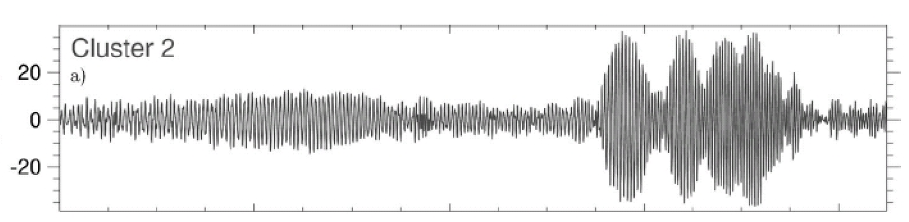
Envelope localized modes in electrostatic plasma waves

Ioannis Kourakis [1] and Padma Kant Shukla [2]
Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr–Universität Bochum, D–44780 Bochum, Germany
email: [1] ioannis@tp4.rub.de , [2] ps@tp4.rub.de

1. Introduction

Modulational instability (MI), a well-known mechanism of energy localization dominating wave propagation in nonlinear dispersive media, has been widely investigated in the past, with respect to plasma electrostatic modes, e.g. ion-acoustic waves (IAW), and experiments have confirmed those studies [1].

The purpose of this study is to provide a *generic* methodological framework for the study of the nonlinear (self-)modulation of the amplitude of such electrostatic modes, a mechanism known to be associated with *harmonic generation* and the formation of *localized envelope modulated wave packets*, such as the ones abundantly observed during laboratory experiments and satellite observations, e.g. *in the Earth's magnetosphere*:



(a)

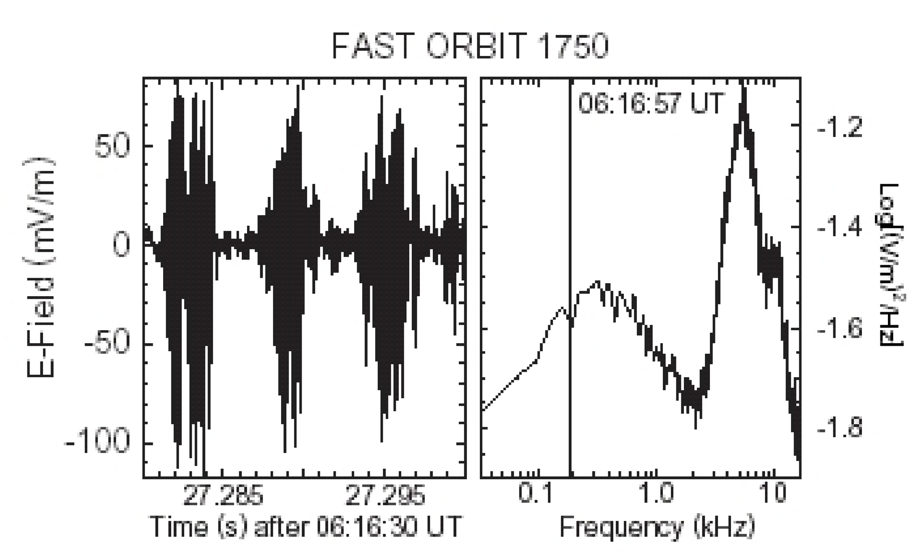
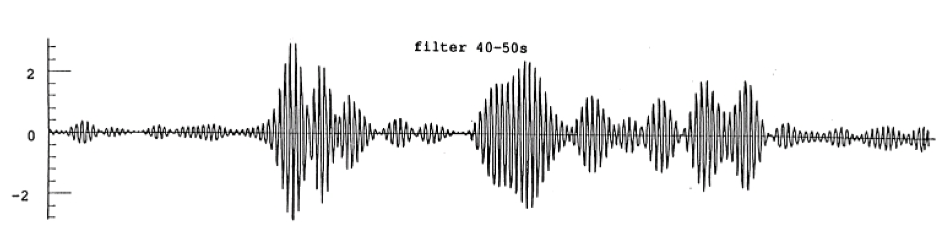
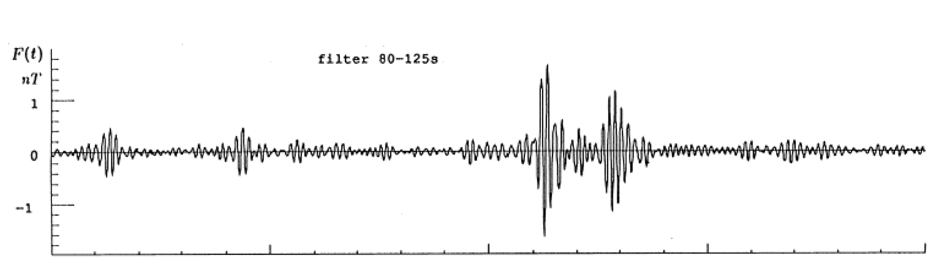


