Intrinsic localized modes in dust lattices *

(Part 2: multi-mode breathers)

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I. THE MODEL

We shall consider the vertical (off-plane, $\sim \hat{z}$) grain displacement in a dust crystal (assumed quasi-one-dimensional: identical grains of charge q and mass M are situated at $x_n = n\, r_0$, where $n = ..., -1, 0, 1, 2, ...$), by taking into account the intrinsic nonlinearity of the sheath electric (and/or magnetic) potential. The in-plane (longitudinal, acoustic, $\sim \hat{x}$ and shear, optical, $\sim \hat{y}$) degrees of freedom are assumed suppressed; this situation is indeed today realized in appropriate experiments, where an electric potential (via a thin wire) [21] or a coherent light (laser) impulse [22–24] is used to trigger transverse dust grain oscillations, while (a) confinement potential(s) ensure(s) the chain's in-plane stability.

A. Equation of motion

The vertical grain displacement obeys an equation in the form [18, 19]

$$\frac{d^2\delta z_n}{dt^2} + \nu \frac{d\delta z_n}{dt} + \omega_0^2 \,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\, \delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,, \qquad (1)$$

where $\delta z_n(t) = z_n(t) - z_0$ denotes the small displacement of the $n-$th grain around the (levitated) equilibrium position z_0, in the transverse ($z-$) direction. The characteristic frequency $\omega_0 = [-q\Phi'(r_0)/(Mr_0)]^{1/2}$ results from the dust grain (electrostatic) interaction potential $\Phi(r)$, e.g. for a Debye-Hückel potential [26]: $\Phi_D(r) = (q/r)\, e^{-r/\lambda_D}$, one has: $\omega_{0,D}^2 = q^2/(Mr_0^3)\,(1+r_0/\lambda_D)\, \exp(-r_0/\lambda_D)$, where λ_D denotes the effective DP Debye radius [1]. The damping coefficient ν accounts for dissipation due to collisions between dust grains and neutral atoms. The gap frequency ω_g and the nonlinearity coefficients α, β are defined via the overall vertical force:

$$F(z) = F_{e/m} - Mg \approx -M[\omega_g^2 \delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3] \, + \mathcal{O}[(\delta z_n)^4]\,, \qquad (2)$$

which has been expanded around z_0 by formally taking into account the (anharmonicity of the) local form of the sheath electric (follow exactly the definitions in Ref. [18], not reproduced here) and/or magnetic [27] field(s), as well as, possibly, grain charge variation due to charging processes [19]. Recall that the electric/magnetic levitating force(s) $F_{e/m}$ balance(s) gravity at z_0. Notice the difference in structure from the usual nonlinear Klein-Gordon equation used to describe one-dimensional oscillator chains — cf. e.g. Eq. (1) in

Ref. [6]: TDLWs (*'phonons'*) in this chain are stable only in the presence of the field force $F_{e/m}$.

For convenience, we may re-scale the time and vertical displacement variables over appropriate quantities, i.e. the characteristic (single grain) oscillation period ω_g^{-1} and the lattice constant r_0, respectively, viz. $t = \omega_g^{-1}\tau$ and $\delta z_n = r_0 q_n$; Eq. (1) is thus expressed as:

$$\frac{d^2 q_n}{d\tau^2} + \epsilon(\, q_{n+1} + \, q_{n-1} - 2\, q_n) + \, q_n + \alpha'\, q_n^2 + \beta'\, q_n^3 = 0\,, \tag{3}$$

where the (dimensionless) damping term, now expressed as $(\nu/\omega_g)dq_n/d\tau \equiv \nu'\dot{q}_n$, will be henceforth omitted in the left-hand side. The coupling parameter $\epsilon = \omega_0^2/\omega_g^2$ measures the strength of the inter-grain interactions (with respect to the single-grain vertical vibrations); this is typically a *small* parameter, in real experiments (see below). The nonlinearity coefficients are now: $\alpha' = \alpha r_0/\omega_g^2$ and $\beta' = \beta r_0^2/\omega_g^2$.

Eq. (3) will be the basis of the analysis that will follow. Note that the primes in α' and β' will henceforth be omitted.

B. The model Hamiltonian

In order to relate our physical problem to existing generic models from solid state physics, it is appropriate to consider the equation of motion (1) as it may be derived from a Hamiltonian function, which here reads:

$$H = \sum_{j=1}^{N}\left[\frac{p_j^2}{2m_j} + V(q_j) - \frac{\epsilon}{2}(q_j - q_j - 1)^2\right]. \tag{4}$$

Here, p_j obviously denotes the (classical) momentum $p_j = M\dot{q}_j$. The substrate potential, related to the sheath plasma environment, is of the form:

$$V(q_j) = \frac{1}{2}\, q_j^2 + \frac{\alpha}{3}\, q_j^3 + \frac{\beta}{4}\, q_j^4\,. \tag{5}$$

The coupling parameter ϵ takes *small* numerical values (cf. below), accounting for the high lattice discreteness anticipated in this study. The minus sign preceding it denotes the inverse dispersive character of (linear excitations propagating in) the system; see the discussion below. Upon setting $\epsilon \to -\epsilon$, the 'traditional' (discretized) *nonlinear Klein-Gordon* model is recovered [28].

