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Theory of nonlinear excitations in dusty plasma crystals

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Outline

A. Introduction
   (i) *Linear waves* in one-dimensional (1d) dust crystals;
   (ii) *Nonlinearity*: Origin and modeling.

B. Nonlinear effects on *transverse* dust-lattice waves (*TDLW*s): amplitude modulation, transverse envelope structures.

C. Nonlinear effects on *longitudinal* dust-lattice waves (*LDLW*s): modulation, longitudinal envelope excitations.

D. *Longitudinal localized excitations* : relation to soliton theories.

E. 1d *Discrete Breathers* (*Intrinsic Localized Modes*) : $\longrightarrow$ poster.

F. Conclusions.

A. Intro.: Linear modes in a *1-dimensional* plasma crystal.

- *Longitudinal Dust Lattice (LDL)* mode:
  – *Horizontal* oscillations ($\sim \hat{x}$): cf. *phonons* in atomic chains;
  – *Acoustic* mode: $\omega(k=0)=0$;
  – *Restoring force* provided by electrostatic interactions.
- *Transverse Dust Lattice (TDL)* mode:
  – *Vertical* oscillations ($\sim \hat{z}$);
  – *Optical* mode:
  $\omega(k=0)=\omega_g \neq 0$
  (center of mass motion);
  – *Single* grain vibrations (propagating $\sim \hat{x}$ for $k \neq 0$):
  Restoring force provided by the *sheath electric potential* (*and* interactions).

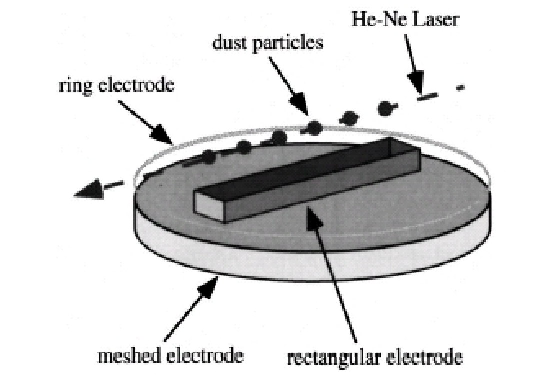


FIG. 10. Experimental configuration for forming a linear dust chain above a long rectangular box on a negatively biased mesh electrode.

- Transverse ($\sim \hat{y}$, in-plane, optical) d.o.f. *suppressed*.
  Also: *No transverse shear- acoustic mode* (cf. 2d lattices).

* Figure from: S. Takamura *et al.*, Phys. Plasmas **8**, 1886 (2001).

Model *Hamiltonian*:

$$H = \sum_n \frac{1}{2} M \left(\frac{d\mathbf{r}_n}{dt} \right)^2 + \sum_{m \neq n} U_{int}(r_{nm}) + \Phi_{ext}(\mathbf{r}_n)$$

where:

– *Kinetic Energy* (1st term);

– $U_{int}(r_{nm})$ is the (binary) *interaction potential energy*;

– $\Phi_{ext}(\mathbf{r}_n)$ accounts for *'external' force fields*:
may account for *confinement potentials* and/or *sheath electric* forces, i.e. $F_{sheath}(z) = -\frac{\partial \Phi}{\partial z}$.

Q.: Nonlinearity: *Origin: where from ?*
Effect: which consequence(s) ?

Nonlinearity: Where does it come from?

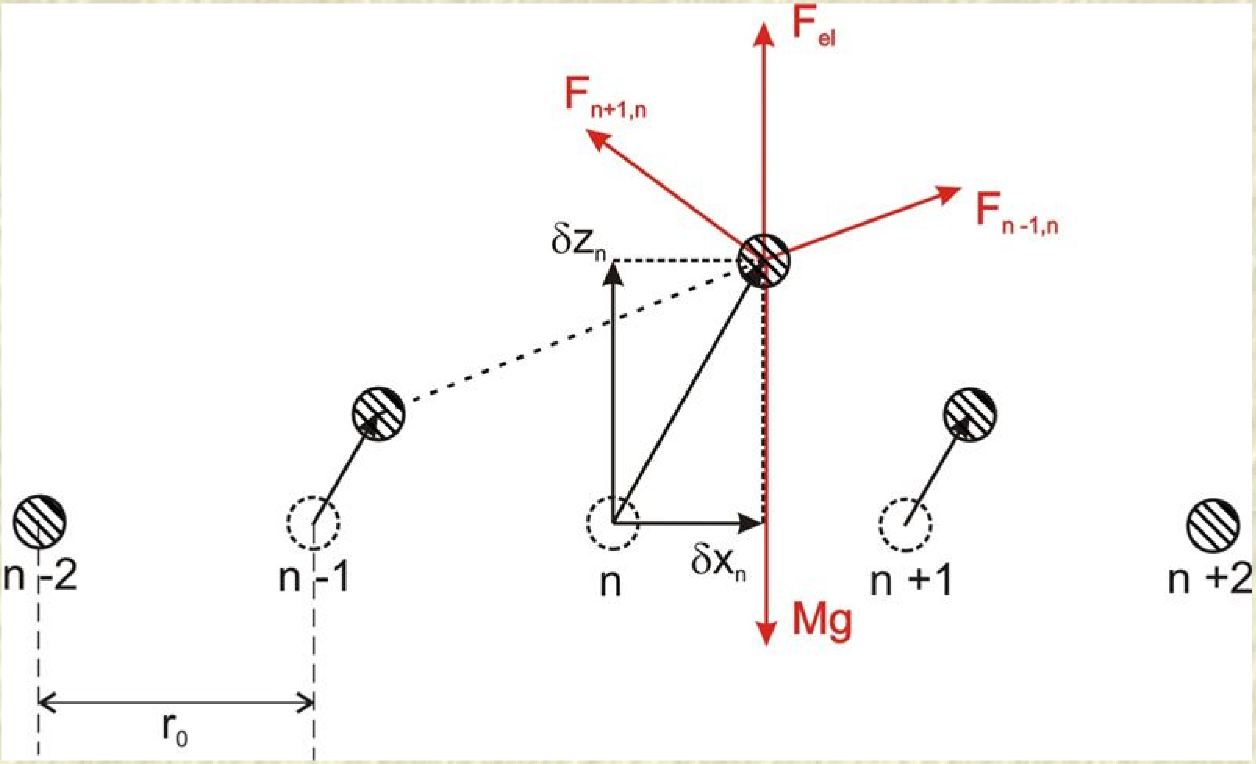
- (i) *Interactions* between grains: Electrostatic character (e.g. repulsive, Debye), long-range, *anharmonicity*:

  Expanding $U_{pot}(r_{nm})$ near equilibrium:
  $\Delta x_n = x_n - x_{n-m} = m r_0, \qquad \Delta z_n = z_n - z_{n-m} = 0$,
  one obtains:

$$
\begin{aligned}
U_{nm}(r) \;\approx\; & \frac{1}{2} M \omega_{L,0}^2 (\Delta x_n)^2 + \frac{1}{2} M \omega_{T,0}^2 (\Delta z_n)^2 \\
& + \frac{1}{3} u_{30} (\Delta x_n)^3 + \frac{1}{4} u_{40} (\Delta x_n)^4 + ... + \frac{1}{4} u_{04} (\Delta z_n)^4 + ... \\
& + \frac{1}{2} u_{12} (\Delta x_n)(\Delta z_n)^2 + \frac{1}{4} u_{22} (\Delta x_n)^2 (\Delta z_n)^2 + ...
\end{aligned}
$$

Nonlinearity: Where from? *(continued ...)*

- (ii) Mode *coupling* also induces non linearity: anisotropic motion, *not* confined along one of the main axes ($\sim \hat{x}$, $\hat{z}$).



[cf. A. Ivlev *et al.*, PRE **68**, 066402 (2003)]

Nonlinearity: Where from? *(continued ...)*

- (iii) *Sheath* environment: *anharmonic* vertical potential:

$$\Phi(z) \approx \Phi(z_0) + \frac{1}{2} M \omega_g^2 (\delta z_n)^2 + \frac{1}{3} M \alpha \, (\delta z_n)^3 + \frac{1}{4} M \beta \, (\delta z_n)^4 + ...$$

cf. experiments [Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)];

$\delta z_n = z_n - z_{(0)}$; α, β, ω_g are defined via $E(z)$, $[B(z)]^\dagger$ and $Q(z)$;

(in fact, functions of n and P) [† V. Yaroshenko *et al.*, NJP 2003; PRE 2004]

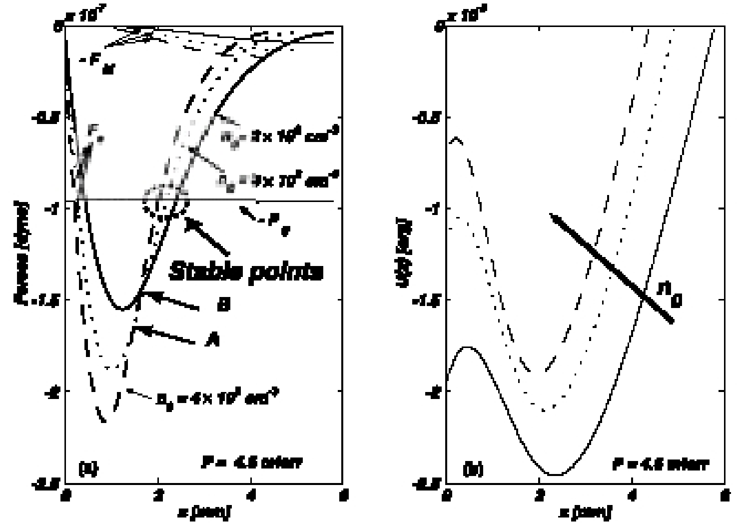


Figure 3: (a) Forces and (b) trapping potential profiles $U(z)$ as function of distance from the electrode for: $n_0 = 2\times10^8 cm^{-3}$ (solid line), $n_0 = 3\times10^8 cm^{-3}$ (dashed line), $n_0 = 4\times10^8 cm^{-3}$ (dotted line). The parameters are: P = 4.6 mtorr, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$.

Source: Sorasio *et al.* (2002).

Part 1: *Transverse* nonlinear oscillations

The vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys

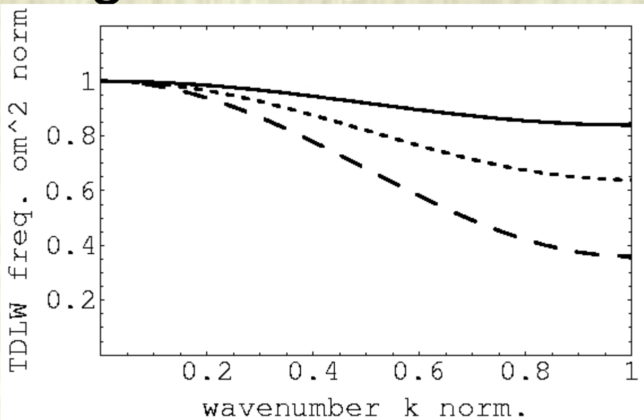
$$\frac{d^2 \delta z_n}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 \, (\, \delta z_{n+1} + \, \delta z_{n-1} - 2 \, \delta z_n) + \omega_g^2 \, \delta z_n$$
$$+ \alpha \, (\delta z_n)^2 + \beta \, (\delta z_n)^3 = 0 \, . \qquad (1)$$

* $\omega_{T,0} \, = \big[-qU'(r_0)/(M r_0) \big]^{1/2} = \omega_{DL}^2 \, \exp(-\kappa) \, (1+\kappa)/\kappa^3 \quad ^{(\dagger)}$

$^{(\dagger)}$ (*for Debye interactions*); $\kappa = r_0/\lambda_D$ is the *lattice parameter*;

* $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$; λ_D is the *Debye length*;

* *Optical* dispersion relation
  (*backward* wave, $v_g < 0$) †:
  $\omega^2 \, = \omega_g^2 - 4\omega_{T,0}^2 \, \sin^2\big(k r_0/2\big)$



* Rem.: ω_g and α, β: see $\Phi(z)$ above.