Figure 2. *Left*: Wave form of broadband noise at base of AKR source. The signal consists of highly coherent (nearly monochromatic frequency of trapped wave) wave packets. *Right*: Frequency spectrum of broadband noise showing the electron acoustic wave (at ∼ 5 kHz) and total plasma frequency (at ∼ 12 kHz) peaks. The broad LF maximum near 300 Hz belongs to the ion acoustic wave spectrum participating in the 3 ms modulation of the electron acoustic waves.

(b)



(c)



(d)

Figure 1. Satellite observations of modulation phenomena: (a) Cluster data, from O. Santolik *et al.*, *J. Geophys. Res.* **108**, 1278 (2003); (b) FAST data, from R. Pottelette *et al.*, *Geophys. Res. Lett.* **26** (16) 2629 (1999); (c), (d) from Ya. Alpert, *Phys. Reports* 339, 323 (2001).

2. The model: a generic description

In general, several known electrostatic plasma modes [2] consist of propagating oscillations of one dynamical plasma constituent, say α (mass m_α, charge $q_\alpha \equiv s_\alpha Z_\alpha e$; e is the absolute electron charge; $s = s_\alpha = q_\alpha/|q_\alpha| = \pm 1$ is the charge *sign*), against a background of one (or more) constituent(s): α' (mass $m_{\alpha'}$, charge $q_{\alpha'} \equiv s_{\alpha'} Z_{\alpha'} e$, similarly); the latter is (are) often assumed to obey a known distribution, e.g. being in a fixed (uniform): $n_{\alpha'} =$ const. or in a thermalized (Maxwellian) state $n_i \approx n_{\alpha',0}\, e^{-q_{\alpha'}\Phi/k_B T_{\alpha'}}$ ($T_{\alpha'}$: temperature, of species $\alpha' = e, i, ...$) for simplicity, depending on the particular aspects (e.g. frequency scales) of the physical system considered.

For instance,
— the *ion-acoustic* (IA) mode refers to ions ($\alpha = i$) oscillating against a Maxwellian electron background ($\alpha' = e$),
— the *electron-acoustic* (EA) mode refers to electron oscillations ($\alpha = e$) against a fixed ion background ($\alpha' = i$),
and so forth [2].

The standard (single) fluid model for the inertial species α provides the moment evolution equations:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\,\mathbf{u}) = 0\,,$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -s\,\nabla\phi - \frac{\sigma}{n}\nabla p\,,$$
$$\frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p = -\gamma\, p\,\nabla\cdot\mathbf{u}\,; \qquad (1)$$

also

$$\nabla^2\phi = \phi - \alpha\,\phi^2 + \alpha'\,\phi^3 - s\,\beta\,(n-1)\,; \qquad (2)$$

i.e. *Poisson's Eq.*: $\nabla^2\Phi = -4\pi \sum_{sp=\alpha,\{\alpha,\alpha'\}} q_{sp}\, n_{sp}$, close to equilibrium.

