Intrinsic localized modes in dust lattices

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1. Introduction

Discrete Breathers (DBs), or *Intrinsic Localized Modes* (ILMs) are highly localized oscillatory modes known to occur in Wigner crystals (e.g. atomic or molecular chains) characterized by nonlinear inter–site coupling and/or substrate potentials and a highly discrete structure [1].

The properties of DB modes have recently gathered an increasing interest in modern nonlinear science [2 - 6].

An exciting paradigm of such a nonlinear chain is provided by dust lattices, i.e. chains of mesoscopic size heavily charged, massive dust particulates, gathered in a strongly coupled arrangements spontaneously formed during plasma discharge experiments [7].

In earth laboratory experiments, these crystals are subject to an (intrinsically nonlinear [8, 9]) external electric and/or magnetic field(s), which balance(s) gravity at the levitated equilibrium position, and are held together by electrostatic (Debye type) interaction forces.

Although the linear properties of these crystals now seem quite well understood, the elucidation of the *nonlinear* mechanisms governing their dynamics is still in a preliminary stage. This study is devoted to an investigation, from first principles, of the existence of DB excitations, associated with transverse (off-plane, optical mode) dust grain oscillations in a dust mono-layer.

2. The model

The vertical (off-plane) grain displacement in a dust crystal (assumed quasi-one-dimensional, composed from identical grains of charge q and mass M, located at $x_n = n\, r_0,\ n \in \mathcal{N}$) obeys an equation of the form

$$\frac{d^2\delta z_n}{dt^2} + \omega_0^2\,(\,\delta z_{n+1} + \delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,, \qquad (1)$$

where

— $\delta z_n = z_n - z_0$ denotes the small displacement of the $n-$th grain around the (levitated) equilibrium position z_0, in the transverse ($z-$) direction;

— the characteristic frequency

$$\omega_0 = \left[-q\Phi'(r_0)/(M r_0)\right]^{1/2}$$

is related to the dust grain (electrostatic) interaction potential $\Phi(r)$, e.g. for a Debye-Hückel potential: $\Phi_D(r) = (q/r)\,e^{-r/\lambda_D}$, one has

$$\omega_{0,D}^2 = q^2/(M r_0^3)\,(1 + r_0/\lambda_D)\,\exp(-r_0/\lambda_D)$$

(where λ_D denotes the effective DP Debye length).

— The gap frequency ω_g and the nonlinearity coefficients α, β are defined via the overall vertical force

$$F(z) = F_{e/m} - Mg \approx -M[\omega_g^2\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3] + \mathcal{O}[(\delta z_n)^4],$$

which has been expanded around z_0 by formally taking into account the (anharmonicity of the) local form of the sheath electric and/or magnetic field(s), as well as, possibly, grain charge variation (refer to the definitions in [8, 9], not reproduced here).

— The electric/magnetic levitating force(s) $F_{e/m}$ balance(s) gravity at z_0.

— Notice the difference in structure from the usual nonlinear Klein-Gordon equation used to describe 1d oscillator chains: 'phonons' in this chain are stable *only* thanks to the field force $F_{e/m}$ (via ω_g).

— Collisions with neutrals and coupling anharmonicity are omitted, at a first step, in this simplified model.

Eq. (1) leads to the discrete dispersion relation

$$\omega^2 = \omega_g^2 - 4\omega_0^2\,\sin^2\!\big(kr_0/2\big)\,. \qquad (2)$$

The wave frequency decreases with increasing wavenumber $k = 2\pi/\lambda$ (or decreasing wavelength λ), implying that transverse vibrations propagate as a *backward wave* (see that $v_g = \omega'(k) < 0$), in agreement with recent experiments [10].

— We notice the *gap frequency* ω_g, as well as the *cutoff frequency* $\omega_{min} = (\omega_g^2 - 4\omega_0^2)^{1/2}$ (obtained at the end of the first Brillouin zone $k = \pi/r_0$), which is *absent in the continuum limit*, viz. $\omega^2 \approx \omega_g^2 - \omega_0^2\,k^2\,r_0^2$ (for $k \ll r_0^{-1}$).

