Oblique wave modulation and localized excitations of dusty plasma electrostatic modes

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Introduction

Amplitude modulation, a well-known nonlinear mechanism dominating wave propagation in nonlinear dispersive media, has been widely investigated in the past, with respect to plasma electrostatic modes, e.g. ion-acoustic waves (IAW), and experiments have confirmed those studies [1]. However, little has been done as far as dusty plasma (DP) [2] is concerned, in this respect [3]. This study aims in partly filling this gap.

A. Dust-acoustic waves

The dust-acoustic wave (DAW) [2] is a very low frequency, *purely* DP mode (i.e. *absent* without dust), representing dust grain oscillations against a thermalized background of electrons and ions. It is characterized by a *very low phase velocity*: $v_{ph,DAW} \ll v_{ph,e}, v_{ph,i}$ and frequency below the dust plasma frequency $\omega_{p,D}$.

The model

We consider a *collisionless, unmagnetized, fully ionized dusty plasma*, consisting of electrons e (mass m, charge e), ions i (mass m_i, charge $q_i = +Z_i e$) and heavy dust grains d (mass m_d and charge $q_d = s\,Z_d e$ assumed constant; $s = sgn\,q_d = \pm 1$).
The dust fluid *moment-Poisson system of equations* reads [4]:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\,\mathbf{u}) = 0\,,$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -s\,\nabla\phi - \frac{\sigma}{n}\nabla p\,,$$
$$\frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p = -\gamma\,p\,\nabla\cdot\mathbf{u}\,;$$

and

$$\nabla^2\phi = \phi - \alpha\,\phi^2 + \alpha'\,\phi^3 - s\,\beta\,(n-1)\,; \qquad (1)$$

Eq. (1) is *Poisson's equation*: $\nabla^2\Phi = -4\pi\sum q_\alpha n_\alpha$, close to the Maxwellian state assumed for both e and i, i.e. $n_e \approx n_{e,0}\,e^{e\Phi/k_B T_e}$, $n_i \approx n_{i,0}\,e^{-Z_i e\Phi/k_B T_i}$ (T_α: temperature, of species $\alpha = e, i$).
Overall neutrality is assumed at equilibrium:

$$n_{e,0} - Z_i n_{i,0} - s Z_d n_{d,0} = 0\,.$$

We have defined the reduced (dimensionless) quantities:
- *dust density*: $n = n_d/n_{d,0}$;
- *dust mean velocity*: $\mathbf{u} = [m_d/(k_B T_e)]^{1/2}\mathbf{u}_d \equiv \mathbf{u_{i}}_d/v_d$;
- *dust pressure*: $p = p_d/p_0 = p_d/(n_{d,0} k_B T_e)$;
- *electric potential*: $\phi = Z_d e\Phi/(k_B T_e)$;
- $\gamma = (f+2)/f = C_P/C_V$ (for f degrees of freedom).

Also, space and time are scaled over:
- the *DP effective Debye length* $\lambda_{D,eff} = (\lambda_{D,e}^{-2} + \lambda_{D,i}^{-2})^{-1/2}$ (where $\lambda_{D,\alpha} = (k_B T_\alpha/4\pi n_{\alpha,0} q_\alpha^2)^{1/2}$, $\alpha = e, i$) and
- the inverse *DP plasma frequency* $\omega_{p,d}^{-1} = (4\pi n_{d,0} q_d^2/m_d)^{-1/2}$.
- The dimensionless parameters appearing in (1) are

$$\alpha = \frac{1}{2Z_d}\,\frac{Z_i^3(\frac{T_e}{T_i})^2\frac{n_{i,0}}{n_{e,0}} - 1}{Z_i^2\frac{T_e}{T_i}\frac{n_{i,0}}{n_{e,0}} + 1}\,, \qquad \alpha' = \frac{1}{6Z_d^2}\,\frac{Z_i^4(\frac{T_e}{T_i})^3\frac{n_{i,0}}{n_{e,0}} + 1}{Z_i^2\frac{T_e}{T_i}\frac{n_{i,0}}{n_{e,0}} + 1}\,,$$

and $\beta = \left(c_D/v_d\right)^2$, where $c_D = \lambda_{Deff}\omega_{p,d}$ is the DA speed [2].
Alternatively, one has: $\alpha \approx \frac{Z_i}{2Z_d}\frac{T_e}{T_i}$, $\alpha' \approx \frac{Z_i^2}{6Z_d^2}\frac{T_e^2}{T_i^2} = \frac{2}{3}\alpha^2$ and $\beta \approx \frac{Z_d^2 n_{d,0}}{Z_i^2 n_{i,0}}\frac{T_i}{T_e}$ for $\mu \ll Z_i\frac{T_e}{T_i}$;
we have defined the *typical dust parameters*

$$\delta = (Z_d n_{d,0})/(Z_i n_{i,0})\,, \qquad \mu = n_{e,0}/(Z_i n_{i,0}) = 1 + s\,\delta\,.$$

Retain: $0 \le \mu < 1$ ($\mu > 1$) corresponds to negative (positive) dust.
Finally, $\sigma = p_0/(n_{d,0}k_B T_e)$ ($= 1$ here, for the above choice for p_0).

Multiple scales (reductive) perturbation method.

Let $\mathbf{S}$ be the state (column) vector $(n,\,\mathbf{u},\,p,\,\phi)^T$;
the *equilibrium state* is $\mathbf{S}^{(0)} = (1,\,\mathbf{0},\,1,\,0)^T$.
We shall consider small deviations by taking ($\epsilon \ll 1$)

$$\mathbf{S} = \mathbf{S}^{(0)} + \epsilon\,\mathbf{S}^{(1)} + \epsilon^2\,\mathbf{S}^{(2)} + \ldots = \mathbf{S}^{(0)} + \sum_{n=1}^{\infty}\epsilon^n\,\mathbf{S}^{(n)}\,.$$

We define the stretched (slow) space and time variables [5]: $\zeta = \epsilon(x - \lambda\,t)$, $\tau = \epsilon^2\,t$ ($\lambda \in \Re$); the *(fast) carrier phase* is $\theta_1 = \mathbf{k}\cdot\mathbf{r} - \omega t$ (*arbitrary propagation direction*), while the harmonic amplitudes vary *slowly along x*:

$$Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(\zeta,\,\tau)\,e^{il(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

($S_{j,-l}^{(n)} = S_{j,l}^{(n)*}$); wavenumber $\mathbf{k}$ is $(k_x,\,k_y) = (k\,\cos\theta,\,k\,\sin\theta)$.

→ *oblique modulation!*

Substituting into (1), one obtains, successively (details in [4]):
- the first harmonics of the perturbation:

$$n_1^{(1)} = s\,\frac{1+k^2}{\beta}\phi_1^{(1)} = \frac{1}{\gamma}p_1^{(1)} = \frac{1}{\omega}\mathbf{k}\cdot\mathbf{u}_1^{(1)} = \frac{k}{\omega\,\cos\theta}u_{1,x}^{(1)}\,, \qquad (2)$$

- the *compatibility condition* (*DAW dispersion relation* [2]):

$$\omega^2 = \frac{\beta\,k^2}{k^2+1} + \gamma\,\sigma\,k^2\,, \qquad (3)$$

- the 2nd order contributions: $\mathbf{S}_{0,1,2}^{(2)}$: → harmonic generation !!!
- the *compatibility condition*, for $n = 2$, $l = 1$:

$$\lambda = v_g(k) = \frac{\partial\omega}{\partial k_x} = \omega'(k)\cos\theta = \frac{k}{\omega}\left[\frac{1}{(1+k^2)^2} + \gamma\sigma\right]\cos\theta\,;$$

λ is therefore the *group velocity* in the modulation ($x-$) direction.

Derivation of the Nonlinear Schrödinger Equation

Proceeding to order $\sim\epsilon^3$, the equations for $l = 1$ yield an explicit compatibility condition in the form of the Nonlinear Schrödinger Equation

$$i\,\frac{\partial\psi}{\partial\tau} + P\,\frac{\partial^2\psi}{\partial\zeta^2} + Q\,|\psi|^2\,\psi = 0\,. \qquad (4)$$

- *Dispersion coefficient* $P = \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2} = \frac{1}{2}\left[\omega''(k)\cos\theta + \omega'(k)\frac{\sin^2\theta}{k}\right]$;
P is related to the *curvature* of the dispersion curve (3).
- *Nonlinearity coefficient* $Q = \sum_{j=0}^{4} Q_j$, due to *carrier wave self-interaction*; $Q_{0/2}$ is due to the 0th/2nd order harmonics and Q_1 is related to the cubic term in (1).
P, Q (*too lengthy!*) can be found in full detail in [4].

Stability analysis

Linearizing around the monochromatic (Stokes wave) solution of the NLSE (4): $\psi = \hat\psi\,e^{iQ|\hat\psi|^2\tau} + c.c.$ i.e. setting

$$\hat\psi = \hat\psi_0 + \epsilon\,\hat\psi_{1,0}\,e^{i(\hat k\zeta - \hat\omega\tau)}$$

we obtain the *(perturbation) dispersion relation*:

$$\hat\omega^2 = P^2\,\hat k^2\left(\hat k^2 - 2\frac{Q}{P}|\hat\psi_{1,0}|^2\right).$$

The wave will be *stable* ($\forall\,\hat k$) if the product PQ is negative.
For *positive* $PQ > 0$, instability sets in for $\hat k_{cr} = \sqrt{2\frac{Q}{P}}|\hat\psi_{1,0}|$;
the instability growth rate $\sigma = |Im\hat\omega(\hat k)|$, reaches its maximum value $\sigma_{max} = |Q|\,|\hat\psi_{1,0}|^2$ for $\hat k = \hat k_{cr}/\sqrt{2}$.

Localized envelope excitations

→ Bright-type solitons (pulses) for $PQ > 0$:

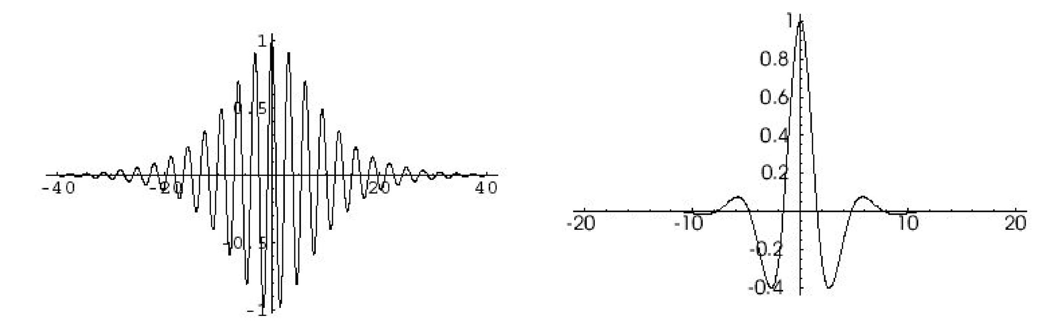

Figure 1. Bright type (pulse) soliton solution of the NLS equation, for two different parameter sets ($PQ > 0$).

→ Dark/grey type solitons (holes) for $PQ < 0$:

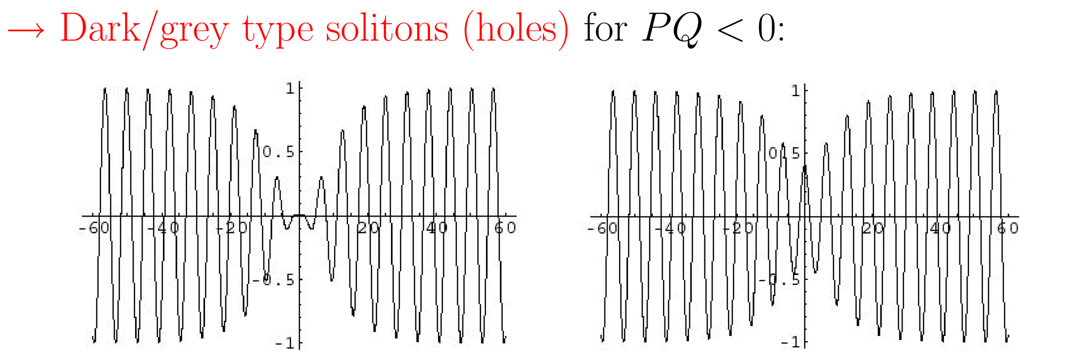

Figure 2. Soliton solutions of the NLS equation for $PQ < 0$ (holes); these excitations are of the: (a) dark type, (b) grey type. Notice that the amplitude never reaches zero in (b).

So, essentially:
- $PQ > 0$: Unstable linear wave, bright-type excitations;
- $PQ < 0$: Stable linear wave, dark/grey-type excitations.

Numerical results

→ Existence of two critical wavenumber $k_{cr,1,2}$, between which instability may occur (see Figs. 4).
→ Dramatic modulation obliqueness effect!: $k_{cr,1,2}$ depend on θ.
→ Important temperature effect on $k_{cr,1,2}$; see Fig. 4.
→ Influence of dust concentration and sign *on the stability profile* and the *soliton features*.
→ Small–angle modulated waves are stable for long wavelengths.

See figures 3, 4, where: $\alpha = 5\cdot 10^{-3}$, $\alpha' \approx 2\alpha^2/3 \approx 1.6\cdot 10^{-5}$ and $\beta \approx 100$, corresponding to $Z_d/Z_i = 10^3$ and $T_e/T_i = 10$.

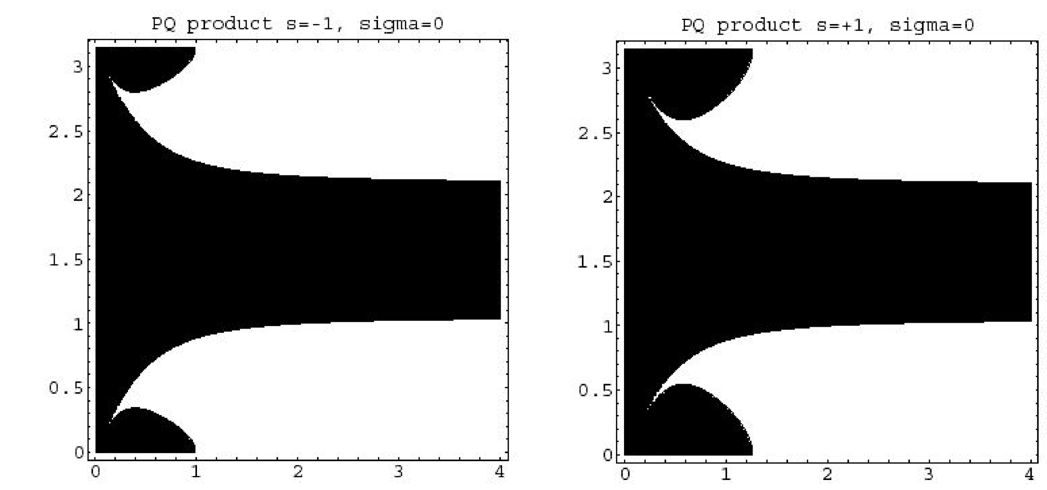

Figure 3. The DAW coefficient product $PQ = 0$ curve is represented against normalized wavenumber k/k_D (in abscissa) and angle θ (between 0 and π); the area in black (white) represents the region in the $(k - \theta)$ plane where PQ is negative (positive); instability occurs for values inside the white area. Here $\sigma = 0$ (cold dust). (a) negative dust ($s = -1$). (b) positive dust ($s = +1$).

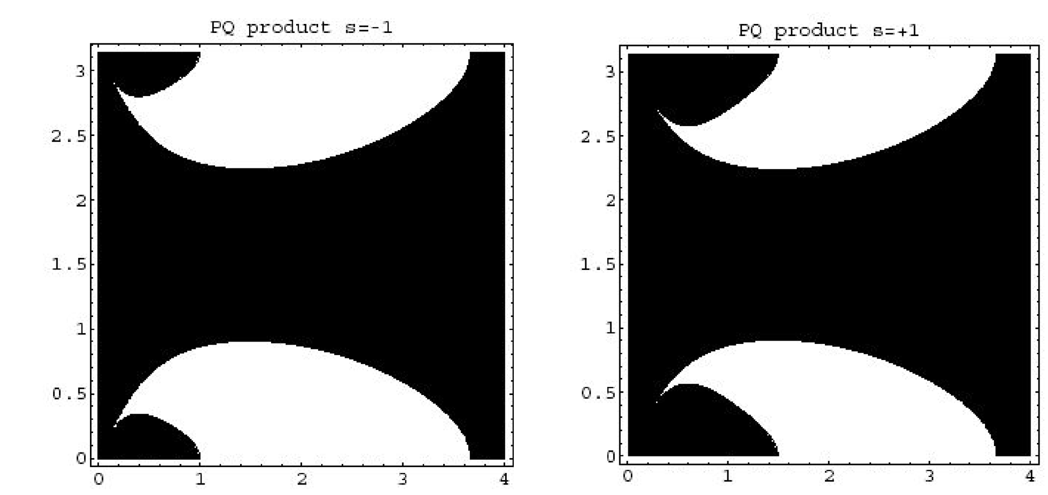

Figure 4. Same as Fig. 3, taking $\sigma = 1$ (hot dust DAW model).

B. Dust-ion acoustic waves (DIAW)

Dust-ion acoustic waves (DIAW) [2] are the DP analogue of the *ion-acoustic electrostatic wave* (IAW), [6], where inertial ions oscillate against a background of thermal electrons and massive dust grains. The DIAW is characterized by $v_{th,i} \ll v_{ph} \ll v_{th,e}$ and $\omega_{p,d} \ll \omega_{DIAW} \ll \omega_{p,i}$. The model equations are identical to (1) (setting $\alpha \to -\tilde\alpha$ & $s = 1$ therein [7, 8]), with the new definitions:
- *ion density*: $n = n_i/n_{i,0}$; *ion pressure*: $p = p_i/(n_{i,0}k_B T_e)$;
- *ion mean velocity*: $\mathbf{u} = [m_i/(k_B T_i)]^{1/2}\mathbf{u}_i \equiv \mathbf{u_{i}}_d/c_s$;
- *electric potential*: $\phi = Z_i e\Phi/(k_B T_e)$; $\sigma = T_i/T_e$

Also, space and time are scaled over:
- the electron Debye length $\lambda_{D,e} = (k_B T_e/4\pi n_{e,0}e^2)^{1/2}$ and
- the characteristic time-scale $\lambda_{D,e}/c_s \equiv \omega_{p,e}^{-1}\frac{m_i}{m_e}$.
- Now, the dimensionless parameters in (1) are
$\tilde\alpha = 1/(2Z_i)$, $\alpha' = 1/(6Z_i^2)$ and $\beta = Z_i^2 n_{i,0}/n_{e,0} = Z_i/\mu$.

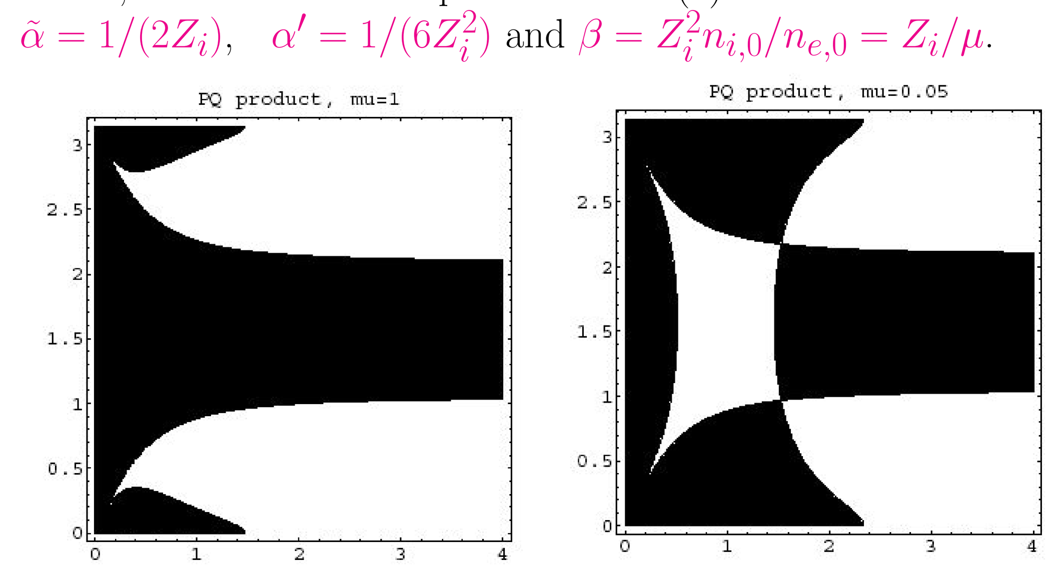

Figure 5. DIAW: similar to Figs. 3, 4, for two limit cases: (a) $\mu = 1$ i.e. in the *dust-free*(e–i plasma) limit; (b) $\mu = 0.05$ (*high* negative dust concentration): notice the generation of unstable regions for high θ. Here $\sigma = 0$ (cold ions).

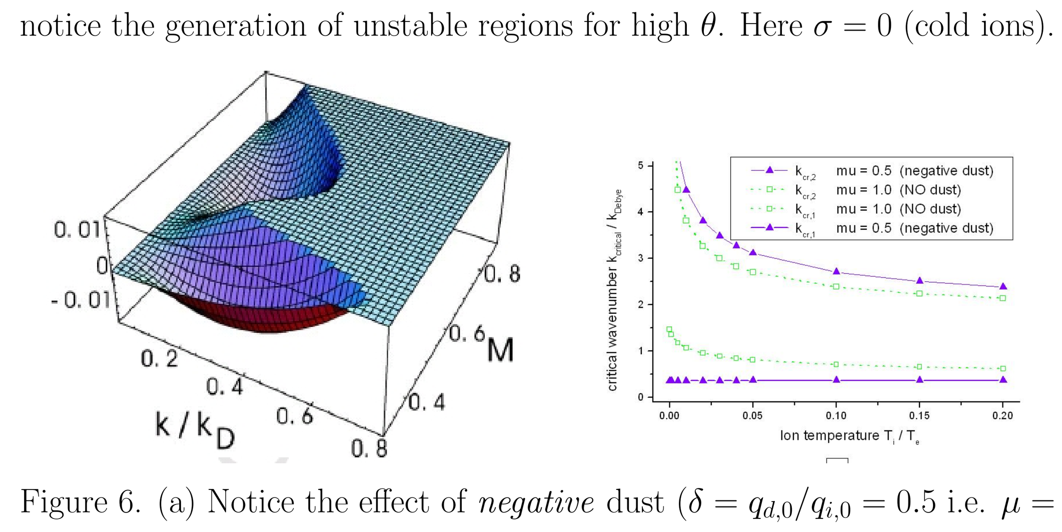

Figure 6. (a) Notice the effect of *negative* dust ($\delta = q_d/q_i = 0.5$ i.e. $\mu = 0.5$ here): lower $k_{cr,1}$ and finite temperature: lower $k_{cr,2}$ ($\sigma = 0.05$ here: warm ions); cf. Fig. 5a where $\mu = 1$, $\sigma = 0$. (b) The two critical wavenumbers $k_{cr,1}/k_{cr,2}$ are depicted against normalized ion temperature $\sigma = T_i/T_e$, for DP with $q_d < 0$.

References

[1] For a brief review, see the Introduction and exhaustive reference list in [4, 7].

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