Overall neutrality is assumed at equilibrium:

$$\sum_{sp=\alpha,\{\alpha,\alpha'\}} q_{sp}\, n_{sp,0} = 0 \quad\Rightarrow\quad s_\alpha Z_\alpha\, n_{\alpha,0} + s_{\alpha'} \sum_{\alpha'} s_{\alpha'} Z_{\alpha'}\, n_{\alpha',0}\,.$$

We have defined the reduced (dimensionless) quantities:
- *particle density*: $n = n_\alpha/n_{\alpha,0}$;
- *mean (fluid) velocity*: $\mathbf{u} = [m_\alpha/(k_B T_*)]^{1/2}\mathbf{u}_d \equiv \mathbf{u}_\alpha/c_*$;
- *dust pressure*: $p = p_\alpha/p_0 = p_\alpha/(n_{\alpha,0} k_B T_*)$;
- *electric potential*: $\phi = Z_\alpha e\Phi/(k_B T_*) = |q_\alpha|\Phi/(k_B T_*)$;
- $\gamma = (f+2)/f = C_P/C_V$ (for f degrees of freedom).

Also, time and space are scaled over:
- t_0, e.g. the inverse *DP plasma frequency*

$$\omega_{p,\alpha}^{-1} = (4\pi n_{\alpha,0} q_\alpha^2/m_\alpha)^{-1/2}$$

and
- $r_0 = c_* t_0$, e.g. an *effective Debye length*

$$\lambda_{D,eff} = (k_B T_*/m_\alpha \omega_{p,\alpha}^2)^{1/2}$$

.
- The dimensionless parameters α, α' and β appearing in (2) should be determined exactly for any specific problem. They incorporate all the essential dependence on the plasma parameters.

Finally, $\sigma = T_\alpha/(n_{d,0} k_B T_*)$ is the temperature (ratio).

3. Multiple scales (reductive) perturbation method.

Let $\mathbf{S}$ be the state (column) vector $(n,\, \mathbf{u},\, p,\, \phi)^T$;
the *equilibrium state* is $\mathbf{S}^{(0)} = (1,\, \mathbf{0},\, 1,\, 0)^T$.
We shall consider small deviations by taking ($\epsilon \ll 1$)

$$\mathbf{S} = \mathbf{S}^{(0)} + \epsilon\,\mathbf{S}^{(1)} + \epsilon^2\,\mathbf{S}^{(2)} + ... = \mathbf{S}^{(0)} + \sum_{n=1}^{\infty} \epsilon^n\,\mathbf{S}^{(n)}\,.$$

We define the stretched (slow) space and time variables [3, 4]: $\zeta = \epsilon(x - \lambda\, t)\,, \quad \tau = \epsilon^2\, t$ ($\lambda \in \Re$); the *(fast) carrier phase* is $\theta_1 = \mathbf{k}\cdot\mathbf{r} - \omega t$ (*arbitrary propagation direction*), while the harmonic amplitudes vary *slowly along x*:

$$Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(\zeta,\,\tau)\, e^{il(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

($S_{j,-l}^{(n)} = S_{j,l}^{(n)\,*}$); wavenumber $\mathbf{k}$ is $(k_x,\, k_y) = (k\,\cos\theta,\, k\,\sin\theta)$.

→ *oblique modulation!*

Substituting into (2), one obtains, successively (details in [5]):
- the first harmonics of the perturbation:

$$n_1^{(1)} = s\,\frac{1+k^2}{\beta}\phi_1^{(1)} = \frac{1}{\gamma}\,p_1^{(1)} = \frac{1}{\omega}\mathbf{k}\cdot\mathbf{u}_1^{(1)} = \frac{k}{\omega\,\cos\theta}\,u_{1,x}^{(1)}\,, \qquad (3)$$

- the *compatibility condition* (*dispersion relation*):

$$\omega^2 = \frac{\beta\,k^2}{k^2+1} + \gamma\,\sigma\,k^2\,, \qquad (4)$$

- the 2nd order contributions: $\mathbf{S}_{0,1,2}^{(2)}$: → harmonic generation !!!
- the *compatibility condition*, for $n = 2$, $l = 1$:

$$\lambda = v_g(k) = \frac{\partial\omega}{\partial k_x} = \omega'(k)\cos\theta = \frac{k}{\omega}\left[\frac{1}{(1+k^2)^2} + \gamma\sigma\right]\cos\theta\,;$$

λ is therefore the *group velocity* in the modulation ($x-$) direction.