II. LINEAR WAVES

Retaining only the linear contribution and considering oscillations of the type, $\delta z_n \sim \exp[i\,(knr_0 - \omega t)] + c.c.$ (complex conjuguate) in Eq. (1), one obtains the well known transverse dust lattice (TDL) wave optical-mode-like dispersion relation

$$\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2\left(\frac{kr_0}{2}\right), \tag{6}$$

i.e.

$$\tilde{\omega}^2 = 1 - 4\epsilon \sin^2(\tilde{k}/2). \tag{7}$$

See that the wave frequency $\omega \equiv \tilde{\omega}\omega_g$ *decreases* with increasing wavenumber $k = 2\pi/\lambda \equiv \tilde{k}/r_0$ (or decreasing wavelength λ), implying that transverse vibrations propagate as a *backward wave*: the group velocity $v_g = \omega'(k)$ and the phase velocity $\omega_{ph} = \omega/k$ have opposite directions (this behaviour has been observed in recent experiments). The modulational

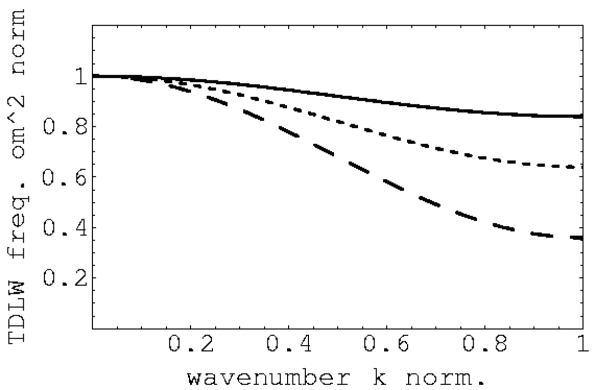

FIG. 1: The dispersion relation of the TDL excitations: frequency ω (normalized over ω_g) versus wavenumber k. The value of ω_0/ω_g ($\sim$ coupling strength) increase from top to bottom. Note that upper (less steep, continuous) curve is more likely to occur in a real (weakly-coupled) DP crystal.

stability profile of these linear waves (depending on the plasma parameters) was investigated in Refs. [18, 19]. Notice the natural *gap frequency* $\omega(k=0) = \omega_g = \omega_{max}$, corresponding to an overall motion of the chain's center of mass, as well as the *cutoff frequency* $\omega_{min} = (\omega_g^2 - 4\omega_0^2)^{1/2} \equiv \omega_g\,(1 - 4\epsilon^2)^{1/2}$ (obtained at the end of the first Brillouin zone $k = \pi/r_0$) which is *absent in the continuum limit*, viz. $\omega^2 \approx \omega_g^2 - \omega_0^2\,k^2\,r_0^2$ (for $k \ll r_0^{-1}$); obviously, the study of wave propagation in this ($k \lesssim \pi/r_0$) region invalidates the continuum treatment employed so far in literature. The essential feature of discrete dynamics, to be retained here, is the (narrow) bounded TDLW (*'phonon'*) frequency band, limited in the interval $\omega \in [(\omega_g^2 - 4\omega_0^2)^{1/2}, \omega_g]$; note that one thus naturally obtains the stability constraint: $\omega_0^2/\omega_g^2 = \epsilon < 1/4$ (so that $\omega \in \Re \quad \forall k \in [0, \pi/r_0]$).

III. EXISTENCE OF DISCRETE BREATHERS - ANALYSIS

We are interested in the (possibility for the) existence of multi-mode breathers, i.e. localized (discrete) excitations in the form:

$$q_n(\tau) = \sum_{m=-\infty}^{\infty} A_n(m) \exp(im\omega\tau) \,, \tag{8}$$

with $A_n(m) = A_n^*(-m)$ for reality and $|A_n(m)| \to 0$ as $n \to \pm\infty$, for localization.

