Theory of solitary waves in complex plasma lattices [*]

Ioannis Kourakis[†], Padma Kant Shukla[‡] and Bengt Eliasson[§]
Institut für Theoretische Physik IV,
Fakultät für Physik und Astronomie,
Ruhr–Universität Bochum, D-44780 Bochum, Germany

(Dated: October 20, 2004)

Abstract

A comprehensive analytical theory for nonlinear excitations related to horizontal (longitudinal, acoustic mode) as well as vertical (transverse, optical mode) motion of charged dust grains in a dust crystal is presented. Different types of localized excitations, similar to those well known in solid state physics, are reviewed and conditions for their occurrence and characteristics in dusty plasma crystals are discussed. By employing a continuum approximation (i.e. assuming a long variation scale, with respect to the inter-particle distance) a dust crystal is shown to support nonlinear kink-shaped supersonic solitary excitations, associated with longitudinal dust grain displacement, as well as modulated envelope localized modes associated with either longitudinal or transverse oscillations. Although a one-dimensional crystal is considered for simplicity, the results in principle apply to a two-dimensional lattice if certain conditions are satisfied. The effect of mode-coupling is also briefly considered. The relation to previous results on atomic chains, and also to experimental results on strongly-coupled dust layers in gas discharge plasmas, is briefly discussed.

---

[*] *International Conference on Plasma Physics - ICPP 2004*, Nice (France), 25 - 29 Oct. 2004;
Poster contribution P3-042 (session P3/topic C: Fundamental Plasma Physics);
Paper available online at: `http://hal.ccsd.cnrs.fr/ccsd-00001890/en/`.

[†] On leave from: U.L.B. - Université Libre de Bruxelles, Physique Statistique et Plasmas C. P. 231, Boulevard du Triomphe, B-1050 Brussels, Belgium; also: Faculté des Sciences Apliquées - C.P. 165/81 Physique Générale, Avenue F. D. Roosevelt 49, B-1050 Brussels, Belgium;
Electronic address: `ioannis@tp4.rub.de`

[‡] Electronic address: `ps@tp4.rub.de`

[§] Electronic address: `bengt@tp4.rub.de`

I. INTRODUCTION

Dust contaminated plasmas (dusty plasmas, DP) have been attracting significant interest recently. Particularly important are dust quasi-lattices, which are typically formed in the sheath region above the negative electrode in discharge experiments, horizontally suspended at a levitated equilibrium position at $z = z_0$, where gravity and electric (and/or magnetic) forces balance. The linear regime of low-frequency oscillations in DP crystals, in the longitudinal (acoustic mode) and transverse (in-plane, shear acoustic mode and vertical, off-plane optical mode) direction(s), is now quite well understood. However, the *nonlinear* behaviour of DP crystals is still mostly unexplored, and has lately attracted experimental [1 - 3] and theoretical [1 - 9] interest.

Recently [5], we considered the coupling between the horizontal ($\sim \hat{x}$) and vertical (off-plane, $\sim \hat{z}$) degrees of freedom in a dust mono-layer; a set of nonlinear equations for longitudinal and transverse dust lattice waves (LDLWs, TDLWs) was thus rigorously derived [5]. Here, we review the nonlinear dust grain excitations which may occur in a DP crystal (here assumed quasi-one-dimensional and infinite, composed from identical grains, of equilibrium charge q and mass M, located at $x_n = n\,r_0,\ n \in \mathcal{N}$). Ion-wake and ion-neutral interactions (collisions) are omitted, at a first step. This study complements recent experimental investigations [1-3] and may hopefully motivate future ones.

II. *TRANSVERSE* ENVELOPE STRUCTURES.

The vertical (off-plane) $n-$th grain displacement $\delta z_n = z_n - z_0$ in a dust crystal obeys the equation [10, 11]

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \,\omega_{T,0}^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \quad (1)$$