4. Derivation of the Nonlinear Schrödinger Equation

Proceeding to order $\sim\epsilon^3$, the equations for $l = 1$ yield an explicit compatibility condition i.e. the Nonlinear Schrödinger Equation

$$i\,\frac{\partial\psi}{\partial\tau} + P\,\frac{\partial^2\psi}{\partial\zeta^2} + Q\,|\psi|^2\,\psi = 0\,. \qquad (5)$$

— *Dispersion coefficient* $P = \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2} = \frac{1}{2}\left[\omega''(k)\cos\theta + \omega'(k)\frac{\sin^2\theta}{k}\right]$;
P is related to the *curvature* of the dispersion curve (4).
— *Nonlinearity coefficient* $Q = \sum_{j=0}^{4} Q_j$, due to *carrier wave self-interaction*;
— $Q_{0/2}$ are due to the 0th/2nd order harmonics,
— Q_1 is related to the cubic term in (2),
— $Q_{3/4}$ are due to the temperature effect (via σ).
An expression for Q (*too lengthy!*) can be found in detail in [5].

5. Modulational stability analysis

Linearizing around the monochromatic solution of Eq. (5): $\psi = \hat\psi\, e^{iQ|\hat\psi|^2\tau} + c.c.$ i.e. setting $\hat\psi = \hat\psi_0 + \epsilon\,\hat\psi_{1,0}\, e^{i(\hat k\zeta - \hat\omega\tau)}$, we obtain the *(perturbation) dispersion relation*:

$$\hat\omega^2 = P^2\,\hat k^2\left(\hat k^2 - 2\frac{Q}{P}|\hat\psi_{1,0}|^2\right).$$

The wave will be *stable* ($\forall\,\hat k$) if the product PQ is negative.

For *positive* $PQ > 0$, instability sets in for $\hat k_{cr} = \sqrt{2\frac{Q}{P}}|\hat\psi_{1,0}|$;
the instability growth rate $\sigma = |Im\hat\omega(\hat k)|$, reaches its maximum value $\sigma_{max} = |Q|\,|\hat\psi_{1,0}|^2$ for $\hat k = \hat k_{cr}/\sqrt{2}$.

6. Localized envelope excitations

We finally obtain a *localized modulated wave packet* in the form:

$$\psi = \epsilon\psi_0\,\cos(kx - \omega t + \Theta)$$

$[+\mathcal{O}(\epsilon^2)]$, where the *slowly varying amplitude* $\psi_0(\epsilon x, \epsilon t)$ and *phase correction* $\Theta(\epsilon x, \epsilon t)$ are determined by (solving) Eq. (5) for $\psi = \psi_0\exp(i\Theta)$ (see [6] for details).

→ Bright-type solitons (pulses) for $PQ > 0$:

$$\psi_0 = \left(\frac{2P}{QL^2}\right)^{1/2} \mathrm{sech}\left(\frac{X - v_e T}{L}\right), \quad \Theta = \frac{1}{2P}\left[v_e X + \left(\Omega - \frac{v_e^2}{2}\right)T\right] \qquad (6)$$

where
– v_e is the envelope velocity;
– L is the pulse's *spatial width*;
– L and Ω is the pulse's time *oscillation* (at rest) *frequency*;
– L and ψ_0 satisfy $L\psi_0 = (2P/Q)^{1/2} =$ constant;
– the maximum amplitude ψ_0 is *independent* from the velocity v_e; [cf. the Korteweg-deVries (KdV) solitons, where $L^2\psi_0 =$ const. and ψ_0 grows with v].

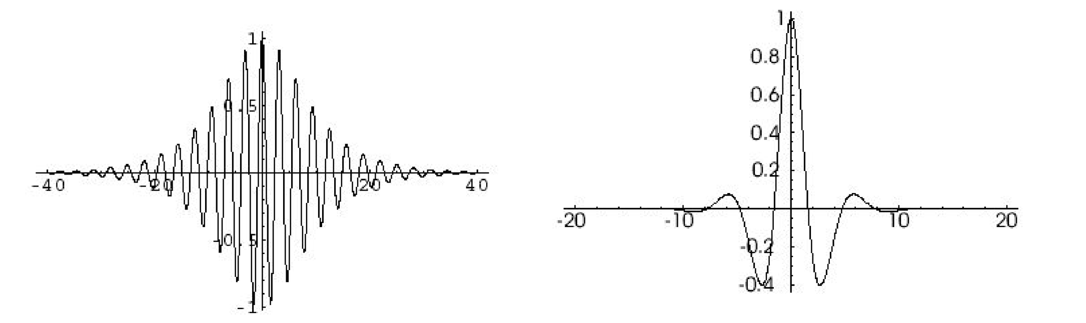


Figure 2. Bright type (pulse) soliton solution of the NLS equation, for two different parameter sets ($PQ > 0$).