† Cf. experiments: T. Misawa *et al.*, *PRL* **86**, 1219 (2001); B. Liu *et al.*, *PRL* **91**, 255003 (2003).

Large amplitude oscillations - envelope structures

A reductive perturbation (multiple scale) technique, viz.
$t \to \{t_0, t_1 = \epsilon t, t_2 = \epsilon^2 t, \dots\}$, $x \to \{x_0, x_1 = \epsilon x, x_2 = \epsilon^2 x, \dots\}$
yields ($\epsilon \ll 1$; damping omitted):

$$\delta z_n \approx \epsilon\,(A\,e^{i\phi_n} + \text{c.c.}) + \epsilon^2 \left[-\frac{2|A|^2}{\omega_g^2} + \left(\frac{A^2}{3\omega_g^2}\,e^{2i\phi_n} + \text{c.c.} \right) \right] + \dots$$

($\phi_n = nkr_0 - \omega t$); the harmonic amplitude $A(X, T)$:

– depends on the *slow* variables $\{X, T\} = \{\epsilon(x - v_g t), \epsilon^2 t\}$;

– obeys the *nonlinear Schrödinger equation* (NLSE):

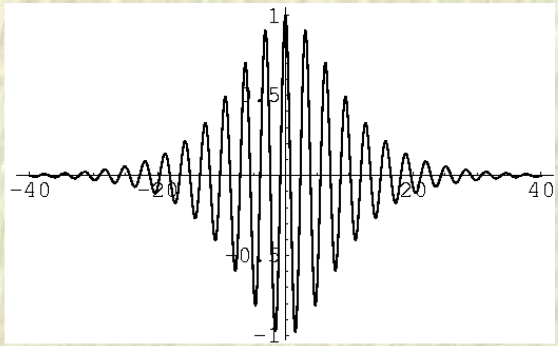
$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \qquad (2)$$

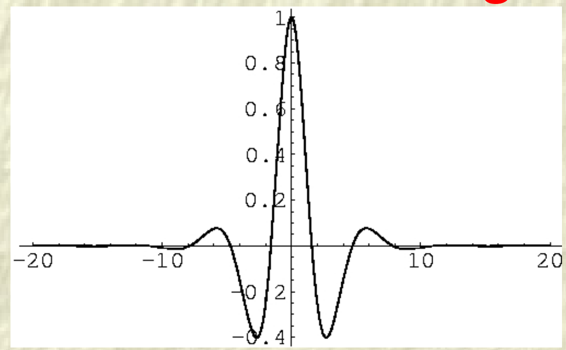
– *Dispersion coefficient*: $P = \omega''(k)/2$ $\to$ see dispersion relation;
– *Nonlinearity coefficient*: $Q = \left[10\alpha^2/(3\omega_g^2) - 3\,\beta\right]/2\omega$.

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 2322 (2004); also *PoP*, in press (Aug. 2004).]

Modulational stability analysis & envelope structures

- $PQ > 0$: *Modulational instability* of the carrier, *bright solitons*:



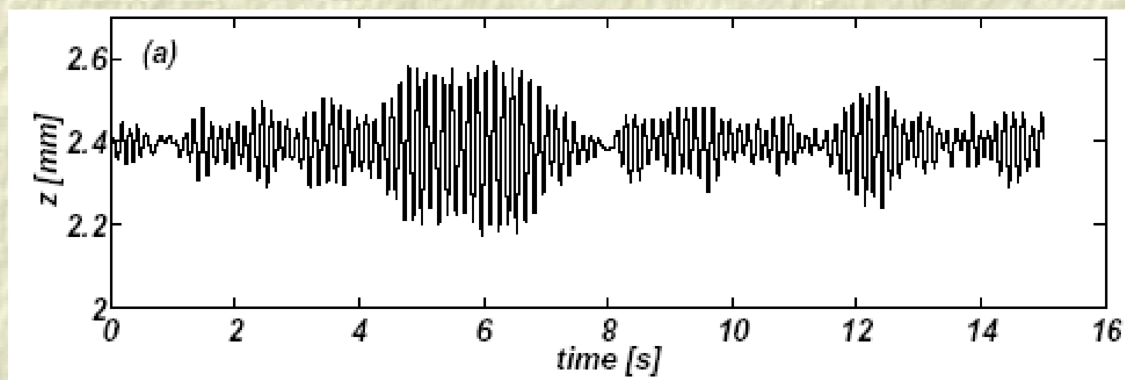


→ *TDLW*s: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

Rem.: $Q > 0$ for *all* known experimental values of α, β.

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)]

Figure 9: Dust grain oscillations induced by a 1% fluctuation in plasma density. The simulation parameters are: $P = 0.9\ mtorr$, $n_0 = 0.8 \times 10^8\ cm^{-3}$, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$, $\varsigma_t = 0.06$, $\varsigma_p = 1\% n_0$



Source: Sorasio *et al.* (2002).