The characteristic frequency

$$\omega_{T,0}\, = [-qU'(r_0)/(Mr_0)]^{1/2}$$

is related to the interaction potential $U(r)$ [e.g. for a Debye-Hückel potential: $U_D(r) = (q/r)\,e^{-r/\lambda_D}$, one has

$$\omega_{0,D}^2\, = \omega_{DL}^2\,\exp(-\kappa)\,(1+\kappa)/\kappa^3\,, \quad (2)$$

where $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$ is the characteristic dust-lattice frequency scale; λ_D is the Debye length; $\kappa = r_0/\lambda_D$ is the DP lattice parameter]. The *gap frequency* ω_g and the nonlinearity coefficients α, β are defined via the potential

$$\Phi(z) \approx \Phi(z_0) + M\Big[\frac{1}{2}\omega_g^2\delta z_n^2 + \frac{\alpha}{3}\,(\delta z_n)^3 + \frac{\beta}{4}\,(\delta z_n)^4\Big] + \mathcal{O}[(\delta z_n)^5] \quad (3)$$

(formally expanded near z_0, taking into account the electric and/or magnetic field inhomogeneity and charge variations [12]), i.e. leading to an overall vertical force

$$F(z) = F_{el/m}(z) - Mg \equiv -\partial\Phi(z)/\partial z \approx -M[\omega_g^2\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3] + \mathcal{O}[(\delta z_n)^4]\,.$$

Recall that $F_{e/m}(z_0) = Mg$.

Linear transverse dust-lattice excitations, viz. $\delta z_n \sim \cos\phi_n$ (here $\phi_n = nkr_0 - \omega t$) obey the *optical*-like *discrete* dispersion relation [14]:

$$\omega^2\, = \omega_g^2\, - 4\omega_{T,0}^2\,\sin^2(kr_0/2) \equiv \omega_T^2\,. \quad (4)$$

The TDLW dispersion curve is depicted in Fig 2. Transverse vibrations propagate as a *backward wave* [see that $v_{g,T} = \omega_T'(k) < 0$] – for any form of $U(r)$ – cf. recent experiments [2]. Notice the lower cutoff $\omega_{T,min}\, = (\omega_g^2\, - 4\omega_{T,0}^2)^{1/2}$ (at the edge of the Brillouin zone, at $k = \pi/r_0$), which is *absent in the continuum limit.* (for $k \ll r_0^{-1}$).

Allowing for a slight departure from the small amplitude (linear) assumption, one obtains:

$$\delta z_n \approx \epsilon\,(A\,e^{i\phi_n} + \text{c.c.}) + \epsilon^2\left[-\frac{2|A|^2}{\omega_g^2} + \left(\frac{A^2}{3\omega_g^2}\,e^{2i\phi_n} + \text{c.c.}\right)\right] + \ldots\,. \quad (5)$$

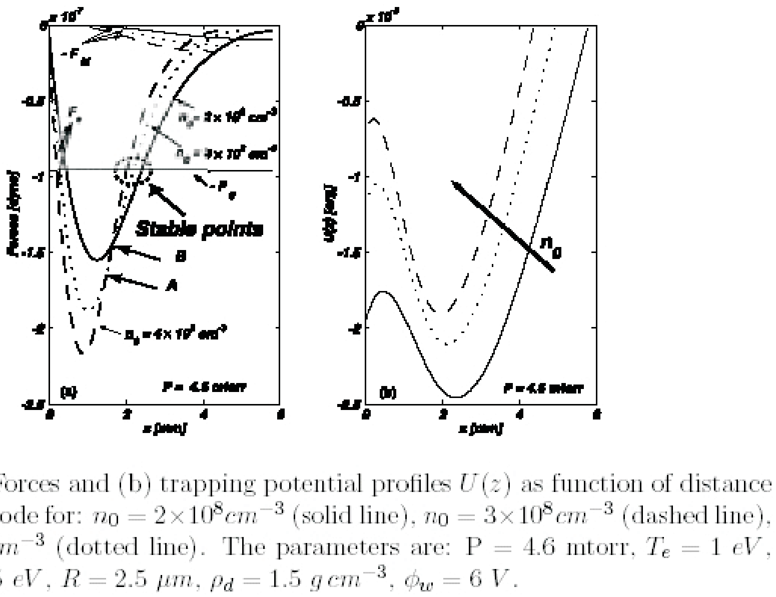

Figure 3: (a) Forces and (b) trapping potential profiles $U(z)$ as function of distance from the electrode for: $n_0 = 2\times10^8 cm^{-3}$ (solid line), $n_0 = 3\times10^8 cm^{-3}$ (dashed line), $n_0 = 4\times10^8 cm^{-3}$ (dotted line). The parameters are: P = 4.6 mtorr, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g cm^{-3}$, $\phi_w = 6\ V$.

