# Nonlinear transverse wave modulation in dusty plasma crystals

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## Introduction

Modulational instability (MI) is a well-known mechanism of energy localization occurring during wave propagation in nonlinear dispersive media. It has been thoroughly studied in the past, mostly in solid state systems, where nonlinearities of the substrate potential and/or particle coupling may destabilize waves and possibly lead to localized excitations. However, no such study has been carried out in the case of dusty plasma (DP) crystals [1], i.e. strongly-coupled Coulomb systems of heavy charged dust grains [2, 3].

## Transverse dust-lattice linear oscillations

In addition to *longitudinal dust-lattice waves* (LDLW) [2, 4], DP crystals support low-frequency optical-mode-like oscillations in the *transverse* (*off plane*) direction (TDLW) [2, 3, 5]:

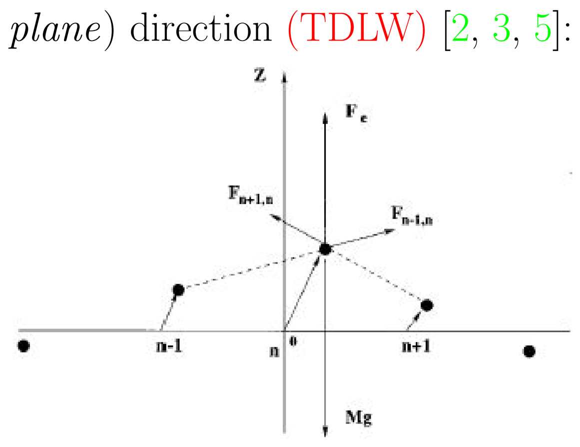

Figure 1. Off-plane dust grain motion in a single dust lattice.

The motion of a dust grain (mass $M$ and charge $Q$ assumed constant; lattice constant $r_0$) obeys the equation:

$$M\,\frac{d^2\delta z_n}{dt^2} = M\,\omega_0^2\,(2\,\delta z_n - \delta z_{n-1} - \delta z_{n+1}) + F_e - Mg \qquad (1)$$

where $\delta z_n = z_n - z_0$ denotes the small displacement of the $n$−th grain around the *equilibrium position* $z_0$, in the transverse direction ($z$−), propagating in the longitudinal ($x$−) direction;
$\omega_0$ is e.g. the DP transverse oscillation 'eigenfrequency':

$$\omega_0^2 = \frac{Q^2}{Mr_0^3}\left(1+\frac{r_0}{\lambda_D}\right)e^{-r_0/\lambda_D} \qquad (2)$$

resulting from the Debye interaction potential [6].
Solving Poisson's equation, one obtains the electric field, which is due to the sheath potential *and* the wake potential generated by supersonic ion flow towards the electrode. The total field $E(z)$ can be developed around $z_0$, so the *electric force* $F_e$ reads:

$$F_e(z) \approx F_e(z_0) + \gamma_{(1)}\,\delta z + \gamma_{(2)}\,(\delta z)^2 + \gamma_{(3)}\,(\delta z)^3 + \mathcal{O}((\delta z)^4)\,.$$

All coefficients are defined via derivatives of the exact field form [3].
The zeroth-order term balances gravity at $z_0$;
$-\gamma_{(1)} = \gamma \equiv M\,\omega_g^2$ is the effective width of the potential well.
Considering phonons of the type: $x_n = A_n\,exp[i\,(knr_0 - \omega t)] + c.c.$, an optical-mode-like dispersion relation is obtained:

$$\omega^2 = \omega_g^2 - 4\omega_0^2\,\sin^2\frac{kr_0}{2} \qquad (3)$$

We do not go into further details concerning this *linear* regime, since it is covered in the references.

## Nonlinear wave modulation - harmonic generation

Limiting ourselves to the continuum (long wavelength $\lambda$) limit (i.e. $k\,r_0 \ll 1$), Eq. (1) takes the form:

$$\frac{d^2u}{dt^2} + c_0^2\,\frac{d^2u}{dx^2} + \omega_g^2\,u + \alpha\,u^2 + \beta\,u^3 = 0 \qquad (4)$$

where we have set $\delta z \equiv u(x,t)$ for simplicity;
$c_0 = \omega_0\,r_0$ is a characteristic propagation velocity related to the Debye potential (see (2));
the nonlinearity coefficients $\alpha$, $\beta$ are related to the electric field: $\alpha = -\gamma_{(2)}/M$, $\beta = -\gamma_{(3)}/M$.
Remember that inter-particle interactions are *repulsive*, hence the difference from the 'ordinary' nonlinear Klein-Gordon equation used to describe 1d oscillator chains. Phonons in this chain are stable *only* in the presence of the electric field (i.e. for $\gamma \neq 0$).
We may now consider small-amplitude oscillations:

$$u = \epsilon\,u_1 + \epsilon^2\,u_2{}^2 + \ldots$$

at each site. Assuming the existence of *multiple scales* in time and space, i.e. $X_n = \epsilon^n\,x$, $T_n = \epsilon^n\,t$ (n = 0, 1, 2, ...), we develop the derivatives in (4) in powers of a smallness parameter $\epsilon$ and then collect the terms arising in successive orders. The equation thus obtained in each order can be solved and substituted to the subsequent order, and so forth [7].
This procedure leads to a solution of the type:

$$u(x,t) = \epsilon\,(A\,e^{i\,\theta} + c.c.) + \epsilon^2\,\alpha\left(-\frac{2\,|A|^2}{\omega_g^2} + \frac{A^2}{3\omega_g^2}\,e^{2i\,\theta} + c.c.\right) + \mathcal{O}(\epsilon^3) \qquad (5)$$

($\theta = kx - \omega t$) where $\omega$ obeys a dispersion law of the form:

$$\omega^2 = \omega_g^2 - c_0^2\,k^2 \qquad (6)$$

(i.e. (3) linearized around $k \approx 0$).

## A Nonlinear Schrödinger Equation

The slowly-varying amplitude $A = A(X_1 - v_g\,T_1)$ moves at the (negative) group velocity $v_g = d\omega/dk = -c_0^2\,k/\omega$ i.e. in the direction *opposite to* the phase velocity (this so called *backward wave* has been observed experimentally: see the discussion in [8]); it is found to obey the *Nonlinear Schrödinger (NLS) Equation*:

$$i\,\frac{dA}{dT} + P\,\frac{d^2A}{dX^2} + Q\,|A|^2\,A = 0 \qquad (7)$$

where the 'slow' variables $\{X, T\}$ are $\{X_1 - v_g\,T_1, T_2\}$ respectively. The *dispersion coefficient* $P$ is related to the curvature of the phonon dispersion curve (6) and the *nonlinearity coefficient* $Q$ is related to electric field nonlinearities:

$$P = \frac{1}{2}\,\frac{d^2\omega}{dk^2} = -\frac{c_0^2\omega_g^2}{2\omega^3}\,, \qquad Q = \frac{1}{2\omega}\left(\frac{10\alpha^2}{3\omega_g^2} - 3\,\beta\right) \qquad (8)$$

Notice that $P < 0$ here, given the parabolic form of $\omega(k)$[9].

## Modulational instability

In a generic manner, a modulated wave whose amplitude obeys the NLS equation (7), is unstable to perturbations if $P \cdot Q > 0$, i.e. from (8) if: $10\,\gamma_{(2)}^2 - 9\,\gamma_{(1)}\,\gamma_{(3)} < 0$. To see this, one may first check that the NLSE accepts the solution (Stokes' wave):

$$A(X,T) = A_0\,e^{iQ|A_0|^2T} + c.c.$$

The standard (linear) stability analysis then shows that a linear perturbation of frequency $\Omega$ and wavenumber $\kappa$ will obey:

$$\Omega^2(\kappa) = P^2\,\kappa^2\left(\kappa^2 - 2\frac{Q}{P}\,|A_0|^2\right) \qquad (9)$$

and is therefore expected to grow, for $\kappa \geq \kappa_{cr} = (Q/P)^{1/2}\,|A_0|$ at a rate attaining a maximum value of:

$$\sigma_{max} = Q\,|A_0|^2$$

until the wave collapses. If $P \cdot Q < 0$, this will never occur. This mechanism is known as the *Benjamin-Feir instability* [10]

## Localized envelope excitations

It is known that the NLSE (7) supports pulse-shaped localized solutions (*envelope solitons*) of the bright ($PQ > 0$) or dark/grey ($PQ < 0$) type [9]. The former (*continuum breathers*) are:

$$A = (2D/PQ)^{1/2}\,sech\big[(2D/PQ)^{1/2}\,(X - v_e\,T)\big]\,\times \exp\big[i\,v_e\,(X - v_c T)/2P\big] + c.c. \qquad (10)$$

where $v_e$ ($v_c$) is the envelope (carrier) velocity and $D = (v_e^2 - 2\,v_e\,v_c)/(4P^2)$; the latter (*holes*) are physically irrelevant here (they correspond to infinite energy stored in the lattice).

