Theory of nonlinear excitations in dusty plasma crystals

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Abstract

The nonlinear aspects of horizontal (longitudinal, acoustic mode) and vertical (transverse, optical mode) motion of charged dust grains in a (1d) dusty plasma monolayer are discussed. Different types of localized excitations (solitary waves) are reviewed and their characteristics (and conditions for occurrence) are discussed.

1. Introduction. Recent studies of collective processes in dusty plasmas (DP) have been of significant interest in relation with experimental observations. Of particular importance are dust quasi-lattices, typically formed in the sheath region above the negative electrode in discharge experiments, horizontally suspended at an equilibrium position at $z = z_0$, where gravity and electric (and/or magnetic) forces balance. The linear regime of low-frequency oscillations in DP crystals, in the longitudinal (acoustic mode) and transverse (in-plane, shear acoustic mode and vertical, off-plane optical mode) direction(s), is now quite well understood. However, the *nonlinear* behaviour of DP crystals is still mostly unexplored, and has lately attracted experimental [1 - 3] and theoretical [1 - 8] interest.

Recently [4], we considered the coupling between the horizontal ($\sim \hat{x}$) and vertical (off-plane, $\sim \hat{z}$) degrees of freedom in a dust mono-layer; a set of nonlinear equations for longitudinal and transverse dust lattice waves (LDLWs, TDLWs) was thus rigorously derived [4]. Here, we review the nofnlinear dust grain excitations which may occur in a DP crystal (here assumed quasi-one-dimensional and infinite, composed from identical grains, of equilibrium charge q and mass M, located at $x_n = n\, r_0,\ n \in \mathcal{N}$). Ion-wake and ion-neutral interactions (collisions) are omitted, at a first step. This study complements recent experimental investigations [1-3] and may hopefully motivate future ones.

2. *Transverse* envelope structures. The vertical (off-plane) $n-$th grain displacement $\delta z_n = z_n - z_0$ in a dust crystal obeys the equation[1,2]

$$\frac{d^2 \delta z_n}{dt^2} + \nu\, \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\, (\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\, \delta z_n + \alpha\, (\delta z_n)^2 + \beta\, (\delta z_n)^3 = 0\,. \quad (1)$$

The characteristic frequency $\omega_{T,0} = [-qU'(r_0)/(Mr_0)]^{1/2}$ is related to the interaction potential $U(r)$ [e.g. for a Debye-Hückel potential: $U_D(r) = (q/r)\, e^{-r/\lambda_D}$, one has $\omega_{0,D}^2 = \omega_{DL}^2 \exp(-\kappa)\,(1+\kappa)/\kappa^3\,;\ \omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$ is the characteristic dust-lattice frequency; λ_D is the Debye length; $\kappa = r_0/\lambda_D$ is the DP lattice parameter]. The *gap frequency* ω_g and the nonlinearity coefficients α, β are defined via the potential $\Phi(z) \approx \Phi(z_0) + M[\omega_g^2 \delta z_n^2/2 + \alpha\,(\delta z_n)^3/3 + \beta\,(\delta z_n)^4/4] + \mathcal{O}[(\delta z_n)^5]$ (formally expanded near z_0, taking into account the electric and/or magnetic field inhomogeneity and charge variations[3]), i.e. leading to an overall vertical force $F(z) = F_{el/m}(z) - Mg \equiv -\partial\Phi(z)/\partial z$ [recall that $F_{e/m}(z_0) = Mg$]. Linear excitations, viz. $\delta z_n \sim \cos\phi_n$ (here $\phi_n = nkr_0 - \omega t$)

obey the *optical*-like *discrete* dispersion relation[4]: $\omega^2 = \omega_g^2 - 4\omega_{T,0}^2 \sin^2(kr_0/2) \equiv \omega_T^2$. Transverse vibrations propagate as a *backward wave* [see that $v_{g,T} = \omega_T'(k) < 0$] – for any form of $U(r)$ – cf. recent experiments [2]. Notice the lower cutoff $\omega_{T,min} = (\omega_g^2 - 4\omega_{T,0}^2)^{1/2}$ (at the edge of the Brillouin zone, at $k = \pi/r_0$), which is *absent in the continuum limit.*