FIG. 1: The (anharmonic) sheath (a) force $F(z)$, and (b) force potential $V(z)$, depicted vs. the vertical distance z from the negative electrode, in plasma discarge experiments; fig. from [13].

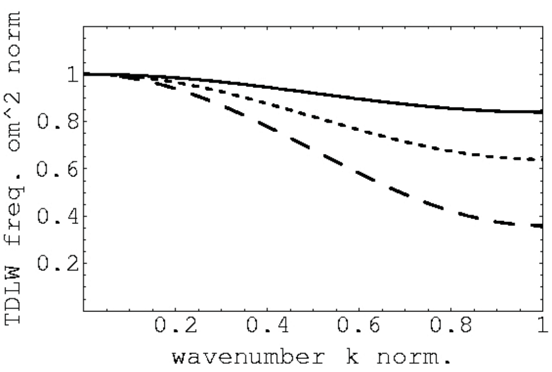

FIG. 2: The TDLW dispersion relation: frequency (square) ω_T^2 vs. wavenumber k.

Notice the generation of higher phase harmonics due to nonlinearity. The (slowly varying) amplitude $w_1^{(1)} \equiv A[\epsilon(x - v_g t), \epsilon^2 t]$ obeys a *nonlinear Schrödinger equation* (NLSE) in the form [7]:

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \tag{6}$$

where $\{X, T\}$ are the *slow* variables $\{\epsilon(x - v_g t), \epsilon^2 t\}$. The *dispersion coefficient* P is related to the curvature of $\omega(k)$ as $P_T = \omega_T''(k)/2$ is negative/positive for low/high values of k. The *nonlinearity coefficient*

$$Q = \frac{1}{2\omega_T}\Big(\frac{10\alpha^2}{3\omega_g^2} - 3\,\beta\Big) \tag{7}$$

is positive for *all* known experimental values of the anharmonicity coefficients α, β [3]. For long wavelengths [i.e. $k < k_{cr}$, where $P(k_{cr}) = 0$], the theory [7] predicts that TDLWs will be modulationally stable, and may propagate in the form of dark/grey envelope excitations (*hole* solitons or *voids*; see Fig. 3a,b). On the other hand, for $k > k_{cr}$, *modulational instability* may lead to the formation of bright (*pulse*) envelope solitons (see Fig. 3c). Analytical expressions for these excitations can be found in [7]. It may be noted that the modulation

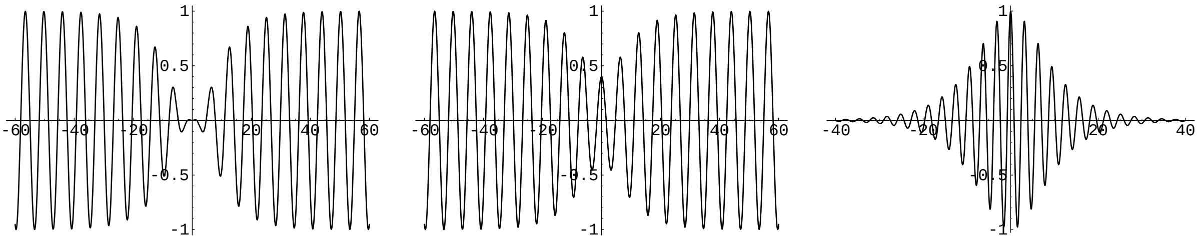

FIG. 3: TDL envelope solitons of the (a) *dark*, (b) *grey*, and (c) *bright* type.