## Coupled DP layers

The above picture is strongly modified if a set of coupled DP lattices is considered [12]. For two such coupled chains, (see fig. 2):

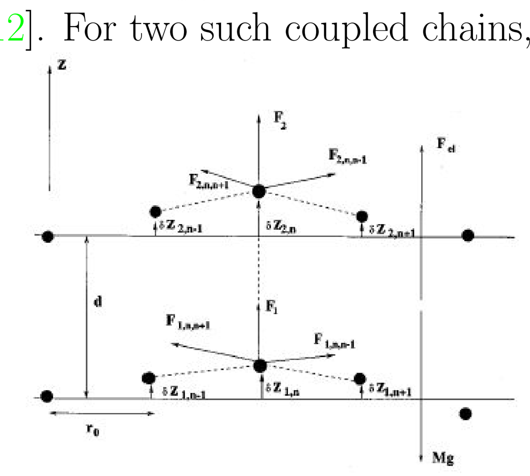

Figure 2. Off-plane dust grain motion in coupled dust lattices.

the *equations of motion* (lower/upper grain: 1/2) read [13, 14]:

$$M\,\frac{d^2\delta z_{1,n}}{dt^2} = M\,\omega_0^2\,(2\,\delta z_{1,n} - \delta z_{1,n-1} - \delta z_{1,n+1}) - M\,\omega_g^2\,\delta z_{1,n} + \Gamma_{11}\,(\delta z_{2,n} - \delta z_{1,n}) + \Gamma_{12}\,(\delta z_{2,n} - \delta z_{1,n})^2$$

$$M\,\frac{d^2\delta z_{2,n}}{dt^2} = M\,\omega_0^2\,(2\,\delta z_{2,n} - \delta z_{2,n-1} - \delta z_{2,n+1}) - M\,\omega_g^2\,\delta z_{2,n} + \Gamma_{21}\,(\delta z_{2,n} - \delta z_{1,n}) + \Gamma_{22}\,(\delta z_{2,n} - \delta z_{1,n})^2\,; \qquad (11)$$

$\Gamma_{ij}$ are functions of the interchain distance $d$, related to the electric potential $\Phi_1(z)$ ($\Phi_2(z)$) felt by the lower (upper) grain:

$$\Gamma_{11} = Q\,\frac{d^2\Phi_1(|z|)}{d|z|^2}\bigg|_{|z|=d}\,, \qquad \Gamma_{21} = -Q\,\frac{d^2\Phi_2(|z|)}{d|z|^2}\bigg|_{|z|=d}$$

Note that the two potentials are *not* symmetric: $\Phi_2(z)$ acting on the upper particles due to the lower ones is a simple Debye-Hückel-type potential, but $\Phi_1(z)$ felt by the lower particles due to the upper ones is modified by downwards ion flow [12]. The dispersion relation obtained in the linear limit now consists of two distinct dispersion branches, both given by (3) with gap frequencies equal to $\omega_{g,1} = \omega_g$ and $\omega_{g,2} = \sqrt{\omega_g^2 + \frac{\Gamma_{11}}{M} + \frac{\Gamma_{21}}{M}}$ [12, 13, 14]).

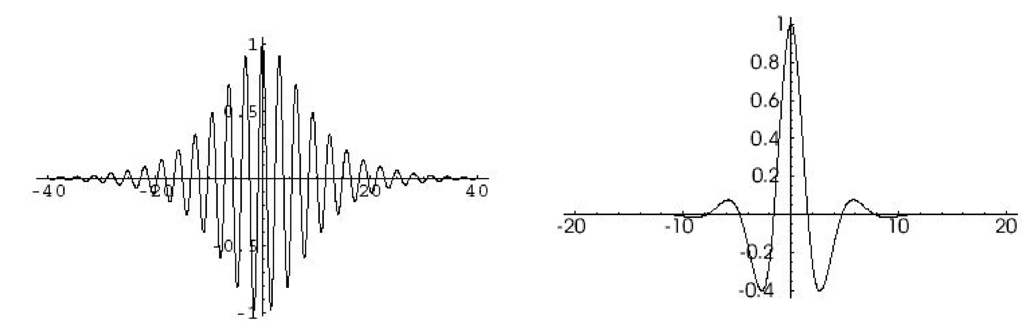

Figure 3. Bright type (pulse) soliton solution of the NLS equation, for two different parameter sets ($PQ < 0$). The second type, where the envelope width is not very different from the carrier wavelength, is the continuum analogue of the (discrete) breathing modes studied in molecular chains [11].

