Envelope solitons in a magnetized pair plasma

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Introduction

Electron-positron (e-p) plasmas are widely encountered in astrophysical situations and are believed to be sources of intense electromagnetic radiation from pulsar magnetospheres [1], in active galactic nuclei [2] and in models for the early universe [3]. Recent experiments have demonstrated the possibility of creating a non-relativistic e-p plasma in the laboratory and several experimental studies are now being carried out [4]. Interestingly, e-p plasma was recently also shown to occur in large fusion machines (tokamaks) [5].

The characteristics of the propagation of electromagnetic (em) waves in electron-positron plasmas are strongly modified with respect to electron-ion plasmas, essentially due to the absence of distinct time and length scales. One generic nonlinear mechanism which remains unexplored in e-p plasmas is the amplitude modulation of em waves, either due to self-interaction of the carrier wave or due to parametric interaction effects. This is a long-known characteristic mechanism governing nonlinear wave propagation in dispersive media, and is related to effects like energy localization, harmonic generation, modulational instabilities and localized envelope structure formation, in contexts as diverse as solid state physics, nonlinear optics and plasma physics [6, 7].

This work is dedicated to a study, from first principles, of the modulation of the electric field amplitude of em waves propagating in e-p plasmas, in correlation with background density and magnetic field variations. The modulational stability profile of em waves will be examined and the possibility for the occurrence of localized envelope excitations will be investigated.

The model

The system we are interested in studying is a collisionless plasma consisting of electrons (rest mass m, charge $-e$, equilibrium density n_0) and positrons (rest mass m, charge e, equilibrium density n_0). There is a magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$ present. In this plasma we look at the amplitude modulation of circularly polarized electromagnetic (cyclotron) waves propagating along the magnetic field. The electric field is given by $\mathbf{E} = \frac{1}{2}\left[E(\mathbf{e}_x - i\mathbf{e}_y)e^{i(kz-\omega t)} + \text{c.c.}\right]$. The linear dispersion relation for these waves is

$$N^2 \equiv \frac{k^2c^2}{\omega^2} = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_c^2}, \qquad (1)$$

where where ω_p is the plasma frequency $\omega_p = (4\pi n_0 e^2/m)^{1/2}$ and ω_c is the cyclotron frequency $\omega_c = eB_0/mc$. Note that these waves can be left-hand or right-hand circularly polarized but that makes no difference in the dispersion relation. It has fast and slow mode solutions given by

$$\omega^2 = \frac{k^2c^2 + \omega_H^2 \pm \sqrt{(k^2c^2+\omega_H^2)^2 - 4k^2c^2\omega_c^2}}{2}, \qquad (2)$$

with the upper-hybrid frequency ω_H defined as $\omega_H = (2\omega_p^2 + \omega_c^2)^{1/2}$.

The standard wave modulation formalism [8, 9], accounting for the coupling with the background slow motion and for weakly relativistic mass variation, leads in the WKB approximation, i.e. for a slowly varying amplitude, and for weak nonlinearities, to an electric field obeying the nonlinear Schrödinger-type equation

$$i\left(\frac{\partial E}{\partial t} + v_g\frac{\partial E}{\partial z}\right) + P\frac{\partial^2 E}{\partial z^2} - \Delta E = 0, \qquad (3)$$

where v_g and P are the group velocity and group dispersion given by

$$v_g = \omega'(k) = \frac{kc^2}{\omega\left[1 + \frac{2\omega_c^2\omega_p^2}{(\omega^2-\omega_c^2)^2}\right]} > 0, \qquad (4)$$

$$P = \frac{\omega''(k)}{2} = \frac{v_g}{2k}\left\{1 + \frac{v_g^2}{c^2}\left[\frac{2\omega_c^2\omega_p^2(3\omega^2+\omega_c^2)}{(\omega^2-\omega_c^2)^3} - 1\right]\right\}. \qquad (5)$$