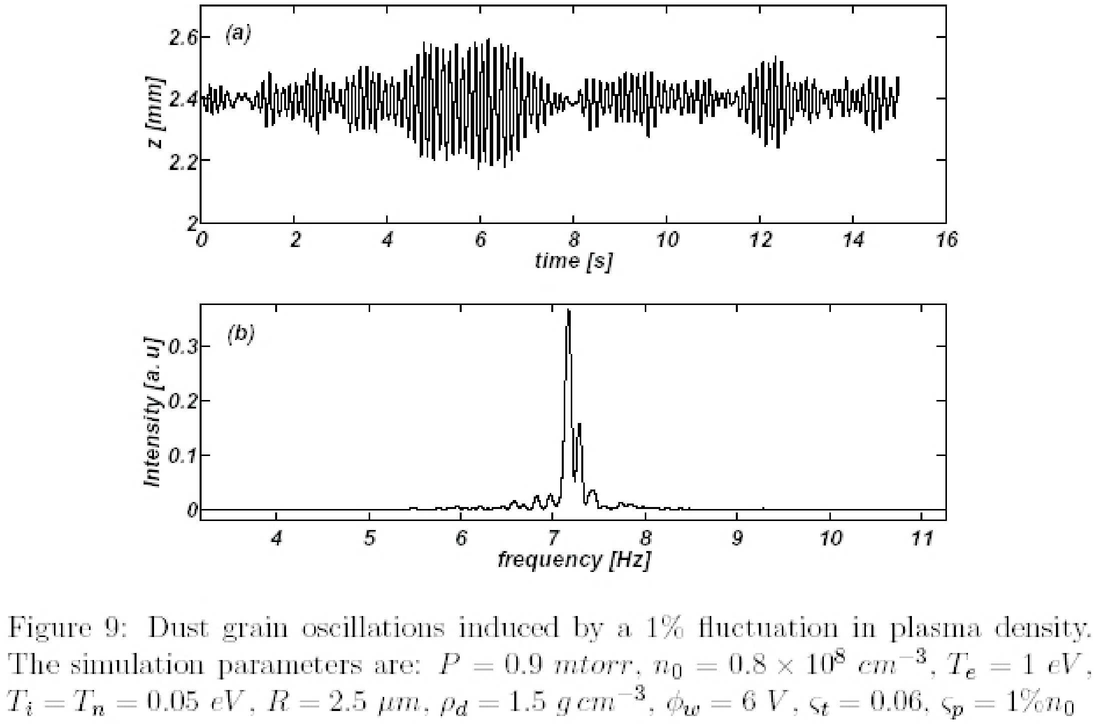

Figure 9: Dust grain oscillations induced by a 1% fluctuation in plasma density. The simulation parameters are: $P = 0.9\ mtorr$, $n_0 = 0.8 \times 10^8\ cm^{-3}$, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$, $\varsigma_t = 0.06$, $\varsigma_p = 1\% n_0$

FIG. 4: Amplitude modulation of transverse dust lattice oscillations; simulation data provided in the embedded caption; figure reprinted from [13].

of transverse dust oscillations clearly appears in numerical simulations [13]; see e.g Fig. 4.

III. *LONGITUDINAL* ENVELOPE EXCITATIONS.

The longitudinal dust grain displacements $\delta x_n = x_n - nr_0$ are described by the *nonlinear* equation of motion [8, 10]:

$$\frac{d^2(\delta x_n)}{dt^2} + \nu \frac{d(\delta x_n)}{dt} = \omega_{0,L}^2 (\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n)$$
$$-a_{20} \left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2\right] + a_{30} \left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3\right]. \quad (8)$$

The resulting linear mode [14] obeys the *acoustic* dispersion relation:

$$\omega^2 = 4\omega_{L,0}^2 \sin^2(kr_0/2) \equiv \omega_L^2 \,, \quad (9)$$

where $\omega_{L,0} = [U''(r_0)/M)]^{1/2}$; in the Debye case, $\omega_{L,0}^2 = 2\,\omega_{DL}^2 \exp(-\kappa)\,(1 + \kappa + \kappa^2/2)/\kappa^3$. The LDLW dispersion curve is depicted in Fig 5.

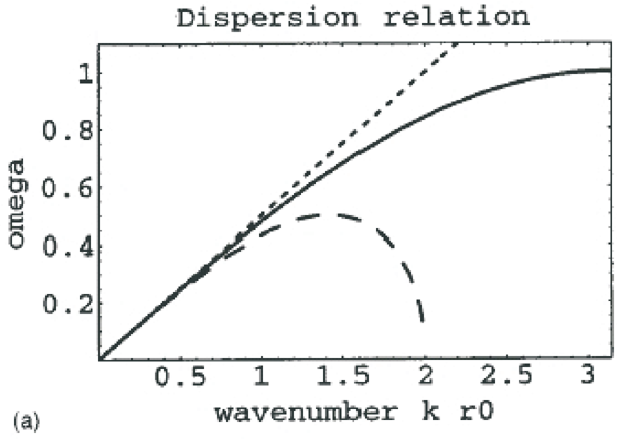

FIG. 5: The LDLW dispersion relation: frequency ω_L vs. wavenumber k (solid curve). We have also depicted: the continuous approximation (dashed curve) and the acoustic (tangent) curve at the origin.