Slightly departing from the small amplitude (linear) assumption, one obtains: $\delta z_n \approx \epsilon\,(w_1^{(1)}\, e^{i\phi_n} + \text{c.c.}) + \epsilon^2\,[w_0^{(2)} + (w_2^{(2)}\, e^{2i\phi_n} + \text{c.c.})] + ...$ (where $w_0^{(2)} \sim |A|^2$, $w_2^{(2)} \sim A^2$); notice the generation of higher phase harmonics due to nonlinearity. The amplitude $w_1^{(1)} \equiv A$ obeys a *nonlinear Schrödinger equation* (NLSE) in the form [5]:

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \tag{2}$$

where $\{X, T\}$ are the *slow* variables $\{\epsilon(x - v_g t), \epsilon^2 t\}$. The *dispersion coefficient* $P_T = \omega_T''(k)/2$ is negative/positive for low/high values of k. The *nonlinearity coefficient* $Q = [10\alpha^2/(3\omega_g^2) - 3\,\beta]/2\omega_T$ is positive for *all* known experimental values of α, β [3]. For long wavelengths [i.e. $k < k_{cr}$, where $P(k_{cr}) = 0$], the theory [5] predicts that TDLWs will be modulationally stable, and may propagate in the form of dark/grey envelope excitations (*hole* solitons or *voids*; see Fig. 1a,b). On the other hand, for $k > k_{cr}$, *modulational instability* may lead to the formation of bright (*pulse*) envelope solitons (see Fig. 1c). Analytical expressions for these excitations can be found in [5] (also see Paper P5.058).

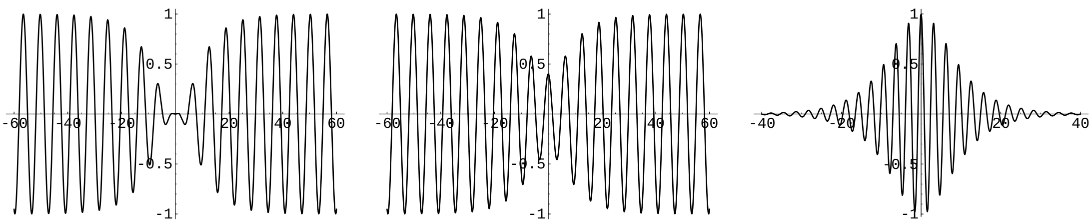

Figure 1: TDL envelope solitons of the (a) *dark*, (b) *grey*, and (c) *bright* type.

3. *Longitudinal* envelope excitations. The *nonlinear* equation of motion[1]:

$$\frac{d^2(\delta x_n)}{dt^2} + \nu\,\frac{d(\delta x_n)}{dt} = \omega_{0,L}^2\,(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n)$$
$$-a_{20}\,[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2] + a_{30}\,[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3] \tag{3}$$

describes the longitudinal dust grain displacements $\delta x_n = x_n - nr_0$. The resulting *acoustic* linear mode[4] obeys: $\omega^2 = 4\omega_{L,0}^2 \sin^2(kr_0/2) \equiv \omega_L^2$, where $\omega_{0,L} = [U''(r_0)/M)]^{1/2}$; in the Debye case, $\omega_{L,0}^2 = 2\,\omega_{DL}^2\, \exp(-\kappa)\,(1+\kappa+\kappa^2/2)/\kappa^3$. The multiple scales (reductive perturbation) technique (cf. above) now yields ($\sim \epsilon$) a *zeroth*-harmonic mode, describing a constant displacement, viz. $\delta x_n \approx \epsilon\,[u_0^{(1)} + (u_1^{(1)}\, e^{i\phi_n} + \text{c.c.})] + \epsilon^2\,(u_2^{(2)}\, e^{2i\phi_n} + \text{c.c.}) + ...$, where the amplitudes now obey the coupled equations [6]:

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L\,\frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\,|u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\,u_1^{(1)}\,\frac{\partial u_0^{(1)}}{\partial X} = 0\,, \tag{4}$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - \omega_{L,0}^2 r_0^2}\,\frac{\partial}{\partial X}\,|u_1^{(1)}|^2\,, \tag{5}$$

where $v_{g,L} = \omega_L'(k)$; $\{X, T\}$ are *slow* variables (as above). The description involves the definitions: $p_0 = -r_0^3 U'''(r_0)/M \equiv 2a_{20} r_0^3$ and $q_0 = U''''(r_0) r_0^4/(2M) \equiv 3a_{30} r_0^4$ (both positive quantities of similar order of magnitude for Debye interactions; see in [4, 7]). Eqs. (4), (5) may be combined into a closed equation, which is identical to Eq. (2) (for $A = u_1^{(1)}$, here). Now, here $P = P_L = \omega_L''(k)/2 < 0$, while the form of $Q > 0$ (< 0) [6] prescribes stability (instability) at low (high) k. Envelope excitations are now *asymmetric*, i.e. rarefactive bright or compressive dark envelope structures (see Figs.).

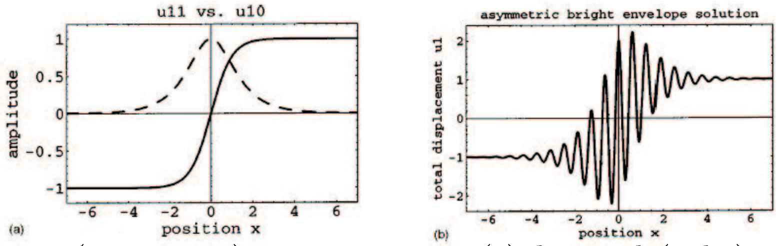

Figure 2: *Bright* LDL (asymmetric) envelope solitons: (a) the zeroth (pulse) and first harmonic (kink) amplitudes; (b) the resulting asymmetric wavepacket.

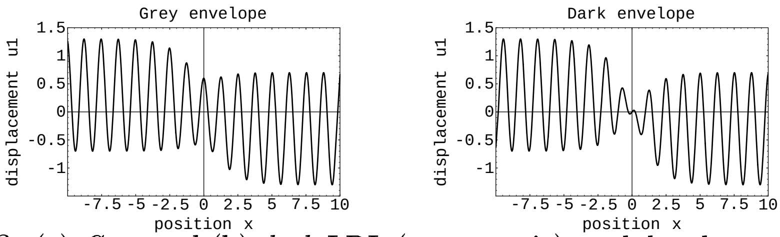

Figure 3: (a) *Grey* and (b) *dark* LDL (asymmetric) modulated wavepackets.

4. *Longitudinal* solitons. Equation (3) is identical to the equation of motion in an atomic chain with anharmonic springs, i.e. in the celebrated FPU (*Fermi-Pasta-Ulam*) problem. Inspired by methods of solid state physics, one may opt for a continuum description at a first step, viz. $\delta x_n(t) \to u(x, t)$. This may lead to different nonlinear evolution equations (depending on simplifying assumptions), some of which are critically discussed in [7]. What follows is a summary of the lengthy analysis carried out therein.

Keeping lowest order nonlinear and dispersive terms, the continuum variable u obeys[1]:

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}\,, \qquad (6)$$

where $(\cdot)_x \equiv \partial(\cdot)/\partial x$; $c_L = \omega_{L,0}\,r_0$; p_0 and q_0 were defined above. Assuming *near-sonic propagation* (i.e. $v \approx c_L$), and defining the relative displacement $w = u_x$, one has

$$w_\tau - a\,w\,w_\zeta + \hat{a}\,w^2\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0 \qquad (7)$$

