Existence of multibreathers in systems with an inverse dispersion law Application in dusty plasma lattice oscillations * †

V. Koukouloyannis[1,‡], I. Kourakis[2,§]

[1] *Aristoteleion University of Thessaloniki, Theoretical Mechanics, 54124 Thessaloniki, Greece*

[2] *Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr Universität Bochum, D-44780 Bochum, Germany*

(Dated: May 27, 2005)

The existence of highly localized multi-site oscillatory structures (discrete multibreathers) in a nonlinear Klein-Gordon chain which is characterized by an inverse dispersion law is proven. The neighborhood of possible initial conditions for the multibreather solutions is evaluated, and their stability is investigated. The methods relies on an idea originally proposed by Poincaré, and elaborated in [V. Koukouloyannis and S. Ichtiaroglou, Phys. Rev. E **66**, 066602 (2002)]. The results are applied in the description of vertical (transverse, off-plane) dust grain motion in dusty plasma crystals, by taking into account the lattice discreteness and the sheath electric and/or magnetic field nonlinearity. Explicit values from experimental plasma discharge experiments are considered. The possibility for the occurrence of multibreathers associated with vertical charged dust grain motion in strongly-coupled dusty plasmas (dust crystals) is thus established.

PACS numbers: 52.27.Lw, 52.35.Fp, 52.25.Vy, 63.20.Pw

Keywords: Discrete Breathers, Multibreathers, Intrinsic Localized Modes, Dust crystals, Dusty Plasma.

---

* *Nonlinear Physics: Condensed Matter, Dynamical Systems and Biophysics*, May 30th-31st, 2005, Paris; poster contribution.

† *For information and/or reprints, please contact I.K. at:* `ioannis@tp4.rub.de` , *or provide your contact coordinates (name, email) in the blank space above.*

‡ Electronic address: `vkouk@physics.auth.gr` ; `http://users.auth.gr/∼vkouk` .

§ Electronic address: `ioannis@tp4.rub.de` ; `www.tp4.rub.de/∼ioannis` .

I. INTRODUCTION

Periodic lattices of interacting particles are known from solid state physics to sustain, apart from propagating vibrations (*phonons*), a variety of localized excitations, due to a mutual balance between the intrinsic nonlinearity of the medium and mode dispersion. Such structures, "traditionally" sought for and investigated in a continuum approximation (i.e. assuming that the typical spatial variation scale far exceeds the typical lattice scale, e.g. the lattice constant r_0), include non-topological *solitons* (pulses), *kinks* (i.e. shocks or dislocations) and localized modulated envelope structures (*envelope solitons*). Various generic nonlinear theories have been developed in order to investigate their occurrence in different physical contexts [1, 2]. In addition to these (continuum) theories, which neglect discreteness for the sake of analytical tractability, attention has been paid since more than a decade ago to highly localized vibrating structures [*discrete breathers* (DBs) or *intrinsic localized modes* (ILMs)], which owe their very existence to the lattice discreteness itself. Following some pioneering ILM related works in the late 80's e.g. [3–7], the breakthrough in the theoretical study of DBs took place with the first breather existence proofs, by MacKay and Aubry [8, 9] (who introduced the notion of the anticontinuous limit) and Flach [10] (using a homoclinic orbit approach). A large number of studies has then followed, elucidating many aspects involved in the spontaneous formation, mobility and interaction of DBs, both theoretically and experimentally; see in Refs. [11–14] for a review.

Recent studies of collective processes in a dust-contaminated plasma (DP) [15] have revealed a variety of new linear and nonlinear collective effects, which are observed in laboratory and space dusty plasmas. An issue of particular importance in DP research is the formation of strongly coupled DP crystals by highly charged dust grains, typically in the sheath region above a horizontal negatively biased electrode in experiments [15, 16]. Typical low-frequency oscillations are known to occur [16] in these mesoscopic dust grain quasi-lattices in the longitudinal (in-plane, acoustic mode), horizontal transverse (in-plane) and vertical transverse (off-plane, inverse dispersive optic-like mode) directions.

Even though nonlinearity is an intrinsic feature of dust crystal dynamics, due to inter-grain (Debye-type, screened electrostatic) nonlinear interactions, to mode coupling [17] or to the sheath environment, which is intrinsically nonlinear. Despite this fact, present day knowledge of nonlinear mechanisms related to dust lattice modes is admittedly still in a preliminary stage. Small amplitude localized longitudinal excitations (described by a *Boussinesq* equation for the longitudinal grain displacement u, or a *Korteweg-deVries* equation for the density $\partial u/\partial x$) were considered in Refs. [18] and generalized in Ref. [19]. Also, the amplitude modulation of longitudinal [20, 21] and transverse (vertical, off-plane) [22, 23] dust lattice waves (LDLW, TDLW, respectively) was recently considered. All of these studies have relied on a quasi-continuum description of the dust lattice dynamics.

The *discrete* character of dust-lattice oscillations has, to our best knowledge, not yet been studied, let alone a recent first investigation which was restricted to single-mode transverse dust-breathers [28]. This study has examined the properties of vertical (*transverse*) dust lattice vibrations. Most interestingly, the transverse (linear) dust lattice mode is known to obey an *inverse dispersion* law: therefore the group velocity $v_g = \omega'(k)$ and the phase velocity $v_{ph} = \omega/k$ point towards opposite directions. The *anharmonic* character of the vertical on-site potential (confirmed experimentally [24, 25]), in combination with the high *discreteness* of dust crystals, clearly suggested by experiments [26, 27], may play an important role in mechanisms like energy localization, information storage and response to external excitations. However, rather surprisingly, these aspects have hardly been investigated yet.

In this study, we are interested in investigating the conditions for the occurrence of discrete multi-site lattice excitations (*multibreathers*) in a nonlinear (infinite sized) Klein-Gordon-like chain, which is characterized by an inverse dispersion law. Nonlinearity is assumed to be supplied by a (non harmonic) on-site potential, while inter-particle interactions are take to be linear. A negative coupling coefficient ("spring constant") value is assumed, in account of an inverse dispersion. Our results will eventually be applied in a description of real transverse dust-lattice excitations, as observed in plasma discharge experiments.

II. EXISTENCE OF MULTIBREATHERS

We shall prove the existence of multibreather excitations in the system described above. The method we adopt is based on the notion of analytical continuation from a suitable solution in the anticontinuous limit, as e.g. in [29]. The formalism used is described in Ref. [30]. A brief outline of the method is provided in the following.

Consider the Hamiltonian

$$H = H_0 + \epsilon H_1 = \sum_{i=-\infty}^{\infty} \left(\frac{1}{2} p_i^2 + V_i(x_i) \right) + \frac{\epsilon}{2} \sum_{i=-\infty}^{\infty} (x_{i+1} - x_i)^2 \tag{1}$$

with $V'(0) = 0$ and $V''(0) = \omega_p^2 > 0$, which leads to the equations of motion

$$\ddot{x}_i = -V_i'(x_i) + \epsilon\,(x_{i+1} - 2\,x_i + x_{i-1}) \qquad \forall\, i \in \mathbb{Z}\,. \tag{2}$$

This is the classical Klein-Gordon chain, which is well known to support multibreather solutions.

As a matter of fact, the multibreather existence theorems, based on a continuation from a suitable anticontinuous limit [30, 34] hold for an ϵ-neighborhood around zero, and are thus valid either for $\epsilon > 0$ or for $\epsilon < 0$, provided that $|\epsilon|$ is sufficiently smaller than 1. Consider the integrable anticontinuous limit ($\epsilon = 0$) where all the oscillators lie in equilibrium apart from $n + 1$ "central" ones which lie on periodic orbits satisfying the resonance condition

$$\frac{\omega_0}{k_0} = \cdots = \frac{\omega_n}{k_n} = \omega\,, \qquad\qquad k_i \in \mathbb{Z}. \tag{3}$$

At this limit, the motion of the central oscillators is described by

$$\begin{aligned} w_i &= \omega_i t + \vartheta_i \\ J_i &= \text{const.} \qquad\qquad i = 0, \ldots, n\,, \end{aligned}$$

where (w, J) are the action angle-variables of the uncoupled oscillators, ϑ is the initial angle and ω is the corresponding angular frequency. The $T = 2\pi/\omega$-periodic motion, which is described by (3), can be continued for $\epsilon > 0$ small enough, to form a T-periodic $(n + 1)$-site breather, provided that the following conditions hold:
1) The *anharmonicity condition* of the individual oscillators, i.e. $\mathrm{d}\omega_i/\mathrm{d}J_i \neq 0$, at least in the neighbourhood of the specific periodic orbit.
2) The *nonresonance condition*: $\omega_p \neq m\,\omega, \quad \forall m \in \mathbb{N}$, where ω_p denotes the linear (phonon) spectrum of the system (in terms of a wave number p).
However, even if both of these conditions hold, not all the states of the anticontinuous limit will be continued to a multibreather. In addition, the phases of the oscillators in this limit must be such that the system of equations

$$\frac{\partial \langle H_1 \rangle}{\partial z_i} = 0 \quad i = 1 \ldots n \tag{4}$$

has simple zeros, i.e. it is required that $\det\left|\partial^2\langle H_1\rangle/\partial z_i \partial z_j\right| \neq 0$, where $z_i = k_i\vartheta_{i-1} - k_{i-1}\vartheta_i$; this is a generalization of the notion of phase difference between the successive oscillators, in order to include resonances other that the 1 : 1. Here,

$$\langle H_1 \rangle = \int_0^T H_1 \mathrm{d}t \tag{5}$$

is the average value of the perturbative term of the Hamiltonian calculated along a periodic orbit over a time-period.

As it is thoroughly explained in Ref. [31], Eq. (4) can be written as

$$\frac{\partial \langle H_1 \rangle}{\partial z_i} = 0 \qquad \Leftrightarrow \qquad \sum_{i=1}^{n}\sum_{m=1}^{\infty} m A_{i-1,k_i m} A_{i,k_{i-1} m} \sin m z_i = 0 \tag{6}$$

where $A_{i,j}$is the j_{th} Fourier coefficient of the i_{th} oscillator. From Eq. (6) we conclude that $z_i = 0, \pi$ always satisfy (4) while, if special symmetry conditions hold, one could also obtain additional solutions.

If the action-angle canonical transformation is known, one could search for these solutions in (4) or its equivalent (6). However, in the generic case where the explicit form of the action-angle variables is *not* known, a method to calculate the necessary quantities has been developed in Ref. [32]. According to this method, the system of equations (4) is equivalent to the following one:

$$\int_0^T \frac{\partial H_1}{\partial x_i} p_i \mathrm{d}t = 0\,, \qquad\qquad i = 1 \ldots n\,. \tag{7}$$

This system can easily be solved numerically, as will be later shown in a specific example.

Besides the existence of the multibreather-solutions, the phase difference between the oscillators determines also its linear stability, as shown in Refs. [31, 33].

The linear stability of a periodic orbit (which in the specific case is the multibreather), is defined by the eigenvalues of the corresponding Floquet matrix λ_i. For $\epsilon = 0$, these eigenvalues lie in two complex conjugate bundles at $e^{\pm i\omega_p T_b}$, except the $2n + 2$ eigenvalues which correspond to the $n + 1$ central oscillators which lie at unity. For $|\epsilon| \neq 0 \ll 1$, the eigenvalues of the non-central oscillators move along the unit circle being of the same Krein kind, while the ones of the central oscillators are given by

$$\lambda_i = e^{\sigma_i T}\,, \tag{8}$$

where σ_i are the corresponding characteristic exponents. As it was proven in Ref. [34] (and also stated, in the present formalism, in Ref. [30]), these exponents are given in the leading order of approximation by

$$\sigma_i = \pm\sqrt{\epsilon}\sigma_{j1} + O(\epsilon)\,, \tag{9}$$

while σ_{j1}^2 coincide with the eigenvalues of the stability matrix

$$E = -A \cdot B\,,$$

with

$$A = \left(\frac{\partial^2 \langle H_1 \rangle}{\partial\vartheta_i \partial\vartheta_j} \right), \qquad B = \left(\frac{\partial^2 H_0}{\partial J_i \partial J_j} \right).$$

Therefore, if the various values of σ_{i1}^2, i.e. the eigenvalues of E, are negative and distinct, the multibreather is linearly stable. If there are no other solutions than the standard ones the corresponding linear stability is well defined by the knowledge of the resonant angles z_i, the kind of potential anharmonicity — i.e. hardening ($\partial\omega_i/\partial J_i > 0$) or softening ($\partial\omega_i/\partial J_i < 0$) — and the sign of ϵ. Let us now apply this method in a specific example, namely the equation of transverse dust grain motion in a dust crystal.

III. TRANSVERSE DUST GRAIN MOTION IN A DUST CRYSTAL

We shall consider the vertical (off-plane, $\sim \hat{z}$) charged grain displacement in a dust crystal (assumed quasi-one-dimensional, of infinite length: identical grains of charge q and mass M are situated at $x_n = n\,r_0$, where $n = ..., -2, -1, 0, 1, 2, ...$), by taking into account the intrinsic nonlinearity of the sheath electric (and/or magnetic) potential. The in-plane (longitudinal, acoustic, $\sim \hat{x}$ and shear, $\sim \hat{y}$) degrees of freedom are assumed suppressed; this situation is indeed today realized in appropriate experiments [26, 27], where a laser impulse triggers transverse dust grain oscillations, while a confinement potential ensures the chain's in-plane stability.

A. Equation of motion

The vertical grain displacement obeys an equation in the form [22, 23]

$$\frac{d^2\delta z_n}{dt^2} + \nu\frac{d\delta z_n}{dt} + \omega_0^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,, \tag{10}$$

where $\delta z_n(t) = z_n(t) - z_0$ denotes the small displacement of the $n-$th grain around the (levitated) equilibrium position z_0, in the transverse ($z-$) direction. The characteristic frequency $\omega_0 = \left[-q\Phi'(r_0)/(Mr_0)\right]^{1/2}$ results from the dust grain (electrostatic) interaction potential $\Phi(r)$, e.g. for a Debye-Hückel potential [35, 36]: $\Phi_D(r) = (q/r)\,e^{-r/\lambda_D}$, one has: $\omega_{0,D}^2 = q^2/(Mr_0^3)\,(1 + r_0/\lambda_D)\,\exp(-r_0/\lambda_D)$, where λ_D denotes the effective DP Debye radius [15]. The damping coefficient ν accounts for dissipation due to collisions between dust grains and neutral atoms. The gap frequency ω_g and the nonlinearity coefficients α, β are defined via the overall vertical force: $F(z) = F_{e/m} - Mg \approx -M[\omega_g^2\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3] + \mathcal{O}[(\delta z_n)^4]$, which has been expanded around z_0 by formally taking into account the (anharmonicity of the) local form of the sheath electric (follow exactly the definitions in Ref. [22], not reproduced here) and/or magnetic [37] field(s), as well as, possibly, grain charge variation due to charging processes [23]. Recall that the electric/magnetic levitating force(s) $F_{e/m}$ balance(s) gravity at z_0. Notice the difference in structure from the usual nonlinear Klein-Gordon equation used to describe one-dimensional oscillator chains — cf. e.g. Eq. (1) in Ref. [6]: TDLWs (*'phonons'*) in this chain are stable only in the presence of the field force $F_{e/m}$.

For convenience, the time and vertical displacement variables may be scaled over appropriate quantities, i.e. the characteristic (single grain) oscillation period ω_g^{-1} and the lattice constant r_0, respectively, viz. $t = \omega_g^{-1}\tau$ and $\delta z_n = r_0 q_n$; Eq. (10) is thus expressed as:

$$\frac{d^2 q_n}{d\tau^2} + \epsilon(\,q_{n+1} + \,q_{n-1} - 2\,q_n) + \,q_n + \alpha'\,q_n^2 + \beta'\,q_n^3 = 0\,, \tag{11}$$

where the (dimensionless) damping term, now expressed as $(\nu/\omega_g)dq_n/d\tau \equiv \nu'\dot{q}_n$, will be henceforth omitted in the left-hand side. The coupling parameter is now $\epsilon = \omega_0^2/\omega_g^2$, and the nonlinearity coefficients are now: $\alpha' = \alpha r_0/\omega_g^2$ and $\beta' = \beta r_0^2/\omega_g^2$.

B. Linear waves

Retaining only the linear contribution and considering oscillations of the type, $\delta z_n \sim \exp[i\,(knr_0 - \omega t)] + c.c.$ (complex conjuguate) in Eq. (10), one obtains the well known transverse dust lattice (TDL) wave optical-mode-like dispersion relation

$$\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2\left(\frac{kr_0}{2}\right), \tag{12}$$

or

$$\tilde{\omega}^2 = 1 - 4\epsilon \sin^2(\tilde{k}/2), \tag{13}$$

See that the wave frequency $\omega \equiv \tilde{\omega}\omega_g$ *decreases* with increasing wavenumber $k = 2\pi/\lambda \equiv \tilde{k}/r_0$ (or decreasing wavelength λ), implying that transverse vibrations propagate as a *backward wave*: the group velocity $v_g = \omega'(k)$ and the phase velocity $\omega_{ph} = \omega/k$ have opposite directions (this behaviour has been observed in recent experiments). The modulational stability profile of these linear waves (depending on the plasma parameters) was investigated in Refs. [22, 23]. Notice the natural *gap frequency* $\omega(k = 0) = \omega_g = \omega_{max}$, corresponding to an overall motion of the chain's center of mass, as well as the *cutoff frequency* $\omega_{min} = (\omega_g^2 - 4\omega_0^2)^{1/2} \equiv \omega_g\,(1-4\epsilon^2)^{1/2}$ (obtained at the end of the first Brillouin zone $k = \pi/r_0$) which is *absent in the continuum limit*, viz. $\omega^2 \approx \omega_g^2 - \omega_0^2\, k^2\, r_0^2$ (for $k \ll r_0^{-1}$); obviously, the study of wave propagation in this ($k \lesssim \pi/r_0$) region invalidates the continuum treatment employed so far in literature. The essential feature of discrete dynamics, to be retained here, is the (narrow) bounded TDLW (*'phonon'*) frequency band, limited in the interval $\omega \in [(\omega_g^2 - 4\omega_0^2)^{1/2}, \omega_g]$; note that one thus naturally obtains the stability constraint: $\omega_0^2/\omega_g^2 = \epsilon < 1/4$ (so that $\omega \in \mathbb{R} \quad \forall k \in [0, \pi/r_0]$).

We needn't go into further details concerning the linear regime, since it is covered in the literature. We shall, instead, see what happens if the *nonlinear* terms are retained, in this discrete description.

C. Multibreathers in dust crystals

Eq. (11) can be generated by a Hamiltonian of the form (1) by considering a quartic polynomial potential of the form

$$V(x) = x^2 + a'x^3 + b'x^4, \tag{14}$$

and considering *negative* values of ϵ (in account of inverse dispersion).

The values of the anharmonicity parameters a' and b' may be deduced from dusty plasma experiments on nonlinear vertical dust lattice oscillations [24–27]. For instance, the Kiel (Germany) experiment by Zafiu *et al.* [25] – using a laser to trigger nonlinear vertical dust grain oscillations – has provided the values: $\alpha/\omega_g^2 = +0.02; +0.016; -0.27\,(\mathrm{mm}^{-1})$ and $\beta/\omega_g^2 = -0.16; -0.17; -0.03\,(\mathrm{mm}^{-2})$ (successively, by gradually increasing the diameter of the dust grains; see Table I in Ref. [25]). In our notation, this implies: $\alpha' \simeq +0.02; +0.016; -0.27$, and $\beta' \simeq -0.16; -0.17; -0.03$ (for a lattice spacing of the order of $r_0 \simeq 1\,\mathrm{mm}$). Note that damping was very low ($\nu' \simeq 0.02$), thus a posteriori justifying neglecting it. These (three) sets of values are shown in table I, for reference.

TABLE I: Experimental data: three sets of sheath potential anharmonicity values, obtained from Ref. [25].

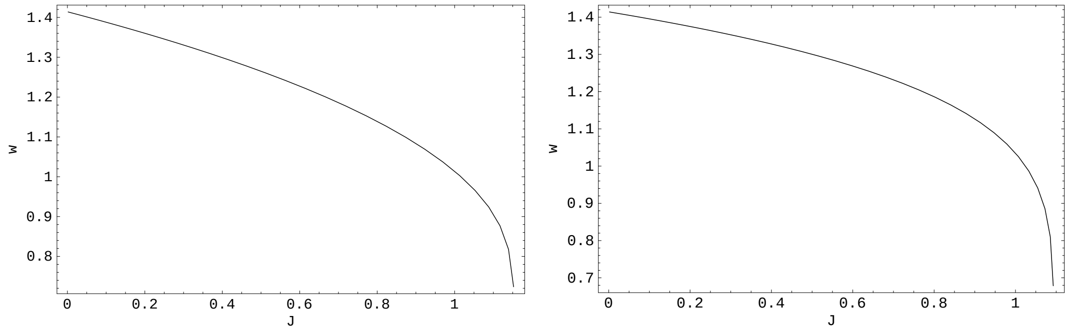
	I	II	III
a	0.02	0.016	-0.27
b	-0.16	-0.17	-0.03

The anharmonicity condition is satisfied in set II and III since $\mathrm{d}\omega/\mathrm{d}J < 0$ in the entire range of allowed values of J as it can be seen in Fig. 1. The computations of $\omega(J)$ has been made numerically since the explicit transformation is not known. For a more detailed description see in Ref. [32].

To make things specific, we choose to have two central oscillators in the anticontinuous limit moving with the same frequency $\omega_1 = \omega_2 = \omega$, which satisfies the nonresonance condition and will also be the frequency of the multibreather.

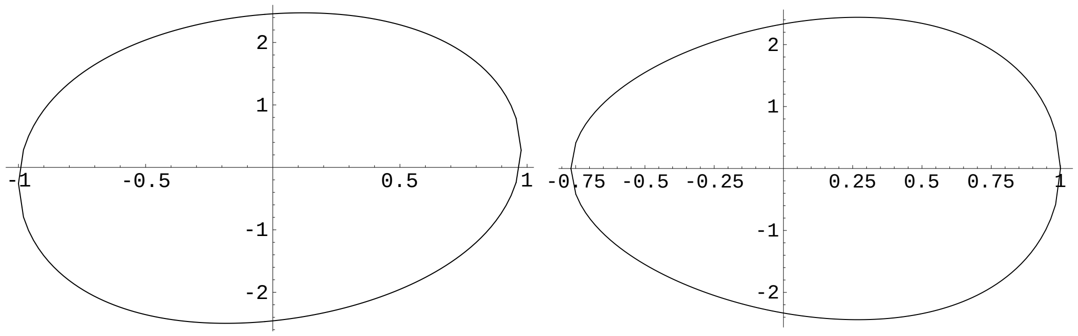
We now have to check condition (4). As already shown above, the $z_i = 0, \pi$ solutions always exist. Since the action-angle transformation is not known, to check for extra solutions, one has to solve the equivalent equation (6)

$$\int_0^T \frac{\partial H_1}{\partial x_2} p_2 \mathrm{d}t = 0\,. \tag{15}$$



(II) (III)

FIG. 1: $\omega(J)$ for sets II and III.



(II) (III)

FIG. 2: $F(x_{20})$ for sets II and III.

This equation defines a relationship between x_1 and x_2. The periodic orbits are thus defined, meaning that the energy E_i end the period T_i of the orbits are known. The only unknown is the set of initial conditions for the orbits x_{10}, p_{10}, x_{20}, p_{20}. We fix $x_{10} = 0$ and choose the specific $p_{10} > 0$ which determines the desired periodic orbit. So, the only free variable is x_{20}, since we can choose p_{20} from the equation of energy. We now need to solve the equation

$$F(x_{20}) = \int_0^T x_1 p_2 \mathrm{d}t = 0\,. \tag{16}$$

This equation is two branched, i.e. yields one branch for each choice of sign for the momentum p_{20}. In fig. 2, these two branches are presented togother in the same diagram for sets II and III, the two roots of $F(x_{20})$ correspond to the standard breather solutions $z = 0, \pi$. As for the stability of these solutions, following the arguments in Refs. [31, 33], the solution with $z = 0$ will be the linearly stable one and, since there are no other solutions besides the ones already mentioned, this solution will be the only linearly stable one. In particular, in Ref. [31] it is shown that

$$\sigma_{11}^2 = -k^2 \frac{\partial \omega}{\partial J} \sum_{m=1}^{\infty} m^2 A_m^2 \cos mz\,, \tag{17}$$

which for $\epsilon < 0$ confirms what has been claimed above.

We have computed this solution for only two central oscillators, but it would be the same for any number n of central oscillators since, as it is shown in [32], the system is consisted by independent equations. In that case, the only linearly stable solution would be $z_i = 0$, for $i = 1, \ldots, n$.

The above mentioned solutions is proven to be linearly stable for small enough ϵ. However, as the absolute value of ϵ increases, the eigenvalues corresponding to the central oscillators will collide to the phonon band and, since they are of opposite Krein sign, they can leave the unit circle forming a complex quadruple; the multibreather thus becomes unstable.

In order for the solutions to be physically relevant the experimentally measured value of the coupling constant ϵ should be inside the stability region. So, the next problem is to determine the value of the coupling constant where the solution bifurcates to become unstable. As can be seen in the diagrams below the point where it bifurcates is $\epsilon \simeq -0.045$ which confirms that this kind of motion can be supported by the specific model.

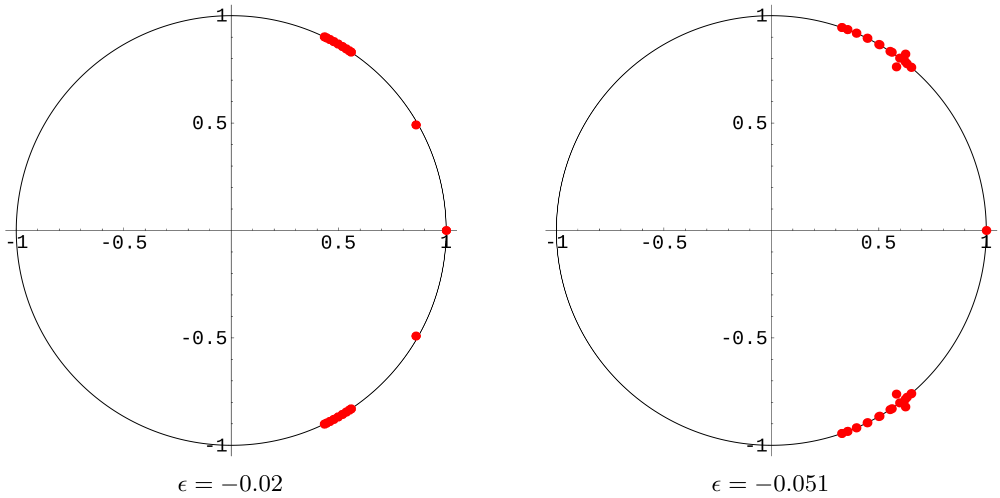


FIG. 3: The eigenvalues of the Floquet matrix for two different values of the coupling constant for the second set

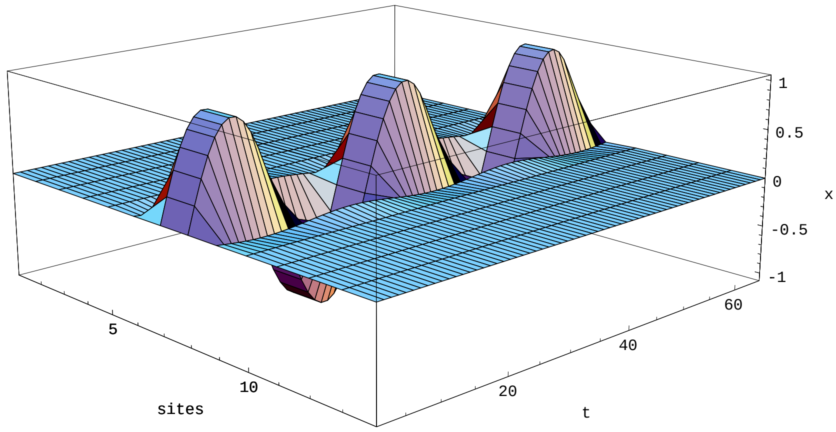


FIG. 4: The evolution of a 2-breather for $\epsilon = -0.02$, for the second set of values above.

[1] M. Remoissenet, *Waves Called Solitons* (Springer, Berlin, 1994).
[2] T. Dauxois and M. Peyrard, *Physics of Solitons*, Cambridge University Press (2005).
[3] S. Takeno, K. Kisoda and A. J. Sievers, Prog. Theor. Phys. Suppl. **94**, 242 (1988).
[4] J. B. Page, Phys. Rev. B **41**, 7835 (1990).
[5] T. Dauxois and M. Peyrard, Phys. Rev. Lett. **70** (25), 3935 (1993).
[6] Yu. Kivshar, Phys. Lett. A **173** (2), 172 (1993).
[7] D.K. Campbell and M. Peyrard, *CHAOS - Soviet American Perspectives on Nonlinear Science*, edited by D.K. Campbell (AIP, Physics New York, 1990).
[8] R. S. MacKay and S. Aubry, Nonlinearity **7**, 1623 (1994).
[9] S. Aubry, Physica D **103**, 201(1997).
[10] S. Flach, Phys. Rev. E **51**, 1503 (1995).
[11] S. Flach, and C. R. Willis, Phys. Rep. **295**, 181 (1998).
[12] R.S. MacKay, Physica A, **288**, 174 (2000).
[13] See various articles in the Volume (Focus Issue): Yu. Kivshar and S. Flach (Eds.), Chaos **13** (2), pp. 586 - 666 (2003).
[14] D. K. Campbell, S. Flach and Yu. S. Kivshar, *Physics Today*, **57** (1) (2004).
[15] P. K. Shukla and A. A. Mamun, *Introduction to Dusty Plasma Physics* (Institute of Physics, Bristol, 2002).
[16] G. E. Morfill, H. M. Thomas and M. Zuzic, in *Advances in Dusty Plasma Physics*, Eds. P. K. Shukla, D. A. Mendis and T. Desai (World Scientific, Singapore, 1997) p. 99.
[17] I. Kourakis and P. K. Shukla, *Physica Scripta* T**113**, 97 - 101 (2004).
[18] F. Melandsø, Phys. Plasmas **3**, 3890 (1996).
[19] I. Kourakis and P. K. Shukla, Eur. Phys. J. D, **29**, 247 (2004).
[20] M. R. Amin, G. E. Morfill and P. K. Shukla, Phys. Plasmas **5**, 2578 (1998); Phys. Scripta **58**, 628 (1998).
[21] I. Kourakis and P. K. Shukla, Phys. Plasmas, **11**, 1384 (2004).
[22] I. Kourakis and P. K. Shukla, Phys. Plasmas, **11**, 2322 (2004).
[23] I. Kourakis and P. K. Shukla, Phys. Plasmas, **11**, 3665 (2004).
[24] A. V. Ivlev, R. Sütterlin, V. Steinberg, M. Zuzic and G. Morfill, Phys. Rev. Lett. **85**, 4060 (2000).
[25] C. Zafiu, A. Melzer and A. Piel, Phys. Rev. E **63**, 066403 (2001).
[26] T. Misawa, N. Ohno, K. Asano, M. Sawai, S. Takamura and P. K. Kaw, Phys. Rev. Lett. **8**6, 1219 (2001).
[27] B. Liu, K. Avinash and J. Goree, *Phys. Rev. Lett.* **91**, 255003 (2003).
[28] I. Kourakis and P. K. Shukla, *Physics of Plasmas*, **12** (1), 014502/1-4 (2005); I. Kourakis, P. K. Shukla and V. Basios, *Proceedings of the 2004 ICPP: International Congress on Plasma Physics* (Nice, France, 2004) (e-Proceedings at: `http://hal.ccsd.cnrs.fr/ccsd-00001892/en/`).
[29] J. - A. Sepulchre and R. S. MacKay, Nonlinearity **10**, 679 (1997); R. S. MacKay and J. - A. Sepulchre, Physica D **119**, 148 (1998).
[30] V. Koukouloyannis and S. Ichtiaroglou, Physical Review E, **66**, 066602 (2002).
[31] V. Koukouloyannis and S. Ichtiaroglou, *A stability Criterion for Multibreathers in Klein-Gordon chains*, to appear in Journal of Bifurcations and Chaos.
[32] V. Koukouloyannis, Physical Review E, **69**, 046613 (2004).
[33] J.F.R. Archilla, J. Cuevasa, B. Sanchez-Rey and A. Alvarez", Physica D,**180**, 235 (2003)
[34] T. Ahn, R.S. MacKay and J-A. Sepulchre, Nonlinear Dynamics, **25**, 157 (2001)
[35] U. Konopka, G. E. Morfill and L. Ratke, *Phys. Rev. Lett.* **84**, 891 (2000).
[36] E.B. Tomme, D. A. Law, B. M. Annaratone and J. E. Allen, Phys. Rev. Lett. **85**, 2518 (2000).
[37] In the *magnetically* levitated dust crystal case, the reader should consider the definitions in Ref. [23], upon setting $K_1 \to \alpha$, $K_2 \to \beta$ and $K_3 \to 0$ therein.