It is straightforward to check numerically that $P < 0$ for the slow mode and $P > 0$ for the fast mode. The nonlinear frequency shift Δ is given by [9]

$$\Delta = \frac{v_g}{kc^2}\frac{\omega^2\omega_p^2}{\omega^2-\omega_c^2}\left[\frac{n_1}{n_0} + \frac{2\omega_c^2}{\omega^2-\omega_c^2}\left(\frac{kV}{\omega} + \frac{B_1}{B_0}\right)\right] + \Delta_r, \qquad (6)$$

where n_1, V, and B_1 are, respectively, the density, fluid velocity, and the compressional magnetic field perturbations (along the z-axis) associated with the plasma slow motion (due to the ponderomotive force and magnetization) and Δ_r is The nonlinear frequency shift Δ_r due to the relativistic particle mass variation Note that quasi-neutrality has been assumed.

Density and magnetic field perturbations

For quasi-static modulations, making use of the ponderomotive force [10] and magnetization [11] due to the em waves, we find [12]

$$\frac{n_1}{n_0} = -\frac{2\omega_p^2}{\omega^2-\omega_c^2}\frac{|E|^2-|E_0|^2}{4\pi n_0 k_B T}, \qquad (7)$$

$$\frac{B_1}{B_0} = -\frac{4\omega_p^4}{(\omega^2-\omega_c^2)^2}\frac{k_BT}{mc^2}\frac{|E|^2-|E_0|^2}{4\pi n_0 k_B T} \qquad (8)$$

where T is the sum of the electron and positron temperatures and k_BT/mc^2 is small but finite. Finally, the nonlinear frequency shift takes the form $\Delta = -Q(|E|^2 - |E_0|^2)$ where the nonlinearity coefficient Q is

$$Q = \frac{v_g}{kc^2}\frac{\omega^2\omega_p^4}{(\omega^2-\omega_c^2)^2}\frac{1}{4\pi n_0 k_B T} \times\left[2 + \frac{4\omega_c^2(\omega^2+2\omega_p^2)+(\omega^2+\omega_c^2)^2}{(\omega^2-\omega_c^2)^2}\frac{k_BT}{mc^2}\right] > 0. \qquad (9)$$

The electric field then obeys the nonlinear Schrödinger equation

$$i\frac{\partial E}{\partial t} + P\frac{\partial^2 E}{\partial \zeta^2} + Q\left(|E|^2 - |E_0|^2\right)E = 0. \qquad (10)$$

Here a Galilean transformation $z \rightarrow \zeta = z - v_g t$ to a frame moving at the group velocity has been introduced. Our equations provide a self-consistent description of the electron-positron plasma, in terms of the electric field E, the magnetic field perturbation B_1 and the density variation n_1.

Modulational stability

We go over to a polar representation, i.e. we write E as $E = \rho e^{i\theta}$ where $\rho(\zeta, t)$ and $\theta(\zeta, t)$ are real functions. Note that this means that $\rho = |E|$ with equilibrium value $\rho_0 = |E_0|$. Equation (10) has the obvious solution $\rho = \rho_0$, $\theta = 0$. When perturbing linearly, i.e. assuming $\rho = \rho_0 + \rho_1 e^{i(K\zeta - \Omega t)}$ and $\theta = \sigma_1 e^{i(K\zeta-\Omega t)}$, we find the dispersion relation

$$\Omega^2 = PK^2(PK^2 - 2Q\rho_0^2) = (PK^2 - Q\rho_0^2)^2 - Q^2\rho_0^4. \qquad (11)$$

We see immediately that if $\eta \equiv P/Q > 0$, Ω^2 becomes negative for values of K below $K_{cr} = (2/\eta)^{1/2}|E_0|$, so that there is a purely growing amplitude mode and the wave gets modulationally unstable. The growth rate $\sigma = \text{Im}(\Omega)$ reaches a maximum $\sigma_{max} = Q|E_0|^2$ for $K = (1/\eta)^{1/2}|E_0|$. On the other hand, for $\eta < 0$, the wave is modulationally stable.