Modulational stability analysis & envelope structures

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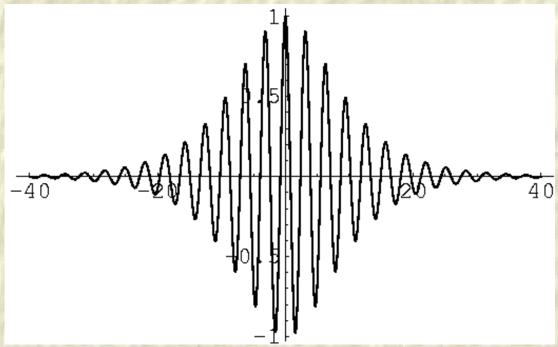
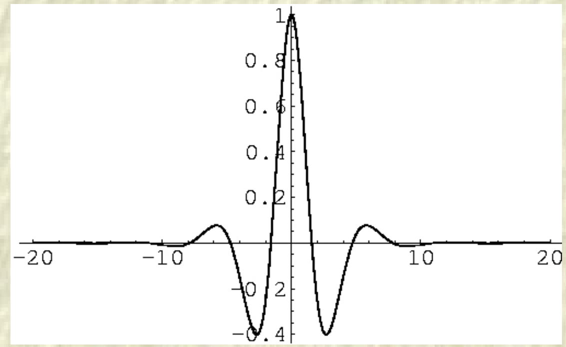
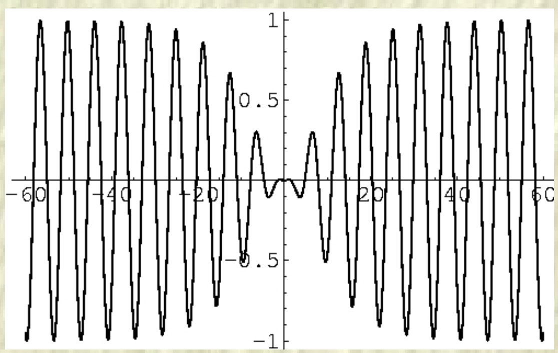
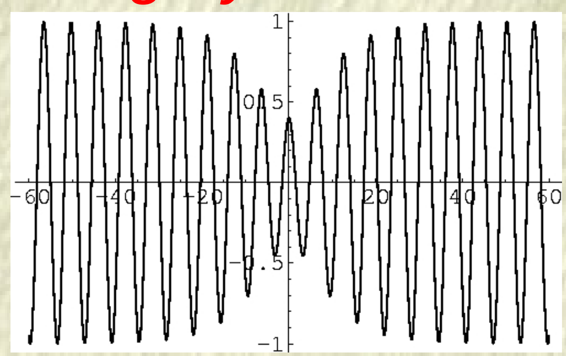




→ *TDLW*s: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

❑ $PQ < 0$: Carrier wave is *stable*, *dark/grey solitons*:





→ *TDLW*s: possible for *long* wavelengths i.e. $k < k_{cr}$.

Rem.: $Q > 0$ for *all* known experimental values of α, β

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)]

Part 2: *Longitudinal* excitations.

The *nonlinear* equation of longitudinal motion reads:

$$\frac{d^2(\delta x_n)}{dt^2} + \nu \frac{d(\delta x_n)}{dt} = \omega_{0,L}^2 \left(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n\right)$$
$$- a_{20} \left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2\right]$$
$$+ a_{30} \left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3\right] \qquad (3)$$

– $\delta x_n = x_n - n r_0$: longitudinal dust grain displacements

– *Acoustic* linear dispersion relation:

$$\omega^2 = 4\omega_{L,0}^2 \sin^2\left(k r_0/2\right) \equiv \omega_L^2(k)$$

– $\omega_{0,L}^2 = U''(r_0)/M) = 2\,\omega_{DL}^2 \exp(-\kappa)\,(1 + \kappa + \kappa^2/2)/\kappa^3 \quad ^{(*)}$

$^{(*)}$ *for Debye interactions*;   Rem.: $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$.

Longitudinal envelope structures.

The reductive perturbation technique (cf. above) now yields:

$$\delta x_n \approx \epsilon \left[u_0^{(1)} + \left(u_1^{(1)}\, e^{i\phi_n} + \text{c.c.}\right)\right] + \epsilon^2 \left(u_2^{(2)}\, e^{2i\phi_n} + \text{c.c.}\right) + \ldots ,$$

[Harmonic generation; Cf. experiments: K. Avinash PoP 2004.

where the amplitudes now obey the coupled equations:

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L\, \frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\, |u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\, u_1^{(1)} \frac{\partial u_0^{(1)}}{\partial X} = 0\, ,$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - c_L^2}\, \frac{\partial}{\partial X}\, |u_1^{(1)}|^2 \equiv R(k) \frac{\partial}{\partial X}\, |u_1^{(1)}|^2$$

– $Q_0 = -\frac{k^2}{2\omega}\left(q_0\, k^2 + \frac{2p_0^2}{c_L^2 r_0^2}\right)$;

– $v_{g,L} = \omega_L{}'(k)$; $\{X, T\}$ are *slow* variables (as above);

– $p_0 = -r_0^3 U'''(r_0)/M \equiv 2a_{20} r_0^3$, $q_0 = U''''(r_0) r_0^4/(2M) \equiv 3a_{30} r_0^4$.

– $R(k) > 0$, since $\forall\, k$ $\quad v_{g,L} < \omega_{L,0}\, r_0 \equiv c_L$ *(sound velocity)*.

Asymmetric *longitudinal* envelope structures.

– The system of Eqs. for $u_1^{(1)}$, $u_0^{(1)}$ may be combined into a closed (*NLSE*) equation (for $A = u_1^{(1)}$, here);

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0$$

– $P = P_L = \omega_L''(k)/2 < 0$;
– $Q > 0$ (< 0) prescribes *stability* (instability) at *low* (high) k.

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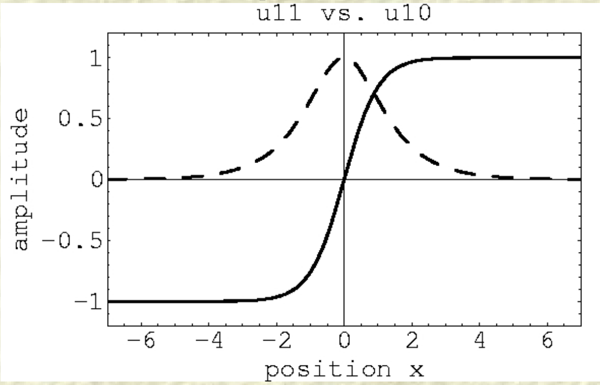
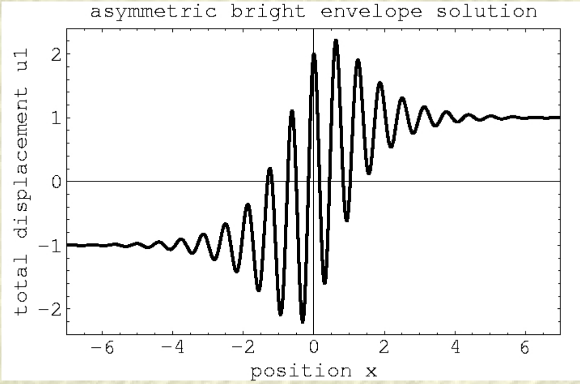
– Envelope excitations are now *asymmetric*:





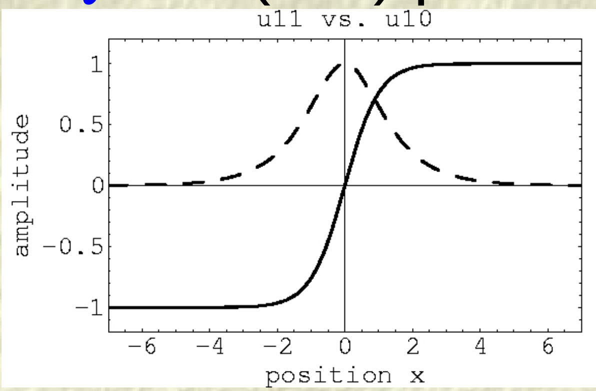
(at high k)

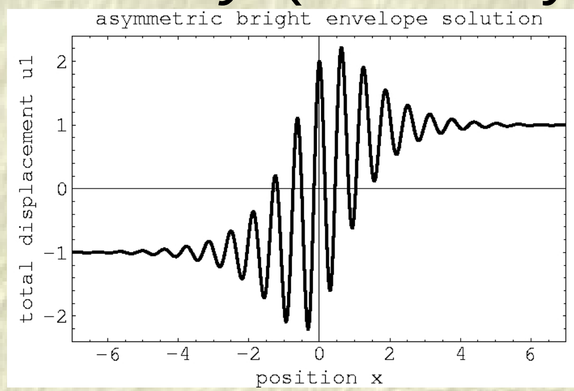
Asymmetric *longitudinal* envelope structures.

– The system of Eqs. for $u_1^{(1)}$, $u_0^{(1)}$ may be combined into a closed (NLSE) equation, which yields *asymmetric* envelope solutions.

– $P = P_L = \omega_L''(k)/2 < 0$;

– $Q > 0$ (< 0) prescribes *stability* (instability) at *low* (high) k.





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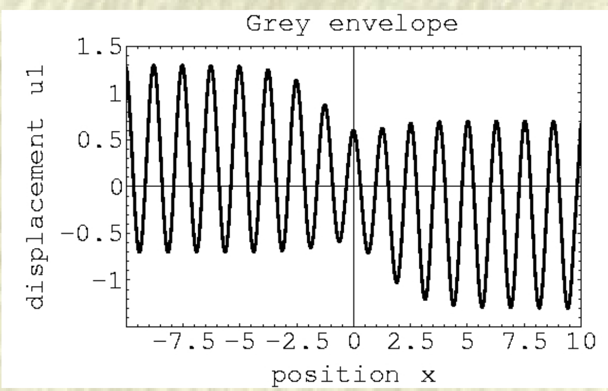
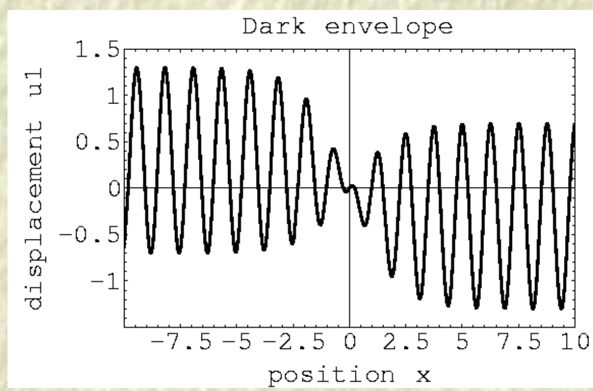




(at low k)

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 1384 (2004).]

Part 3: *Longitudinal* soliton formalism.

Q.: *A link to soliton theories: the Korteweg-deVries Equation.*

– *Continuum approximation*, viz. $\delta x_n(t) \to u(x,t)$.

– *"Standard"* description: keeping *lowest order* nonlinearity,

$$\ddot{u} + \nu\, \dot{u} - c_L^2\, u_{xx} - \frac{c_L^2}{12}\, r_0^2\, u_{xxxx} = -\, p_0\, u_x\, u_{xx}$$

$c_L = \omega_{L,0}\, r_0$; $\omega_{L,0}$ and p_0 were defined above.

– For *near-sonic propagation* (i.e. $v \approx c_L$), slow profile evolution in time τ and defining the *relative displacement* $w = u_\zeta$, one obtains

$$w_\tau - a\, w\, w_\zeta + b\, w_{\zeta\zeta\zeta} = 0$$

(for $\nu = 0$); $\zeta = x - vt$; $a = p_0/(2c_L) > 0$; $b = c_L r_0^2/24 > 0$.

– This KdV Equation yields soliton solutions, ... ($\to$ *next page*)

The KdV description

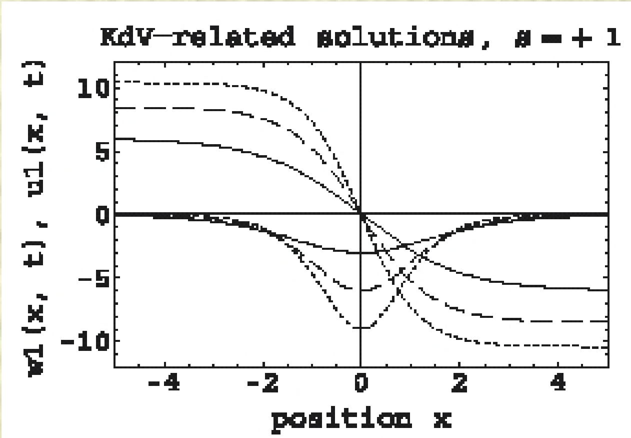
The Korteweg-deVries (KdV) Equation

$$w_\tau - a\,w\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0$$

yields *compressive* (*only*, here) solutions, in the form (here):

$$w_1(\zeta,\tau) = -w_{1,m} sech^2 \Big[(\zeta - v\tau - \zeta_0)/L_0 \Big]$$

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.