The multiple scales (reductive perturbation) technique (cf. above) now yields ($\sim \epsilon$) a *zeroth*-harmonic mode, describing a constant displacement, viz.

$$\delta x_n \approx \epsilon\, [u_0^{(1)} + (u_1^{(1)}\, e^{i\phi_n} + \text{c.c.})] + \epsilon^2\, (u_2^{(2)}\, e^{2i\phi_n} + \text{c.c.}) + \dots .$$

The 1st-order amplitudes obey the coupled equations [6]:

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L \frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\, |u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\, u_1^{(1)} \frac{\partial u_0^{(1)}}{\partial X} = 0 \,, \quad (10)$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - \omega_{L,0}^2 r_0^2} \frac{\partial}{\partial X} |u_1^{(1)}|^2 \,, \quad (11)$$

where $v_{g,L} = \omega_L'(k)$; $\{X, T\}$ are *slow* variables (as above). The description involves the definitions:

$$p_0 = -r_0^3 U'''(r_0)/M \equiv 2a_{20} r_0^3$$

and

$$q_0 = U''''(r_0) r_0^4/(2M) \equiv 3a_{30} r_0^4$$

(both positive quantities of similar order of magnitude for Debye interactions; see in [4, 7]). Eqs. (10), (11) may be combined into a closed equation, which is identical to Eq. (6) (for $A = u_1^{(1)}$, here). Now, here

$$P = P_L = \omega_L''(k)/2 < 0\,,$$

while the form of $Q > 0$ (< 0) [8] prescribes stability (instability) at low (high) k. Envelope excitations are now *asymmetric*, i.e. rarefactive bright or compressive dark envelope structures (see Figs.).

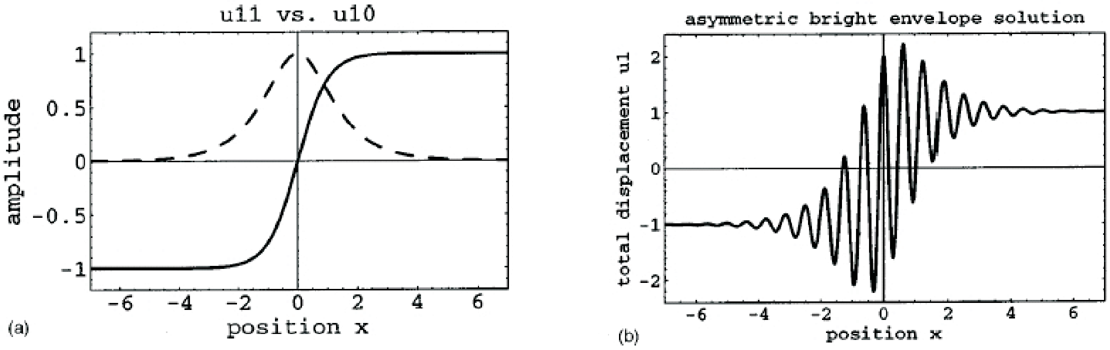

FIG. 6: *Bright* LDL (asymmetric) envelope solitons: (a) the zeroth (pulse) and first harmonic (kink) amplitudes; (b) the resulting asymmetric wavepacket.