3. A discrete envelope evolution equation.

Following Ref. [4] (and drawing inspiration from the quasi-continuum limit [8, 9]), one may adopt the ansatz

$$\delta z_n \approx \epsilon\,(u_{1,n}^{(1)}\,e^{-i\omega_g t} + \text{c.c.}) + \epsilon^2\,[u_{2,n}^{(0)} + (u_{2,n}^{(2)}\,e^{-2i\omega_g t} + \text{c.c.})] + \ldots \qquad (3)$$

where we assume:

— $\omega_g^2,\ \alpha,\ \beta \sim 1$

and

— $d/dt, \omega_0^2 \sim \epsilon^2$,

implying a high ω_g/ω_0 ratio (this condition is clearly satisfied in recent experiments; cf. [10]).

Inserting into Eq. (1), one obtains the discrete nonlinear Schrödinger equation (DNLSE)

$$i\,\frac{du_n}{dt} + P\,(u_{n+1} + u_{n-1} - 2u_n) + Q\,|u_n|^2\,u_n = 0\,, \qquad (4)$$

— $u_n = u_{1,n}^{(1)}(t)$, and

$$u_{2,n}^{(2)} = [\alpha/(3\omega_g^2)]\,u^2\,, \qquad u_{2,n}^{(0)} = -(2\alpha/\omega_g^2)\,|u|^2$$

— *discreteness* coefficient:

$$P = -\omega_0^2/(2\omega_g) < 0\,;$$

— *nonlinearity* coefficient:

$$Q = (10\alpha^2/3\omega_g^2 - 3\beta)/2\omega_g\,;$$

The sign of Q depends on the sheath characteristics and need to be determined from experiments.

4. Harmonic envelope solution and stability analysis

Eq. (4) yields a plane wave solution

$$u_n = u_0\exp(i\theta_n) \equiv u_0\exp[-i(n\tilde{k}r_0 - \tilde{\omega}t)] + \text{c.c.}$$

where the frequency $\tilde{\omega}$ obeys the envelope dispersion relation:

$$\tilde{\omega}(\tilde{k}) = -4P\sin^2(\tilde{k}r_0/2) + Q|u_0|^2\,. \qquad (5)$$

The (envelope) *frequency band* (for $0 \le \tilde{k} \le \pi/2$) is:

$$|Q||u_0|^2\, \le \tilde{\omega}\, \le |Q|u_0|^2 - 4P|\,.$$

The envelope stability may be studied by setting

$$u_0 \to u_0 + \xi\hat{u}_0\exp[-i(n\hat{k}r_0 - \hat{\omega}t)]$$

and

$$\theta_n \to \theta_n + \xi\hat{\theta}_0\exp[-i(n\hat{k}r_0 - \hat{\omega}t)]\,,$$

(where $\xi \ll 1$) and then *linearizing* in ξ; one thus obtains the *perturbation* dispersion relation

$$[\hat{\omega} - 2P\sin\hat{k}r_0\sin\tilde{k}r_0]^2 = 4\,P\,\sin^2(\hat{k}r_0/2)\,\cos\tilde{k}r_0 \times[4P\sin^2(\hat{k}r_0/2)\,\cos\tilde{k}r_0 - 2Q|u_0|^2]\,. \qquad (6)$$

The wave envelope will be *unstable* to external perturbations (viz. $\mathrm{Im}\hat{\omega} \neq 0$) if

(i)

$$PQ\cos\tilde{k}r_0 > 0$$

and (in the same time)

(ii) the amplitude u_0 exceeds some critical value $u_{cr}(\tilde{k})$ [or, in fact, for any $\tilde{k}$, if $u_0 > (2P/Q)^{1/2} = u_{cr,max}$].

Otherwise, the wave envelope will be *stable*.

