{R}_r^\alpha = m_\alpha \int d\mathbf{v}\, v_r\, \mathcal{K}\{f(\mathbf{x},\mathbf{v};\, t)\}$$

Collisional heat-exchange rate:

$$\mathcal{Q}^\alpha = \frac{1}{2} m_\alpha \int d\mathbf{v}\, |\mathbf{v} - \mathbf{u}^\alpha|^2\, \mathcal{K}\{f(\mathbf{x},\mathbf{v};\, t)\}$$

etc. ($\mathcal{K}\{f\}$ is our *new cylindrical-symmetric* collision term !)

9 Conclusions

* Agreement with: [Ghendrih 1988] (influence of the magnetic field)

yet

* contradiction with the standard description used in the past, where the influence of the magnetic field on the collision term is either *under-estimated* [Montgomery et al. 1974 +] or *neglected* [Ghendrih 1987] when discussing the transport properties of plasma.

In conclusion, we have reported new exact formulae for the diffusion coefficients in magnetized plasma. These formulae suggest an explicit dependence on both particle velocity and physical parameters such as plasma temperature, density, and - the point we wanted to focus upon - the magnitude of the magnetic field. A more extended study will be reported soon.

(References: see Poster)

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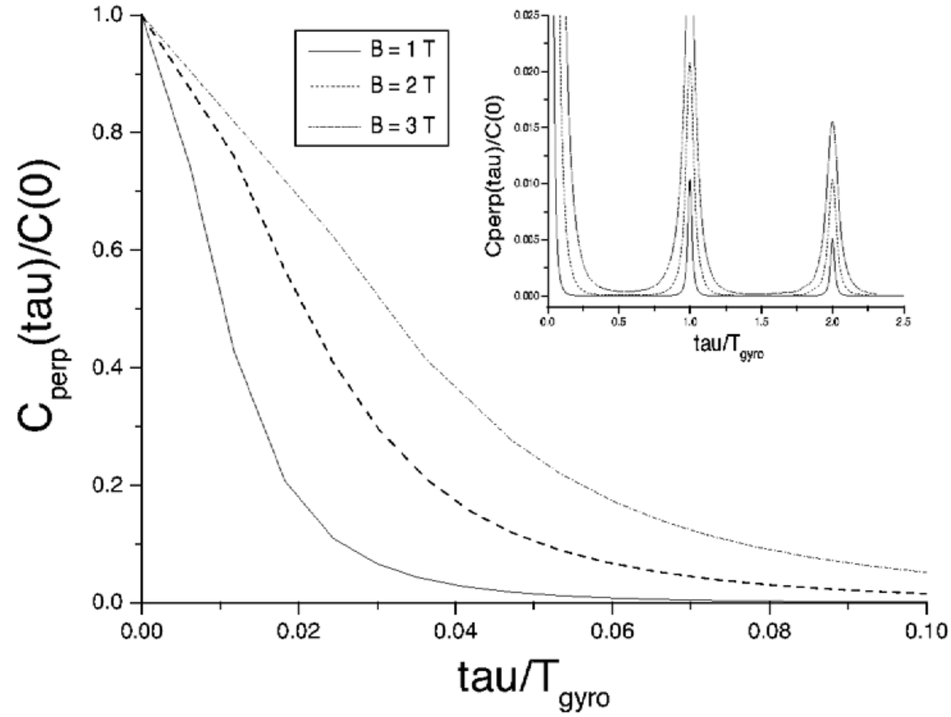

Figure 1: The perpendicular force auto-correlation function $C_{\perp}(\tau; v_{\perp}, v_{\|}, B)$ (normalized over $C_{\perp}(\tau = 0)$) as a function of time τ (scaled over a cyclotron period T_c i.e. $\Omega\tau/2\pi$). In ascending order, the magnitude of the magnetic field is set to $B = 1, 2, 3 \quad T$ respectively. Both velocity components are taken equal to $v_{th} = (T/m)^{1/2}$. $C_{\perp}$ can be seen to decrease very fast in time, still bearing a 'tail' of gradually smoothed out peaks every gyration period T_{gyro} (actually a signature of the magnetic field; see enclosed figure 1b).

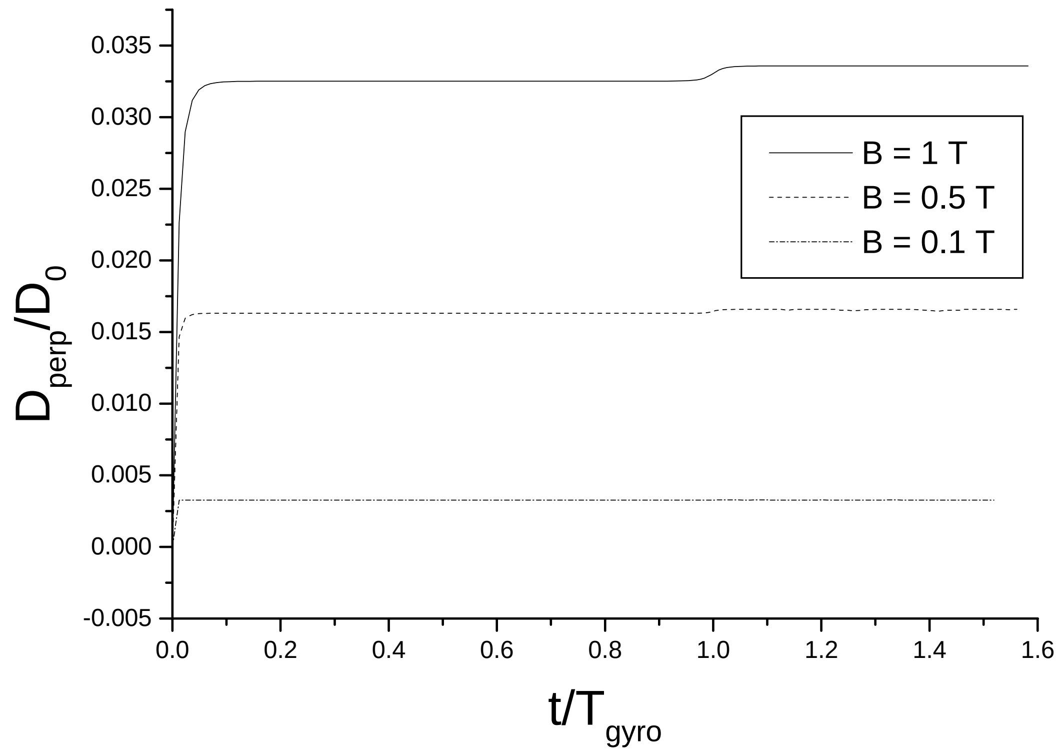

Figure 2: The perpendicular velocity diffusion coefficient $D_{\perp}(t; v_{\perp}, v_{\|}, B)$ versus time t (scaled over T_{gyro}) for values of $B = 1, 2, 3 \quad T$, respectively, in ascending order. $D_{\perp}$ is seen to increase with B. The small 'kinks' at every gyration reflect the form of $C_{\perp}$ (cf. fig. 1b).