1. Introduction

Discrete Breathers (DBs), or Intrinsic Localized Modes (ILMs), are highly localized oscillatory modes known to occur in Wigner crystals (e.g., atomic or molecular chains) characterized by nonlinear inter-site coupling and/or substrate potentials and a highly discrete structure [1].

The properties of DB modes have recently gathered an increasing interest in modern nonlinear science [2–6]. An exciting paradigm of such a nonlinear chain is provided by dust lattices, i.e., chains of monosized dust beams tightly bound, massless dust particulates, gathered in a strongly coupled arrangements spontaneously formed during plasma discharge experiments [7].

In earth laboratory experiments, these crystals are subject to an (intrinsically nonlinear) electric external field(s) and/or magnetic field(s), which balance gravity at the levitated equilibrium position, and are held together by electrostatic (Debye type) interaction forces. Although the linear properties of these crystals now seem well understood, the elucidation of the nonlinear mechanisms governing their dynamics is still in a preliminary stage. This study is devoted to an investigation, from first principles, of the existence of DB excitations, associated with transverse (off-plane, optical mode) dust grain oscillations in a dust mono-layer.

2. The model

The vertical (off-plane) grain displacement in a dust crystal (assumed quasi-one-dimensional, composed from identical grains of charge $q$ and mass $M$, located at $x_n = n r$, $n \in \mathbb{Z}$) obeys an equation of the form

$$\frac{d^2 \delta_x}{dt^2} + (\Delta_0 + \lambda (\Delta_2 + \Delta_3 - 2 \Delta_4) + \omega^2 \delta_x) = 0,$$

where $\Delta_0 = \Delta_2 - \Delta_3$ denotes the small displacement of the $n$-th grain around the (levitated) equilibrium position $x_0$, in the $(x_0)$ direction.

- The characteristic frequency

$$\omega_0 = \sqrt{q^2 F_0 (r/|M_0|)}$$

is related to the dust (deleterious) interaction potential $F_0 (r') = (q/q') e^{-r'/\lambda} F_{00}$. One has

$$\omega_0^2 = q^2 F_0 (r/|M_0|) + \omega^2 \lambda^2 (r/|M_0|),$$

where $F_{00}$ denotes the effective (DP) Debye length.

- The gap frequency $\omega_g$ and the nonlinearity coefficients $\alpha, \beta$ are defined via the overall force $F(x) = F_{Debye} + \alpha \delta x + \beta \delta x^2$.

3. A discrete envelope evolution equation.

Following Ref. [4] (and drawing inspiration from the quasi-continuum limit [8,9]), one may adopt the ansatz

$$\Delta_n \approx \Delta_0 e^{\omega_0 t + \lambda^2 (\Delta_2 + \Delta_3 - 2 \Delta_4) + \omega^2 t} + \cdots$$

(2) where we assume:

$$\omega_0^2 \gg \omega^2, \quad \alpha, \beta \gg \omega^2, \quad \lambda \gg 1 \quad \text{and} \quad \left[ \frac{d}{dt} \omega_0 \approx \left( \frac{d^2 \omega}{dt^2} \right) \omega_0^2 \approx \left( \frac{d^2 \lambda}{dt^2} \right) \ight] \approx \omega_0 \omega^2 \lambda^2 \approx \omega^2 \lambda^2 \approx \omega^2 \lambda^2.$$

implying a high $\omega_0/\omega$ ratio (this condition is clearly satisfied in recent experiments [10]).

Inserting into Eq. (1), one obtains the discrete nonlinear Schrödinger equation (DNLSE)

$$\Delta_{n+1} + P (\Delta_n + \Delta_{n-1}) + Q \Delta_n = 0,$$

where $Q = \omega_0^2/\omega^2$.

- dispersion coefficient:

$$P = -2 \omega_0^2 \Delta_0 < 0;$$

- nonlinearity coefficient:

$$Q = (15 \omega_0^4/2 \lambda^2 - 3/2 \lambda^2 \omega^2).$$

The sign of $Q$ depends on the characteristics and need to be determined from experiments.

4. Harmonic envelope solution and stability analysis

Eq. (4) yields a plane wave solution

$$\Delta_n = y_0 \exp (i \theta_n),$$

where the frequency $\omega$ and the envelope dispersion relation

$$\omega = \omega_0 \sqrt{1 - 4 P^2 \sin^2 (k_0/2) + Q^2 y_0^2},$$

where $Q < 0$.

The envelope stability may be estimated by setting

$$\Delta_n = \Delta_0 + \delta \exp [i (\omega_0 t - \omega n)],$$

and

$$\theta_n = \theta_0 + \delta [\exp (i k_0 x_0) - 1],$$

(where $\delta < 1$ and then linearizing in $\delta$), one thus obtains the perturbation dispersion relation

$$\omega = \omega_0 \sqrt{1 - 4 P^2 \sin^2 (k_0/2) + Q^2 y_0^2}.$$

The wave envelope will be unstable to external perturbations (viz. $\text{Im} \omega \neq 0$).

5. Bright-type localized gap modes.

It is known that modulation instability is here possible e.g. for $Q < 0$ and $k_0 > \pi/2$, in the following section of the linear modes in the frequency gap region [11].

Following Page 3, simple mode of the form $\exp (i \omega_0 t)$ odd-parity localized periodic solutions may be sought in the form

$$\Delta_n (u, \eta, \beta) = \Delta_0 \left( u (\eta, \beta) \right),$$

and the amplitude $u_0$ obeys the following solution of the form

$$u_0 = \frac{Q}{\pi^2} \xi^2 \text{cos} x_0 \sin \xi^2,$$

for $\xi > 0$, and $\text{Im} u_0 \neq 0$.

One thus obtains a localized lattice pattern of the form

$$\Delta_n = \Delta_0 \cos \left( \left[ u_0 (\eta, \beta) \right] \right) \cos x_0 \sin \xi^2.,$$

for $\eta > 0$. (See Fig. 1) where $Q < 0$ and $\eta = -P/(Q^2)$.

6. Dark/grey-type localized gap modes

Dark-type solutions (redu) may also be sought. For $Q < 0$, one should look into the region $k > \pi/2$ (or $\xi > 1$), e.g. near the cutoff frequency $\omega_{c2}$. One thus finds (via the pattern

$$\Delta_n = 2 \Delta \cos \left( \left[ u (\eta, \beta) \right] \right) \cos x_0 \sin \xi^2, \quad \text{for} \quad \eta = 0 < \xi < 1.$$