5. Bright-type localized gap modes.

It is known that modulational instability, here possible e.g. for $Q < 0$ *and* $0 < \tilde{k} < \pi/2r_0$, may result in the formation of localized modes in the linear frequency gap region [11].

Following Page [2], single-mode (fixed ω) *odd-parity* localized periodic solutions may be sought in the form

$$u_n(t) = f_n\,u_0\,\cos\Omega t = f_n\,u_0/2\,(\exp i\Omega t + \text{c.c.})$$

, assuming

$$f_0 = 1\,, \qquad f_{-n} = f_n\,, \qquad |f_n| \ll f_1 \qquad \text{for} \quad |n| > 1\,.$$

One thus obtains a localized lattice pattern of the form:

$$u_n(t) = u_0\,\cos\Omega t\,\{...,0,0,\eta,1,\eta,0,0,...\}\,, \qquad (7)$$

(see Fig. 1) where $\Omega \approx |Q|u_0^2/4$ and $\eta = 4P/(Qu_0^2) \ll 1$ (see that $\eta > 0$). We see that Ω lies outside the (amplitude) frequency band prescribed by Eq. (5).

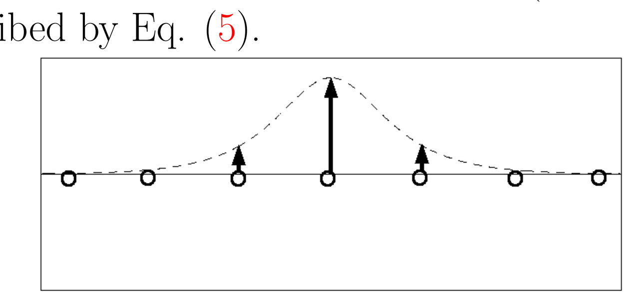

Figure 1. Discrete breather dust lattice excitation of the *bright* type, obtained for $Q < 0$; the successive lattice site displacements are depicted at maximum amplitude: this is an odd-parity solution, as given by Eq. (7).

For $Q > 0$, the modulational stability profile is reversed (instability occurs for $\tilde{k} > \pi/2r_0$). The same procedure then leads to an *even parity* solution of the form

$$u_n(t) = u_0\,\cos\Omega t\,\{...,0,0,-\eta,1,-\eta,0,0,...\}\,, \qquad (8)$$

where $\eta = -P/(Qu_0^2)$ satisfies $0 < \eta \ll 1$ and $\Omega \approx -4P + Qu_0^2$ (see Fig. 2).

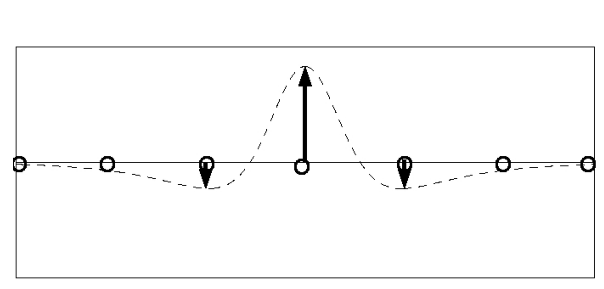

Figure 2. Even-parity bright-type breather solution, given by Eq. (8).

6. Dark/grey-type localized gap modes

Dark-type solutions (*voids*) may also be sought.

For $Q < 0$, for instance, one should look into the region $\tilde{k} > \pi/(2r_0)$, e.g. near the cutoff frequency $\tilde{\omega}_{cr,2}$.

One thus finds [12] the discrete pattern

$$u_n(t) = 2A\,\cos\Omega t\,\{...,1,-1,1,-(1-\eta),\,0,\,1-\eta,-1,1,-1,...\}\,, \qquad (9)$$

where $\Omega = 4P - Q|A|^2$ and $\eta = P/(QA^2)$ satisfies $0 < \eta \ll 1$; see Fig. 3. This *dark-type* discrete lattice excitation [4] is modulated by two opposite narrow shock-like (kink) excitations (see Fig. 3). This is, in fact, an approximate solution of Eq. (4), which fits more accurately for lower values of η.