Bright solitons in the fast mode

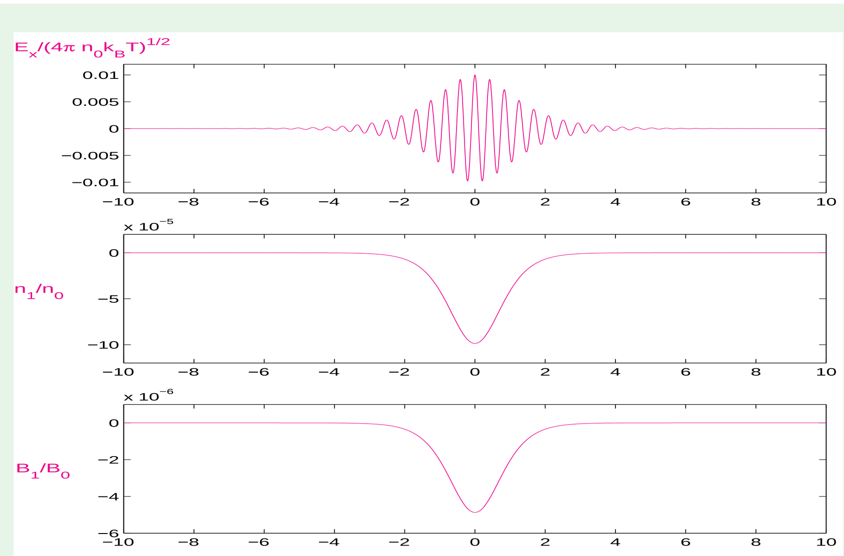

Fig.1: Electric field, density and magnetic field profiles of the *bright* soliton at $t = 0$. We took $\omega_p/\omega_c = 1$ and $k_BT/mc^2 = 0.05$ and are in the slow mode at $kc/\omega_p = 0.2$, $\omega/\omega_c = 1.7398$. We have chosen $A = 0.01$, $v_e = 0$, $\Omega = 0$, $\zeta_0 = 0$ and $\theta_0 = 0$. Care has been taken to meet the requirements of weak nonlinearity, slowly varying amplitude and the group and envelope velocities small compared to the speed of light.

As we saw above, the fast mode has $\eta > 0$, and is thus modulationally unstable. Wave collapse can now lead to the formation of a *bright* soliton [13], i.e. a localised pulse-like envelope modulating the carrier wave. It is of the form $E = \rho e^{i\theta}$ where

$$\rho = \frac{\sqrt{2\eta}}{L}\text{sech}\left(\frac{\zeta - \zeta_0 - v_e t}{L}\right),$$

$$\theta = \frac{1}{2P}\left[v_e\zeta + \left(\Omega - \frac{v_e^2}{2}\right)t\right] + \theta_0. \qquad (12)$$

Here v_e is the envelope velocity, L, Ω and ζ_0 are the pulses spatial width, oscillation frequency and position at $t = 0$, and θ_0 is an arbitrary phase. The equilibrium amplitude is $\rho_0 = 0$.

The maximum amplitude ρ_M is inversely proportional to the width L, i.e. $\rho_M L = (2\eta)^{1/2}$. Note that when going back to the original coordinates, the envelope is moving at the speed $v_g + v_e$. Let us also emphasize that the total phase of the electric field is $kz - \omega t + \theta$. It is found that these *bright* solitons correspond to a reduction of both density and magnetic field, as can be seen in Fig. 1.