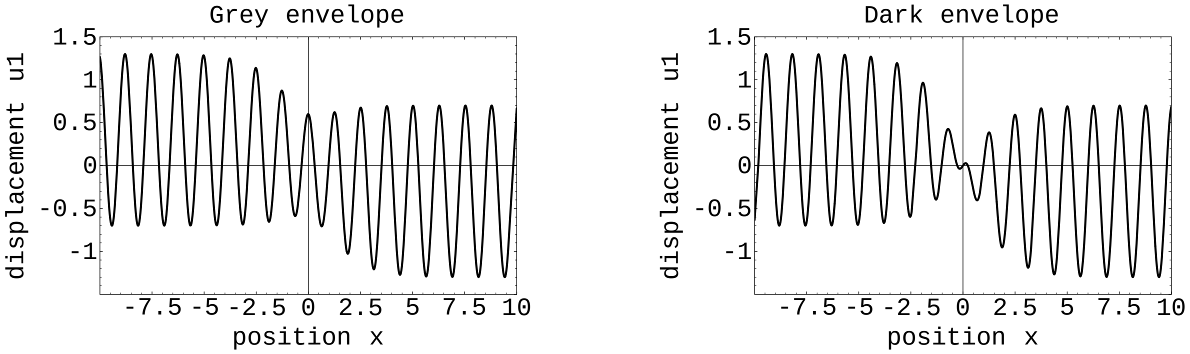

FIG. 7: (a) *Grey* and (b) *dark* LDL (asymmetric) modulated wavepackets.

IV. *LONGITUDINAL* SOLITONS.

Equation (8) is identical to the equation of motion in an atomic chain with anharmonic springs, i.e. in the celebrated FPU (*Fermi-Pasta-Ulam*) problem. Inspired by methods of solid state physics, one may opt for a continuum description at a first step, viz. $\delta x_n(t) \to u(x,t)$. This may lead to different nonlinear evolution equations (depending on simplifying assumptions), some of which are critically discussed in [9]. What follows is a summary of the lengthy analysis carried out therein.

A. Modified KdV Equation

Keeping lowest order nonlinear and dispersive terms, the continuum variable u obeys [10]:

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}\,, \tag{12}$$

where $(\cdot)_x \equiv \partial(\cdot)/\partial x$; $c_L = \omega_{L,0}\,r_0$; p_0 and q_0 were defined above. Assuming *near-sonic propagation* (i.e. $v \approx c_L$), and defining the relative displacement $w = u_x$, one has

$$w_\tau - a\,w\,w_\zeta + \hat{a}\,w^2\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0 \tag{13}$$

(for $\nu = 0$), where $a = p_0/(2c_L) > 0$, $\hat{a} = q_0/(2c_L) > 0$, and $b = c_L r_0^2/24 > 0$. Since the original work of Melandsø [4], various studies have relied on the *Korteweg - deVries* (KdV) equation, i.e. Eq. (13) for $\hat{a} = 0$, in order to gain analytical insight in the *compressive* structures observed in experiments [1]. Indeed, the KdV Eq. possesses *negative* (*only*, here, since $a > 0$) supersonic pulse soliton solutions for w, implying a compressive (anti-kink) excitation for u; the KdV soliton is thus interpreted as a density variation in the crystal, viz. $n(x,t)/n_0 \sim -\partial u/\partial x \equiv -w$. Also, the pulse width L_0 and height u_0 satisfy $u_0 L_0^2 = cst.$, a feature which is confirmed by experiments [1]. Now, here's a crucial point to be made (among others [9]): in a Debye crystal, $\hat{a} \approx 2a$ roughly (for $\kappa \approx 1$), so the KdV approximation (i.e. assuming $\hat{a} \approx 0$) is not valid. Instead, one may employ the *extended KdV* Eq. (eKdV) (13), which accounts for *both* compressive *and* rarefactive lattice excitations (see expressions in [9]; also cf. Fig. 4).

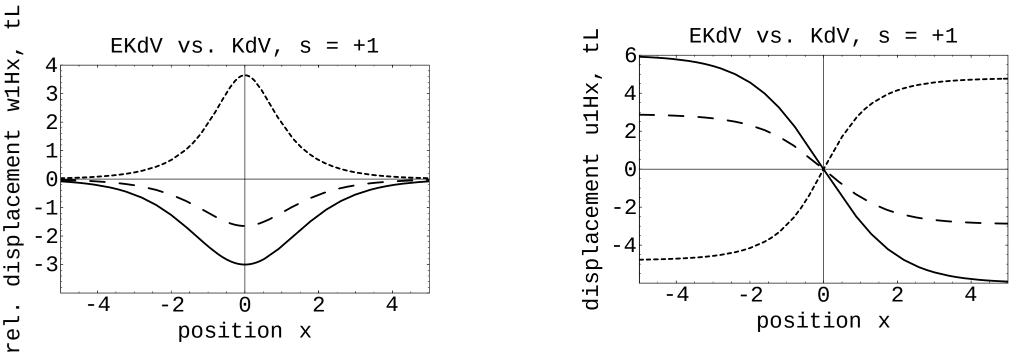

FIG. 8: Solutions of the *extended* KdV Eq. (for $q_0 > 0$; dashed curves) vs. those of the KdV Eq. (for $q_0 = 0$; solid curves): (a) relative displacement u_x; (b) grain displacement u.