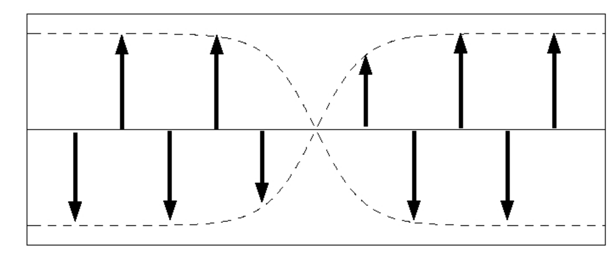

Figure 3. *Dark* type dust lattice excitation: successive dust grain displacements.

Focusing in the middle of the Brillouin zone, i.e. at $k = \pi/2r_0$ (where *odd* sites remain at rest, while *even* ones oscillate out of phase; cf. the analysis in [4, 13]), one obtains [4, 12] the pattern

$$u_n(t) = 2A\,\cos\Omega t\,\{...,1,0,-1,0,1-\eta,1-\eta,0,-1,0,1,...\}\,, \qquad (10)$$

where $\Omega = 2P - Q|A|^2$ and $\eta = P/2QA^2$ satisfies $0 < \eta \ll 1$. This *grey-type* discrete lattice excitation (known in solid lattices [4]), is depicted in Fig. 4.

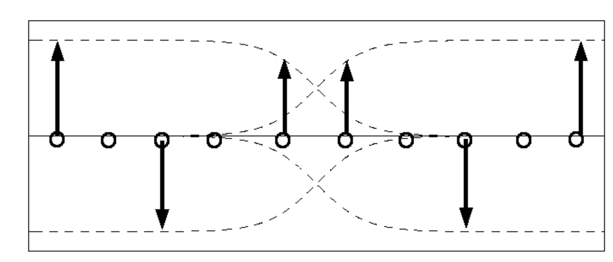

Figure 4. *Grey* type dust lattice excitation: successive dust grain displacements.

7. Breather control.

The stability of a breather excitation may be *controlled* via external feedback, as known from one-dimensional discrete solid chains [5]. The method consists in using the knowledge of a reference state (unstable breather), say $\delta z_n^{(0)} = \hat{z}_n(t)$, e.g. obtained via an investigation of the homoclinic orbits of the 2d map obeyed by the main Fourier component [6], and then perturbing the evolution equation (1) by adding a term $+K[\hat{z}_n(t) - \delta z_n]$ in the right-hand side (*rhs*), in order to stabilize breathers via tuning of the control parameter K. This method relies on the application of the *continuous feedback control* (*cfc*) formalism (see the Refs. in [6]).

Alternatively, as argued in [6], a more efficient scheme should instead involve a term $+Ld[\hat{z}_n(t) - \delta z_n]/dt$ in the *rhs* of Eq. (1) (*dissipative cfc*), whence the damping imposed results in a higher convergence to the desired solution $\hat{z}_n(t)$. Preliminary work in this direction is being carried out and progress will be reported later.

8. Conclusions.

We have shown that a dusty plasma crystal can sustain localized discrete breather (vertical) excitations of either bright- or dark-/grey-type, i.e. extremely localized pulses or voids, respectively.

These localized structures owe their existence to the intrinsic lattice discreteness in combination with the nonlinearity of the plasma sheath. Both are experimentally tunable physical mechanisms, so our results may be investigated (and will hopefully be verified) by appropriately designed experiments, showing the way to potential applications involving pulse localization in dusty plasma crystals.

Finally, the question of the existence of multi-mode breathers, i.e. discrete excitations in the form: $u_n(t) = \sum_{k=-\infty}^{\infty} A_n(k)\exp(ik\omega t)$ with $A_n(k) = A_n^*(-k)$ for reality and $|A_n(k)| \to 0$ as $n \to \pm\infty$, for localization, should be more rigorously addressed in a forthcoming work, which is in preparation.

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