The KdV description

The Korteweg-deVries (KdV) Equation

$$w_\tau - a\,w\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0$$

yields *compressive* (*only*, here) solutions, in the form (here):

$$w_1(\zeta,\tau) = -w_{1,m} sech^2\Big[(\zeta - v\tau - \zeta_0)/L_0\Big]$$

– Pulse amplitude: $w_{1,m} = 3v/a = 6vv_0/|p_0|$;

– Pulse width: $L_0 = (4b/v)^{1/2} = [2v_1^2 r_0^2/(vv_0)]^{1/2}$;

– Note that: $w_{1,m}L_0^2 = \mathrm{constant}$ (cf. experiments)†.

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.

– This is the standard treatment of dust-lattice solitons today ... †

† F. Melandsø 1996; S. Zhdanov *et al.* 2002; K. Avinash *et al.* 2003; V. Fortov *et al.* 2004.

Characteristics of the KdV theory

The *Korteweg - deVries* theory presented above:

– provides a *correct qualitative description* of *compressive* excitations observed in experiments;

– draws benefit from the KdV *"artillery"* of analytical *know-how* obtained in the past: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

Characteristics of the KdV theory

The *Korteweg - deVries* theory presented above:

– provides a *correct qualitative description* of *compressive* excitations observed in experiments;

– benefits from the KdV *"artillery"* of analytical know-how obtained throughout the years: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

but possesses a few drawbacks:

– *approximate* derivation: (i) propagation velocity v near (longitudinal) sound velocity c_L, (ii) time evolution terms omitted *'by hand'*, (iii) higher order nonlinear contributions omitted;

– *only* accounts for *compressive* solitary excitations (for Debye interactions); nevertheless, the existence of *rarefactive* dust lattice excitations is, *in principle*, *not excluded*.

Longitudinal **soliton formalism** ***(continued)***

Q.: *What if we also kept the next order in nonlinearity ?*

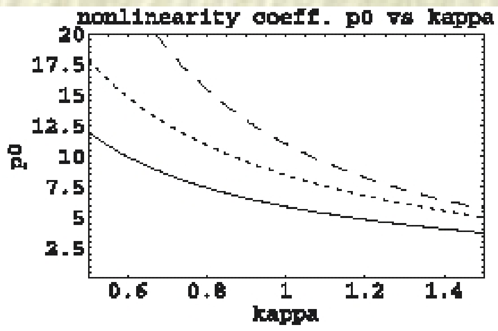
Longitudinal **soliton formalism** ***(continued)***

Q.: *What if we also kept the next order in nonlinearity ?*

– *“Extended”* description: :

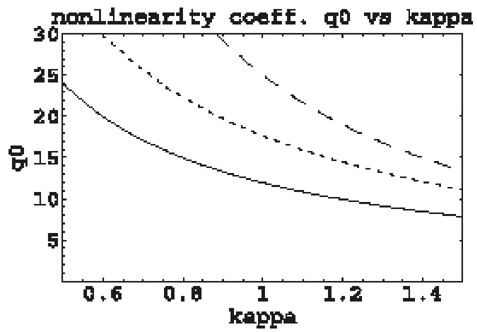
$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}$$

$c_L = \omega_{L,0}\,r_0$; $\quad \omega_{L,0},\ p_0 \sim -U'''(r)$ and $q_0 \sim U''''(r)$ (cf. above).



(b)

Fig. 4. (a) The nonlinearity coefficient p_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.



(b)

Fig. 5. (a) The nonlinearity coefficient q_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.

Rq.: q_0 is *not* negligible, compared to p_0! (instead, $q_0 \approx 2p_0$ practically!)

Longitudinal **soliton formalism** ***(continued)***

Q.: *What if we also kept the next order in nonlinearity ?*

– *"Extended"* description: :

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}$$

$c_L = \omega_{L,0}\,r_0$; $\omega_{L,0}$, p_0 and q_0 were defined above.

– For *near-sonic propagation* (i.e. $v \approx c_L$), and defining the *relative displacement* $w = u_\zeta$, one has

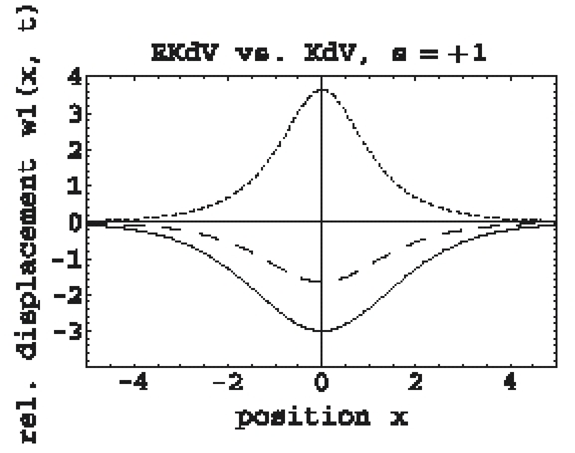
$$w_\tau - a\,w\,w_\zeta + \hat{a}\,w^2\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0 \qquad (4)$$

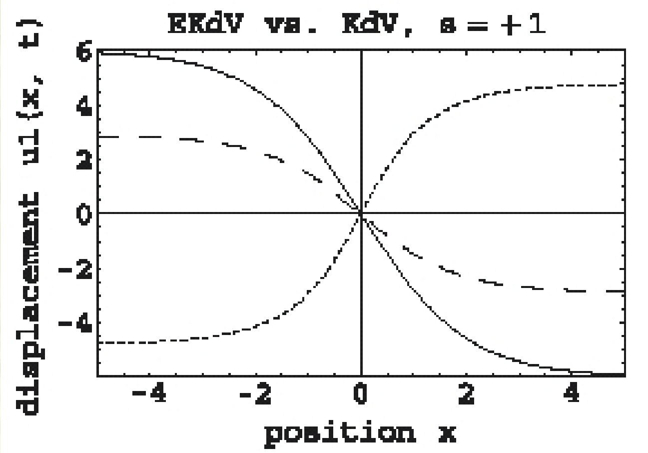
(for $\nu = 0$); $\zeta = x - vt$; $a = p_0/(2c_L) > 0$; $b = c_L r_0^2/24 > 0$; $\hat{a} = q_0/(2c_L) > 0$.

