1. Introduction

A number of recent theoretical studies have been devoted to collective processes in dusty plasmas (DP), in relevance with experimental observations. Dust (quasi-)lattices (DL) are typically formed in the sheath region above the negative electrode in discharge experiments, horizontally suspended at a levitated equilibrium position, at \( z = z_0 \), where gravity and electric (and/or magnetic) forces balance. The linear regime of low-frequency oscillations in DP crystals, in the longitudinal (acoustic mode) and transverse (in-plane, shear acoustic mode and vertical, off-plane optical mode) directions is, in now quite well understood. However, the nonlinear behaviour of DP crystals is little explored, and has lately attracted experimental [1-3] and theoretical [1-8] interest.

Recently [2], we considered the coupling between the horizontal \((\sim \hat{x})\) and vertical (off-plane, \( \sim \hat{z} \)) degrees of freedom in a dust monolayer, as a set of nonlinear equations for longitudinal and transverse dusty lattice waves (LTDW, TDLMs). This was thus rigorously derived [7]. Here, we revisit the nonlinear dust grain excitations which may occur in a DP crystal (assumed quasi-one-dimensional and infinite), composed from identical grains, of equilibrium charge \( q \) and mass \( M \), located at \( x_n = n q / \kappa L \) in \( X \) by- and- or neutral-interactions (collisions) are omitted, for simplicity. This study complements recent experimental investigations [1-3] and may hopefully motivate future ones.

2. Transverse envelope structures (continuum)

Taking into account the intrinsic nonlinearity of the shear elastic (and/or magnetic) potential, the transverse (off-plane) \( n \)-th grain displacement \( z_n = z_n(t) \) in a dust crystal (where \( n = ... , 0, 1, 2, ... \)...) obeys the equation

\[
\frac{d^2z_n}{dt^2} + \nu \frac{dz_n}{dt} + \omega_p^2 \left[ (\Delta z_n + \Delta z_{n-1} + 2 \Delta z_n) + \Delta z_{n+1} \right] + \alpha \frac{dz_n}{dt} \left[ (\delta z_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right] = 0 \tag{1}
\]

(where coupling anharmonicity and second + neighbor interactions are omitted, for simplicity).

The characteristic frequency

\[
\omega_p = \sqrt{-k_i (\nu^2/\omega_p^2) / \omega_p^2} \frac{dz_n}{dt} + \omega_p^2 \left[ (\Delta z_n + \Delta z_{n-1} + 2 \Delta z_n) + \Delta z_{n+1} \right] + \alpha \frac{dz_n}{dt} \left[ (\delta z_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right] = 0 \tag{1}
\]

is related to the electrostatic interaction potential, for a Debye-Hückel potential. \( \omega_p^2 = \sqrt{(q^2/\epsilon r^2) / (\varepsilon_0^2 \mu^2)} \), one has

\[
\frac{d^2z_n}{dt^2} + \omega_p^2 \left( \frac{dz_n}{dt} + \omega_p^2 \left[ (\Delta z_n + \Delta z_{n-1} + 2 \Delta z_n) + \Delta z_{n+1} \right] + \alpha \frac{dz_n}{dt} \left[ (\delta z_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right] \right) = 0 \tag{1}
\]

which is indeed satisfied in all known LTDW experiments [2].

3. Intrinsic transverse Localized Modes (ILMs) – Discrete Breathers (DBs)

ILMs, i.e. highly localized Discrete Breather (DB) and multi-breather type low-frequency vibrations, were also shown to occur in transverse DL systems, and are currently investigated from first principles [7]. These excitations have recently received increased interest among researchers in solid state physics, due to their occurrence in periodic lattices and remarkable physical properties [8]. Remarkably, the existence of such DB structures at a frequency \( \omega_p \) generally requires the non-resonance condition

\[
\omega_p \neq \omega_n (\forall n) \setminus \mathbb{N} \tag{1}
\]

4. Longitudinal envelope excitations

The nonlinear equation of motion

\[
\frac{d^2\delta x_n}{dt^2} + \nu \omega_n \frac{d\delta x_n}{dt} + \delta x_n (\Delta x_n + \Delta x_{n-1} + 2 \Delta x_n) + \delta x_n (\Delta x_{n+1} + \Delta x_{n+2}) + \alpha \frac{d\delta x_n}{dt} \left[ (\delta z_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right] = 0 \tag{1}
\]

is the Debye length; \( \varepsilon_0 \), the plasma frequency; \( \mu \), the magnetic permeability; \( \alpha \), the nonlinearity coefficient; \( \nu \), the damping coefficient; \( \omega_n \), the intrinsic frequency.

\[
\omega_n = \sqrt{\frac{4 \kappa}{\nu^2} \left[ \left( \delta x_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2 \right) \right] \setminus \mathbb{N} \tag{1}
\]

The dispersion coefficient \( \nu \omega_n \) and the phase speed \( \gamma \) are given by

\[
\nu \omega_n = \frac{\left( (\delta x_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right) \setminus \mathbb{N} \tag{1}
\]

\[
\gamma = \frac{\omega_n}{\nu \omega_n} \left( \frac{\omega_n}{\nu \omega_n} \right) \setminus \mathbb{N} \tag{1}
\]

where \( \omega_p = \omega_n / \nu \omega_n \setminus \mathbb{N} \) and \( \nu \omega_n \) are the wavenumber and frequency obeying the opto-like discrete dispersion relation

\[
\omega_n^2 - \frac{\omega_p^2}{\nu^2} \delta x_n (\Delta x_n + \Delta x_{n-1} + 2 \Delta x_n) + \delta x_n (\Delta x_{n+1} + \Delta x_{n+2}) + \alpha \frac{d\delta x_n}{dt} \left[ (\delta z_n^2 + \delta z_{n-1}^2 - \delta z_{n-1}^2) \right] = 0 \tag{1}
\]

5. Longitudinal solitons

Equation (1) is essentially identical to the equation of atomic motion in a chain with anharmonic springs, i.e. in the celebrated FPU (Fermi-Pasta-Ulam) problem. At a first step, one may adopt a continuum description, viz. \( \delta x_n(t) \rightarrow u(x,t) \). This leads to different nonlinear evolution equations (depending on the simplifying hypotheses adopted), some of which are critically discussed in [10].

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Following Melonian ([11]), various studies have relied on the Korteweg – de Vries (KdV) equation, i.e. Eq. (5), for a \( a = 0 \), to gain analytical insight in the compressive structures observed in experiments [1]. Indeed, the KdV Eq. possesses negative (only here, since \( a > 0 \)) pulse soliton solutions for \( a \), implying a compressive (anti-kink) excitation for \( u \). The KdV soliton is thus interpreted as a density variation in the crystal, viz. \( u(x,t)/4 \setminus \mathbb{N} \setminus \mathbb{N} \setminus \mathbb{N} \); the KdV Eq. is the inverse Korteweg – de Vries (iKdV) equation, which accounts for both compressive and rarefactive lattice excitations (exact expressions in [10]). Alternatively, Eq. (7) can be reduced to a Generalized Boussinesq (GBo) equation. Inversion [10], for \( \rho_0 \setminus \mathbb{N} \), one recovers a Boussinesq (Bq) equation, widely studied in solid state.

The (GBq) equation yields, like its KdV counterpart, both compressive and rarefactive (only compressive, respectively) solutions; however, the (compressive) propagation speed now does not have to be close to \( c_p \). The beauty analysis [see Eq. (11) for details] is not reproduced here.

References