(for $\nu = 0$), where $a = p_0/(2c_L) > 0$, $\hat{a} = q_0/(2c_L) > 0$, and $b = c_L r_0^2/24 > 0$. Since the original work of Melandsø [8], various studies have relied on the *Korteweg - deVries* (KdV) equation, i.e. Eq. (7) for $\hat{a} = 0$, in order to gain analytical insight in the *compressive* structures observed in experiments [1]. Indeed, the KdV Eq. possesses *negative* (*only*, here, since $a > 0$) supersonic pulse soliton solutions for w, implying a compressive (anti-kink) excitation for u; the KdV soliton is thus interpreted as a density variation in the crystal, viz. $n(x,t)/n_0 \sim -\partial u/\partial x \equiv -w$. Also, the pulse width L_0 and height u_0 satisfy $u_0 L_0^2 = cst.$, a feature which is confirmed by experiments [1]. Now, here's a crucial point to be made (among others [7]): $\hat{a} \approx 2a$ roughly in a Debye crystal (for $\kappa \approx 1$), thus invalidating the KdV approximation (i.e. for $\hat{a} \approx 0$). Instead, one may employ the *extended KdV* Eq. (eKdV) (7), which accounts for *both* compressive *and* rarefactive lattice excitations (see expressions in [7]; also cf. Fig. 4).

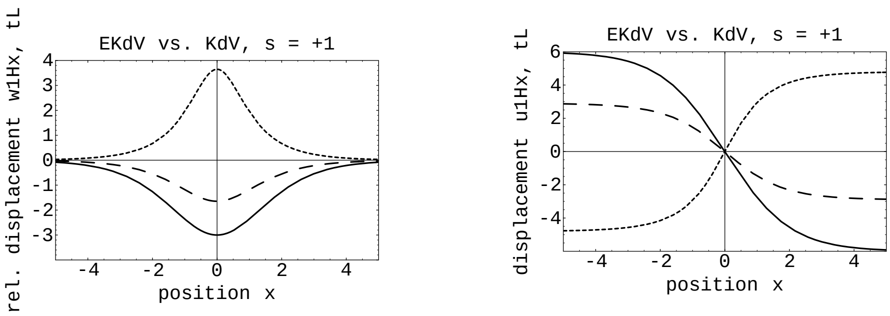

Figure 4: Solutions of the *extended* KdV Eq. (for $q_0 > 0$; dashed curves) vs. those of the KdV Eq. (for $q_0 = 0$; solid curves): (a) relative displacement u_x; (b) grain displacement u.

Alternatively, Eq. (6) can be reduced to a *Generalized Boussinesq* (GBq) Equation

$$\ddot{w} - v_0^2 \, w_{xx} = h \, w_{xxxx} + p \, (w^2)_{xx} + q \, (w^3)_{xx} \qquad (8)$$

($w = u_x$; $p = -p_0/2 < 0$, $q = q_0/3 > 0$); again, for $q \sim q_0 = 0$, one recovers a *Boussinesq* (Bq) equation, e.g. widely studied in solid chains. As physically expected, the GBq (Bq) equation yields, like its eKdV (KdV) counterpart, both compressive and rarefactive (only compressive) solutions; however, the (supersonic) propagation speed v now does *not* have to be close to c_L. A detailed comparative study of (and exact expressions for) all of these soliton excitations can be found in [7] and is too lengthy to reproduce here.

Concluding, we have reviewed recent results on nonlinear excitations (solitary waves) occurring in a (1d) dust mono-layer. Modulated envelope TDL and LDL structures occur, due to sheath and coupling nonlinearity. Both compressive and rarefactive longitudinal excitations are predicted and may be observed by appropriate experiments.

Acknowledgements. Support from the *Deutsche Forschungsgemeinschaft* through *Sonderforschungsbereich 591* as well as from the EU (Contract No. HPRN-CT-2000-00140) is gratefully acknowledged. I.K. is currently on leave from *Université Libre de Bruxelles* (Belgium).

References

[1] Only first neighbor interactions are considered. See in [4] for details and coefficient definitions.

[2] Coupling anharmonicity, i.e. a term $\sim [(\delta z_{n+1} - \delta z_n)^3 - (\delta z_n - \delta z_{n-1})^3]$, is omitted here.

[3] Follow exactly the definitions in [4, 5], not reproduced here.

[4] The damping term is neglected by setting $\nu = 0$ here; for $\nu \neq 0$, an imaginary part appears, in account for damping in both dispersion relation $\omega(k)$ and the resulting envelope equations.

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