Characteristics of the EKdV theory

The *extended Korteweg - deVries* Equation:

– accounts for *both compressive and rarefactive* excitations;





(*horizontal grain displacement* $u(x, t)$)

– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– is previously widely studied, in literature;

Still, ...

– It was derived under the *assumption*: $v \approx c_L$.

One more alternative: the Boussinesq theory

The *Generalized* *Boussinesq* (Bq) Equation (for $w = u_x$):

$$\ddot{w} - c_L^2\, w_{xx} = \frac{c_L^2 r_0^2}{12}\, w_{xxxx} - \frac{p_0}{2}\, (w^2)_{xx} + \frac{q_0}{2}\, (w^3)_{xx}$$

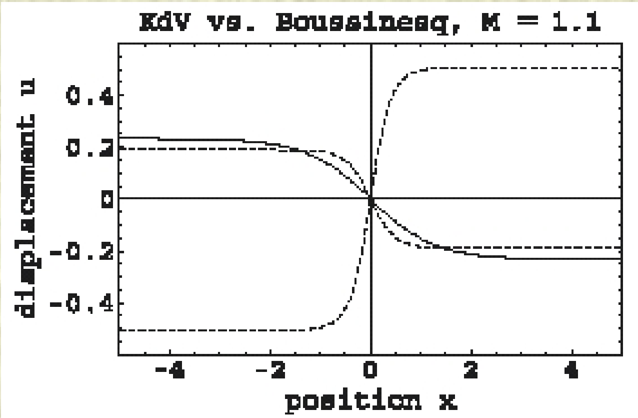
– predicts *both compressive and rarefactive* excitations;

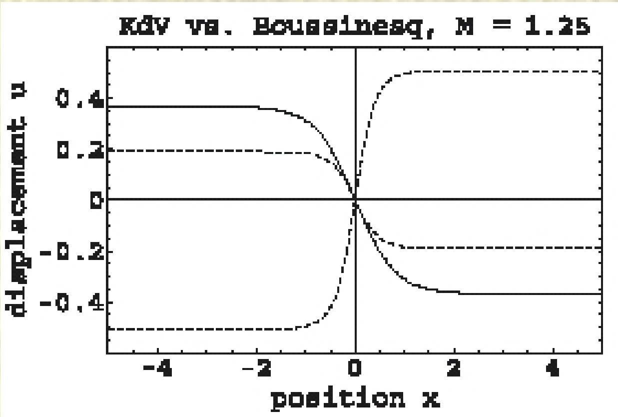
– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– has been widely studied in literature;

and, ...

– relaxes the velocity assumption, i.e. is valid $\forall \quad v > c_L$.





Conclusions

We have seen that:

– *Energy localization* via *modulational instability* leading to the formation of *envelope excitations* is possible in both *transverse* and *longitudinal* directions ;

– Solitary waves can be efficiently modeled by existing soliton theories (e.g. KdV, EKdV, MKdV; more accurately: Boussinesq, EBq) ;

– Compressive *and* rarefactive excitations are predicted ;

– *Urge* for experimental confirmation (technical constraints?) ;

– Future directions: include dissipation (dust-neutral friction, ion drag); particle-wake effects; mode coupling effects; ... *(Realism!)*

Thank You !

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Padma Kant Shukla

Bengt Eliasson

Material from:

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Appendix I: Solutions of the NLSE
Localized envelope excitations 1: *bright solitons*

- The NLSE accepts various soliton solutions: $\psi = \rho \, e^{i\Theta}$; the *total* wavepacket is then: $u \approx \epsilon \, \rho \, \cos(kx - \omega t + \Theta)$ where the *amplitude* ρ and *phase correction* Θ depend on ζ, τ.

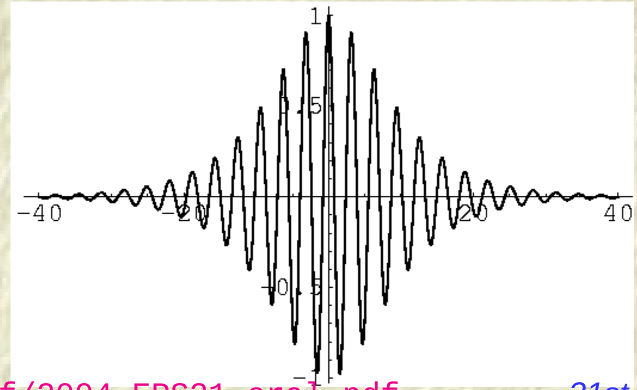
- Bright–type envelope soliton (pulse):

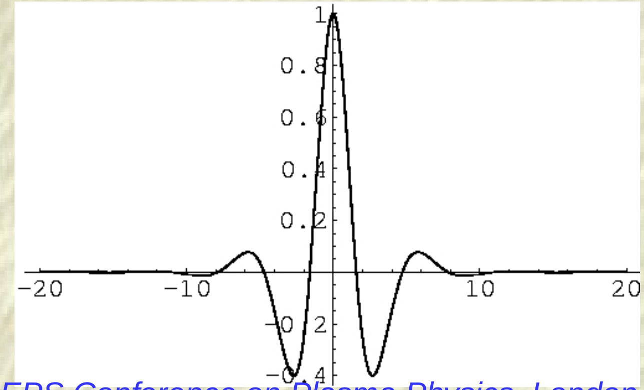
$$\rho = \rho_0 \, sech\left(\frac{X - u_e \, T}{L}\right) , \qquad \Theta = \frac{1}{2P} \left[u_e \, X \, + (\Omega - \frac{1}{2} u_e^2) T \right] . \quad (5)$$