A. The formalism

Inserting Eq. (8) in the equation of motion (3), one obtains a (numerable) set of algebraic equations in the form:

$$\begin{aligned} A_{n+1}(m) + A_{n-1}(m) + C_m A_n(m) &= -\frac{\beta}{\epsilon} \sum_{m_1} \sum_{m_2} \sum_{m_3} A_n(m_1) A_n(m_2) A_n(m_3) \\ &\quad - \frac{\alpha}{\epsilon} \sum_{m_4} \sum_{m_5} A_n(m_4) A_n(m_5) \,, \end{aligned} \tag{9}$$

where the dummy indices m_j ($j = 1, 2, ..., 5$) satisfy $m_1 + m_2 + m_3 = m_4 + m_5 = m$; we have defined:

$$C_m = -\Big(2 - \frac{1 - m^2\omega^2}{\epsilon}\Big) \,. \tag{10}$$

In order to be more precise and gain in analytical tractability (yet somewhat losing in generality), one may assume that the contribution of higher (for $m \geq 2$) frequency harmonics may be neglected. Eq. (8) then reduces to:

$$q_n(t) \approx 2A_n(1) \cos\omega\tau \, + A_n(0) \,. \tag{11}$$

Note the zeroth-harmonic (mean displacement) term, for $n = 0$, which is due to the cubic term ($\sim \alpha$, above), and should vanish for $\alpha = 0$. The system (9) thus becomes (for $m = 0, 1$):

$$\begin{aligned} A_{n+1}(1) + A_{n-1}(1) + C_1 A_n(1) &= -2\frac{\alpha}{\epsilon} A_n(1) A_n(0) - \frac{\beta}{\epsilon} \left[A_n(1) A_n^2(0) + 3A_n^2(1) A_n(-1)\right] \\ A_{n+1}(0) + A_{n-1}(0) + C_0 A_n(0) &= -2\frac{\alpha}{\epsilon} A_n(1) A_n(-1) \, - 6\frac{\beta}{\epsilon} \, A_n(0) A_n(1) A_n(-1) \,, \end{aligned} \tag{12}$$

i.e., setting $A_n(1) = A_n(-1) = A_n$ and $A_n(0) = B_n$, viz. $q_n(t) = 2A_n \cos\omega\tau + B_n$:

$$\begin{aligned} A_{n+1} + A_{n-1} + C_1 A_n &= -2\frac{\alpha}{\epsilon} A_n B_n - \frac{\beta}{\epsilon}(A_n B_n^2 + 3A_n^3) \\ B_{n+1} + B_{n-1} + C_0 B_n &= -2\frac{\alpha}{\epsilon} A_n^2 - 6\frac{\beta}{\epsilon} A_n^2 B_n\,. \end{aligned} \tag{13}$$

We see that the amplitudes A_n (B_n) of the first (zeroth) harmonic terms, corresponding to the $n-$th site, will be given by the iterative solution of Eqs. (13) [or, of Eqs. (9), should higher harmonics m be considered]. In specific, one may express (13) as:

$$\begin{aligned} a_{n+1} &= -c_n - C_1 a_n + 2\frac{\alpha}{\epsilon} a_n b_n + \frac{\beta}{\epsilon}(a_n b_n^2 + 3a_n^3) \equiv f_1(a_n, b_n, c_n, d_n) \\ b_{n+1} &= -d_n - C_0 b_n + 2\frac{\alpha}{\epsilon} a_n^2 + 6\frac{\beta}{\epsilon} a_n^2 b_n \equiv f_0(a_n, b_n, c_n, d_n) \\ c_{n+1} &= a_n \\ d_{n+1} &= b_n\,, \end{aligned} \tag{14}$$

and then iterate, for a given initial condition $(a_1, b_1, c_1, d_1) = (A_1, B_1, A_0, B_0)$, the map defined by (14).

At this stage, one needs to determine whether the fixed point of the 4-dimensional map (14) [or of the complete 4N-dimensional map corresponding to (9), in general] is hyperbolic, and examine the dimensionality of its stable and unstable manifolds. It is known [12, 13] that the existence of discrete breathers is associated with homoclinic orbits, implying a saddle point at the origin.

Let us now linearize the map (14) near the fixed point $(a_1, b_1, c_1, d_1) = (0, 0, 0, 0) \equiv \mathbf{0}_4$, by setting e.g. $(a_n, b_n, c_n, d_n) = (\xi_1, \xi_2, \xi_3, \xi_4)_n^T \equiv \mathbf{\Xi}_n \in \Re^4$, where $\xi_{j,n} \ll 1$ ($j = 1, ..., 4$). One thus obtains the matrix relation:

$$\mathbf{\Xi}_{n+1} = \mathbf{M}\,\mathbf{\Xi}_n\,, \tag{15}$$

where $\mathbf{M}$ is the matrix:

$$\mathbf{M} = \begin{pmatrix} -C_1 & 0 & -1 & 0 \\ 0 & -C_0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{16}$$