→ Dark/grey type solitons (holes) for $PQ < 0$:

$$\psi_0 = \pm\psi'_0\tanh\left(\frac{X - v_e T}{L'}\right), \quad \Theta = \frac{1}{2P}\left[v_e X + \left(2PQA_0^2 - \frac{v_e^2}{2}\right)T\right] \qquad (7)$$

(see Fig. 2a); again, $L'\psi'_0 = (2|P/Q|)^{1/2}$ (=cst.).
The *grey* envelope reads [6]:

$$\psi_0 = \psi''_0\,\{1 - d^2\,\mathrm{sech}^2\{[X - v_e\,T]/L''\}\}^{1/2}\,, \qquad (8)$$

$$\Theta = \frac{1}{2P}\left[V_0\,X - \left(\frac{1}{2}V_0^2 - 2PQ\psi''^2_0\right)T + \Theta_0\right]$$
$$-S\,\sin^{-1}\frac{d\,\tanh\left(\frac{X - v_e T}{L''}\right)}{\left[1 - d^2\,\mathrm{sech}^2\left(\frac{X - v_e T}{L''}\right)\right]^{1/2}}\,. \qquad (9)$$

Here
– Θ_0 is a constant phase;
– S denotes the product $S = \mathrm{sign}(P)\,\times\,\mathrm{sign}(v_e - V_0)$;
– The pulse width L'' satisfies $L'' = (|P/Q|)^{1/2}/(d\psi''_0)$
– $0 < d \le 1$; the real parameter d is given by:

$$d^2 = 1 + (v_e - V_0)^2/(2PQ\psi''^2_0) \le 1\,;$$

– $V_0 =$ const. $\in \Re$ satisfies:

$$V_0 - \sqrt{2|PQ|\,\psi''^2_0} \le v_e \le V_0 + \sqrt{2|PQ|\,\psi''^2_0}\,.$$

For $d = 1$ (thus $V_0 = v_e$), one recovers the *dark* envelope soliton (cf. above).

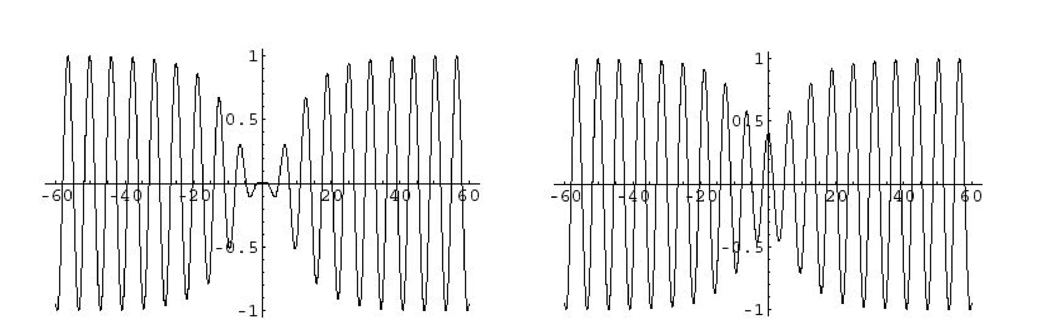


Figure 3. Soliton solutions of the NLS equation for $PQ < 0$ (holes); these excitations are of the: (a) dark type, (b) grey type. Notice that the amplitude never reaches zero in (b).

So, the *essential conclusion* to retain is:
- $PQ > 0$: Unstable linear wave, bright-type excitations;
- $PQ < 0$: Stable linear wave, dark/grey-type excitations.

References

[1] For a brief review, see the Introduction and exhaustive reference list in [5, 7 - 9].
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