In order $\epsilon^3$ one now obtains two *coupled NLS (CNLS) Equations*:

$$i\,\frac{dA}{dT} + P\,\frac{d^2A}{dX^2} + Q_{11}\,|A|^2\,A + Q_{12}\,|B|^2\,A = 0\,,$$
$$i\,\frac{dB}{dT} + P\,\frac{d^2B}{dX^2} + Q_{21}\,|A|^2\,B + Q_{22}\,|B|^2\,B = 0\,; \qquad (12)$$

$A$, $B$ are the first harmonic amplitudes in the two lattices [13].
The dispersion coefficient $P = \omega''(k)/2$ is given by (8);
the expressions for the (non-symmetric) nonlinearity matrix elements $Q_{ij}$, in fact complicated analytic functions related to interaction potentials $\Phi_1$, $\Phi_2$ are omitted here [12].
The *linear stability analysis* around the solution:

$$A(X,T) = A_0\,e^{i(Q_{11}|A_0|^2 + Q_{12}|B_0|^2)\,T} + c.c.$$
$$B(X,T) = B_0\,e^{i(Q_{22}|B_0|^2 + Q_{21}|A_0|^2)\,T} + c.c.$$

now results in the dispersion relation:

$$\Omega^2 = \frac{1}{2}\left[(\Omega_{12}^2 + \Omega_{22}^2) + \sqrt{(\Omega_{11}^2 - \Omega_{22}^2) + 4\,\Omega_{12}^2\Omega_{21}^2}\right] \qquad (13)$$

where:

$$\Omega_{11}^2(\kappa) = P^2\,\kappa^2\left(\kappa^2 - 2\frac{Q_{11}}{P}\,|A_0|^2\right)$$
$$\Omega_{22}^2(\kappa) = P^2\,\kappa^2\left(\kappa^2 - 2\frac{Q_{22}}{P}\,|B_0|^2\right)$$
$$\Omega_{ij}^2(\kappa) = -2\,P\,Q_{ij}\,P\,|A_0||B_0|\,, \qquad (i \neq j = 1, 2) \qquad (14)$$

($\kappa$ is the perturbation wavenumber); see that, for vanishing cross-coupling $Q_{ij}$ terms, dispersion relation (9) is recovered.
The investigation of conditions for $\Omega$ to possess an imaginary part are now more perplex [12]. An *enlarged* instability region in $\kappa$ values is obtained, in terms of $Q_{ij}$. The basic highlight of the analysis is that: *a stable (single-layer) wave mode* (i.e. for $PQ_{ii} < 0$) *may become unstable due to layer coupling*.
Of course, one's task now consists in assuming an explicit form for the electrostatic potentials $\Phi_1(z)$, $\Phi_2(z)$ and deriving exact expressions for the coefficients in the CNLS equations (12) above. The conditions for instability will then be explicitly formulated in terms of intrinsic plasma parameters.

*In conclusion*, we have seen seen that:
(i) *modulational instability* is, in principle, possible in transverse DP lattice waves;
(ii) instability is potentially *enhanced* by inter-layer coupling;
(iii) *energy localization* via localized envelope excitations may occur in a DP lattice.
Appropriate experiments may hopefully confirm these results.

Of course, a more complete description should include factors ignored in this simple model: collisions with neutral particles, dust charge variations and transverse-to-longitudinal mode coupling. Work in this direction is in progress and will be reported soon.

**References**

[1] Also see our neighbouring poster, for MI investigation in dust–acoustic and dust–ion electrostatic waves acoustic propagating in weakly-coupled dusty plasma.

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[5] S. V. Vladimirov, P. V. Shevchenko, N. F. Cramer, Phys. Rev. E **56**, R74 (1997).

[6] Even though Debye interactions are considered in a first step, a more sophisticated approach may be used; see e.g. I. Kourakis and P. Shukla, *Study of the intergrain interaction potential and associated instability of dust-lattice plasma oscillations in the presence of ion flow*, Phys. Lett. **A** 315 (in press).

[7] This *reductive perturbation* technique is a standard procedure in the study of nonlinear wave propagation, often used in the description of optical pulse propagation, prediction of instabilities etc; see e.g. [10].

[8] T. Misawa *et al*, Phys. Rev. Lett. **86**, 7 (2001).

[9] This is only true close to $k = 0$ (continuum case): in the general (discrete) case (see (3)), $P$ changes sign at a critical value of $k$ (*zero-dispersion point*) in the first Brillouin zone.

[10] M. Remoissenet, *Waves Called Solitons*, Springer.

[11] S. Flach and C. Willis, Physics Reports **295** 181 (1998).

[12] I. Kourakis and P. K. Shukla, *Transverse wave amplitude modulation in coupled dusty plasma layers*, in preparation.

[13] The exact form of this expression is valid for *identical* coupled DP layers (same $M$, $Q$) only; a more general treatment is considered in [12].

[14] S. V. Vladimirov, P. V. Shevchenko, N. F. Cramer, Phys. Plasmas **5**, 1 (1998).