Now, it is a trivial algebraic exercise to show that the characteristic polynomial $p(\lambda) \equiv Det(\mathbf{M} - \lambda\mathbf{I})$ of this matrix may be factorized as:

$$p(\lambda) = (\lambda^2 + C_0\lambda + 1)(\lambda^2 + C_1\lambda + 1) \equiv p_0(\lambda)p_1(\lambda)\,,$$

implying the existence of 4 eigenvalues, say $\lambda_{1,2,3,4}$, such that $p_0(\lambda_{1,2}) = p_0(\lambda_{3,4}) = 0$. One may check that the condition for all eigenvalues to be real and different, hence for $\mathbf{0}_4$ to be a saddle point, amounts to the constraint: $|C_{0,1}| > 2$, i.e. $C_0 \notin [-2, 2]$ *and* $C_1 \notin [-2, 2]$. Recalling that

$$C_1 = (1 - 2\epsilon - \omega^2)/\epsilon\,, \qquad C_0 = (1 - 2\epsilon)/\epsilon, \tag{17}$$

from (10), one finds the (simultaneous) constraints: $1-4\epsilon > 0$ and $(1-\omega^2)(1-\omega^2-4\epsilon) > 0$. One immediately sees that the former (i.e. $\epsilon < 1/4$) corresponds to the linear stability condition mentioned above, while the latter amounts to the requirement that the breather frequency should lie outside the 'phonon band', viz. $\omega^2/\omega_g^2 \notin [1 - 4\epsilon, 1]$.

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It is straightforward to show that in case one considers the complete multi-mode map, defined by Eq. (9), one obtains an analogous factorizable characteristic polynomial for the $4N \times 4N$ matrix $\mathbf{M}$, viz. $p(\lambda) = \prod_m p_m(\lambda)$. The same analysis then leads to the hyperbolicity criterion:

$$|C_m| < 2 \qquad m = 0, 1, 2, ...$$

One thus recovers, in addition to the first of the above constraint ($\epsilon < 1/4$), the condition: $m\omega/\omega_g \notin (1 - 4\epsilon, 1)^{1/2}$ ($\forall m = 0, 1, 2, ...$), which coincides with the – physically meaningful – non-breather-phonon-resonance condition found via different analytical methods [8–10]. We see that the breather frequency, as well as all its multiples (harmonics) should lie outside the allowed linear vibration frequency band, otherwise the breather may enter in resonance with the linear TDLW (*'phonon'*) dispersion curve, resulting in its being decomposed into a superposition of linear excitations (and hence de-localized).

B. Numerical analysis

At this stage, one is left with task of finding the numerical values of A_n, B_n [cf. (13)] for a given homoclinic orbit; these may then be used as an initial condition, in order to solve the equation (13) numerically, by considering a given number of particles N and harmonic modes m_{max} (viz. $m = 0, 1, 2, ..., m_{max}$). One thus obtains a given set of numerical values for u_n ($n = 1, 2, ..., N$), which constitute the numerical solution for the anticipated breather excitation. The stability of the solution thus obtained, say $\hat{q}_n$, my be checked by directly substituting with $q_n = \hat{q}_n + \xi_n$ (for $n = -N, ..., 0, ..., N$) into the initial equation of motion (3).

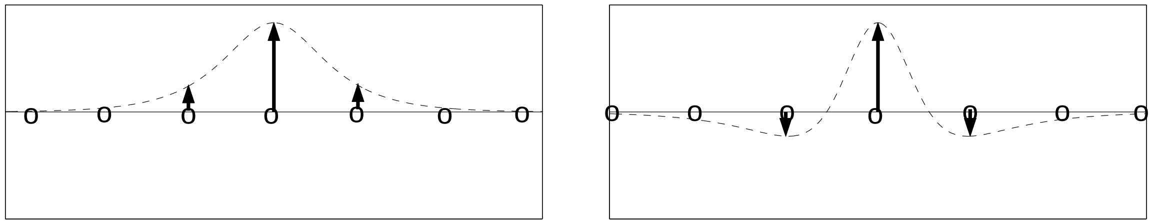

FIG. 2: Localized discrete breather dust lattice excitations; the successive lattice site displacements are depicted at maximum amplitude: (a) odd-parity solution; (b) even-parity solution.

This numerical scheme is now being elaborated, and the detailed results will be reported in an extended paper, in preparation.

IV. CONCLUSIONS - DISCUSSION

We have investigated, from first principles, the possibility of existence of localized discrete breather-type excitations associated with vertical dust grain motion in a dust mono-layer, which is assumed to be one-dimensional. It may be noted that these localized structures owe their existence to the intrinsic lattice discreteness in combination with the nonlinearity of the plasma sheath. Both are experimentally tunable physical mechanisms, so our results may be investigated (and will hopefully be verified) by appropriately designed experiments. The experimental confirmation of their existence in dust crystals appears as a promising field, which may open new directions e.g. in the design of applications.

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