Grey and dark solitons in the slow mode

For the slow mode we have $\eta < 0$, so this one is modulationally stable. Now there are *grey* envelope solitons [13] which represent a localized region of negative electric field density.

$$\rho = \frac{\sqrt{2|\eta|}}{Ld}\sqrt{1 - d^2\text{sech}^2\left(\frac{\zeta - \zeta_0 - v_e t}{L}\right)}, \qquad (13)$$

$$\theta = \frac{1}{2P}\left(V_0\zeta - \frac{V_0^2}{2}t\right) + s\arcsin\frac{d\tanh\left(\frac{\zeta - v_e t}{L}\right)}{\sqrt{1 - d^2\text{sech}^2\left(\frac{\zeta - v_e t}{L}\right)}} + \theta_0.$$

Here the parameter d, laying in the range $0 < d \leq 1$, regulates the modulation depth, $s = \pm 1$, and V_0 is given by the formula $V_0 = v_e + s\frac{2P}{Ld}\sqrt{1-d^2}$. Note that for $d = 1$, one has a *dark* soliton. The (finite) equilibrium amplitude ρ_0 is now inversely proportional to both the width L and the parameter d, i.e. $\rho_0 Ld = (2|\eta|)^{1/2}$. The minimum amplitude ρ_m is given by $\rho_m = \rho_0(1 - d^2)^{1/2}$, which is zero in the *dark* case. These *grey* and *dark* solitons correspond to a decrease in the density and an increase of the magnetic field, as can be seen in Fig. 2.

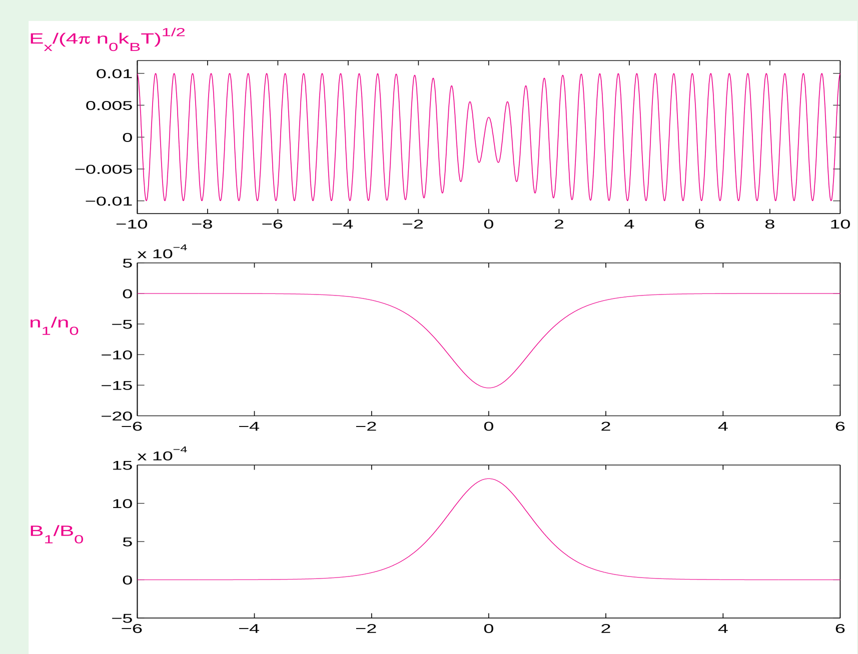

Fig. 2: Electric field, density and magnetic field profiles of the *grey* soliton at $t = 0$. We took again $\omega_p/\omega_c = 1$ and $k_BT/mc^2 = 0.05$ and are in the fast mode at $kc/\omega_p = 4$, $\omega/\omega_c = 0{,}9398$. We have chosen $A = 0.01$, $v_e = 0$, $d = 0.95$, $s = +1$, $\zeta_0 = 0$ and $\theta_0 = 0$.

Conclusions

We studied the amplitude modulation of magnetic field-aligned circularly polarized electromagnetic waves in an electron-positron magnetoplasma. We obtained a set of coupled equations for the electric field envelope and the density and magnetic field perturbations induced by the electromagnetic waves taking into account the ponderomotive force, magnetization and weakly relativistic mass variation. Modulational stability was explored and the possiblities for *bright* and *grey* envelope electromagnetic soliton structures have been investigated.

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