B. Generalized Boussinesq description

Alternatively, Eq. (12) can be reduced to a *Generalized Boussinesq* (GBq) Equation

$$\ddot{w} - v_0^2 \, w_{xx} = h \, w_{xxxx} + p \, (w^2)_{xx} + q \, (w^3)_{xx} \tag{14}$$

($w = u_x$; $p = -p_0/2 < 0$, $q = q_0/3 > 0$); again, for $q \sim q_0 = 0$, one recovers a *Boussinesq* (Bq) equation, e.g. widely studied in solid chains. As physically expected, the GBq (Bq) equation yields, like its eKdV (KdV) counterpart, both compressive and rarefactive (only compressive) solutions; however, the (supersonic) propagation speed v now does *not* have to be close to c_L. A detailed comparative study of (and exact expressions for) all of these soliton excitations can be found in [9].

V. CONCLUSIONS.

Concluding, we have reviewed recent results on nonlinear excitations (solitary waves) occurring in a (1d) dust mono-layer. Modulated envelope TDL and LDL structures occur, due to sheath and coupling nonlinearity. Both compressive and rarefactive longitudinal excitations are predicted and may be observed by appropriate experiments.

Acknowledgments

This work was supported by the *SFB591 (Sonderforschungsbereich) – Universelles Verhalten gleichgewichtsferner Plasmen: Heizung, Transport und Strukturbildung* German government Programme.

---

[1] D. Samsonov *et al.*, Phys. Rev. Lett. **88** 095004 (2002); K. Avinash *et al.*, Phys. Rev. E **68**, 046402 (2003).

[2] T. Misawa *et al.*, Phys. Rev. Lett. **86**, 1219 (2001); B. Liu *et al.*, Phys. Rev. Lett. **91**, 255003 (2003).

[3] C. Zafiu *et al.*, Phys. Rev. E **63** 066403 (2001); A. V. Ivlev *et al.*, Phys. Rev. Lett. **85**, 4060 (2000).

[4] F. Melandsø, Phys. Plasmas **3**, 3890 (1996).

[5] I. Kourakis and P. K. Shukla, *Physica Scripta*, in press (2004); also see: `http://hal.ccsd.cnrs.fr/ccsd-00001893/en/` .

[6] A. V. Ivlev, S. K. Zhdanov, and G. E. Morfill Phys. Rev. E **68**, 066402 (2003).

[7] I. Kourakis and P. K. Shukla, Phys. Plasmas, **11** (5), 2322 (2004); *idem*, Phys. Plasmas, **11** (7), 3665 (2004)

[8] I. Kourakis and P. K. Shukla, Phys. Plasmas, **11**, 1384 (2004).

[9] I. Kourakis and P. K. Shukla, Eur. Phys. J. D, **29**, 247 (2004).

[10] Only first neighbor interactions are considered throughout this paper. See in [5] for details and coefficient definitions.

[11] Coupling anharmonicity, expressed by a term $\sim [(\delta z_{n+1} - \delta z_n)^3 - (\delta z_n - \delta z_{n-1})^3]$, which is omitted in the right-hand side of Eq. (1), may be added at a later stage.

[12] Follow exactly the definitions in [5, 6], not reproduced here.

[13] G. Sorasio, R. A. Fonseca, D. P. Resendes, and P. K. Shukla, in *Dust Plasma Interactions in Space*, Nova Publishers (N.Y, 2002), p. 37.

[14] The damping term is neglected by setting $\nu = 0$ in the following, thus omitting collisions with neutrals; for $\nu \neq 0$, an imaginary part appears, in account for damping in both dispersion relation $\omega(k)$ and the resulting envelope equations.