$$L = \sqrt{\frac{2P}{Q} \frac{1}{\rho_0}}$$

$(X_0 = 0)$

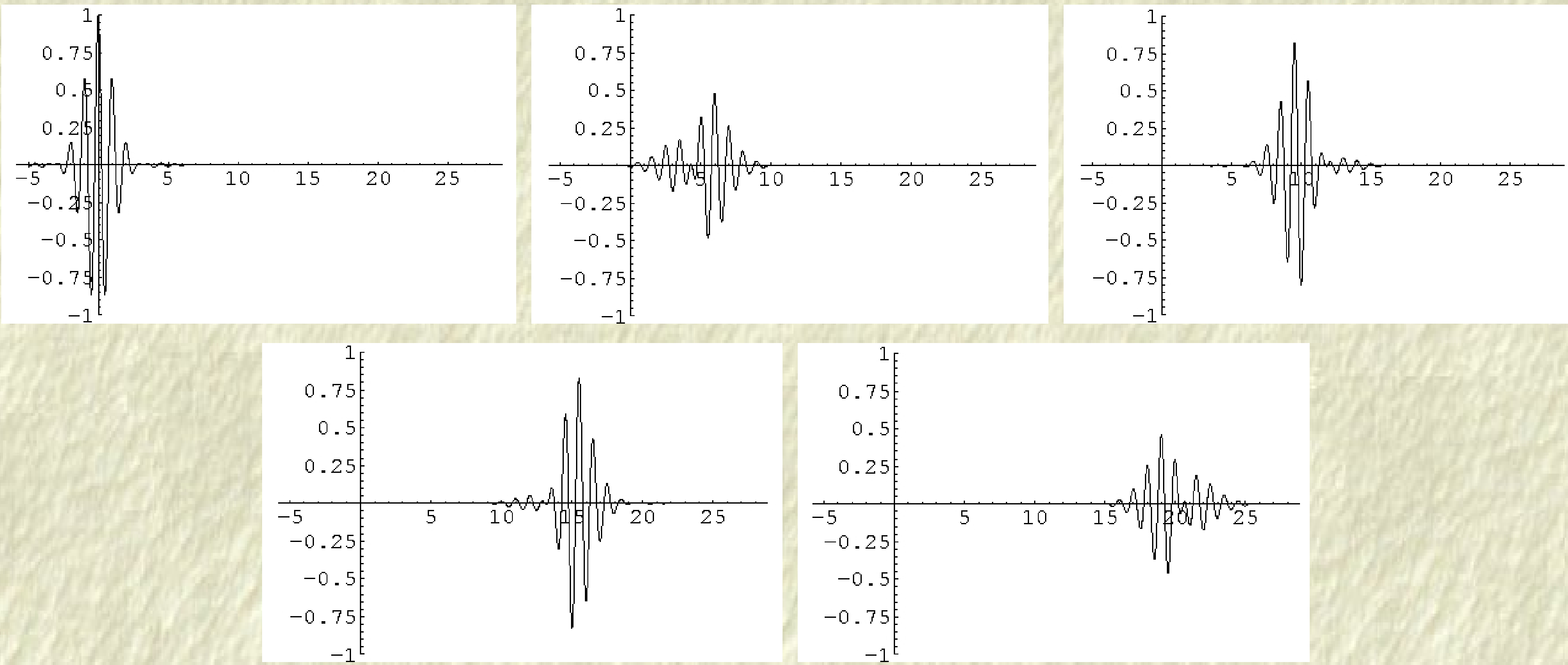
$(\Theta_0 = 0)$





Propagation of a bright envelope soliton (pulse)

This *envelope modulated wavepacket* is essentially *a propagating (and oscillating) localized pulse*, confining the *carrier wave*:



Localized envelope excitations 2: ***dark/grey solitons***

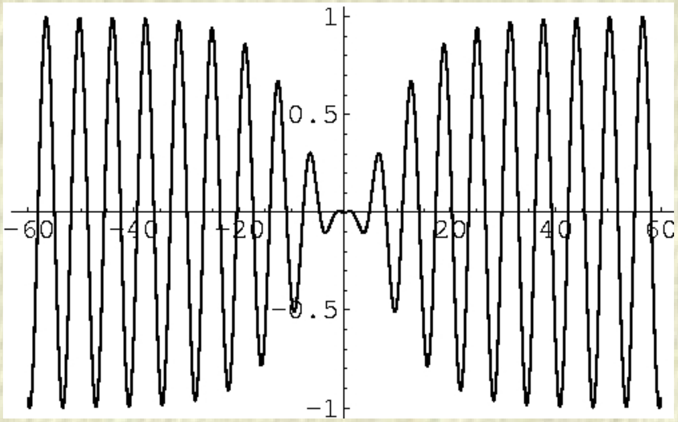
- Dark–type envelope solution (*hole soliton*):

$$\rho = \pm\rho_1 \left[1 - \operatorname{sech}^2\left(\frac{X - u_e T}{L'}\right)\right]^{1/2} = \pm\rho_1 \tanh\left(\frac{X - u_e T}{L'}\right),$$

$$\Theta = \frac{1}{2P}\left[u_e X - \left(\frac{1}{2}u_e^2 - 2PQ\rho_1^2\right)\tau\right]$$

$$L' = \sqrt{2\left|\frac{P}{Q}\right|\frac{1}{\rho_1}} \qquad (X_0 = 0) \qquad (6)$$

This is a *propagating localized hole* (zero density void):



dark/grey solitons (continued...)

- Grey–type envelope solution (*void soliton*):

$$\rho = \pm\rho_2 \left[1 - d^2 \, sech^2 \left(\frac{X - u_e T}{L''}\right)\right]^{1/2}$$

$$\Theta = \ldots$$

$$L'' = \sqrt{2 \left|\frac{P}{Q}\right| \frac{1}{d\rho_2}} \qquad (7)$$

This is a propagating *(non zero-density)* void: