New generalized dispersion relation for low-frequency electromagnetic waves in Hall-magnetohydrodynamic dusty plasmas

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1. Introduction

Within the framework of the ideal magnetohydrodynamic (MHD) theory, Alfvén electromagnetic (EM) waves [1] are governed by the continuity and momentum equations for the plasma mass flow, together with Faraday's law, in which the electric field and the mass flow velocity are related by Ohm's law. The restoring force is provided by the magnetic pressure, while inertia is provided by the ion mass. The Alfvén wave dispersion [2] describes finite frequency ω ($< \omega_{ci}$, where ω_{ci} is the ion gyrofrequency), finite ion Larmor radius, finite ion polarization, and finite electron inertia effects. In dispersive Alfvén waves, the frozen-in field lines are broken, and linear coupling between various modes (e.g. among Alfvén, magnetosonic, shear Alfvén waves, and whistlers) may occur. The dynamics of the dispersive Alfvén waves within the fluid model is governed by the Hall-MHD equations [3], in which one uses the generalized Ohm's law to include the $\mathbf{J}\times\mathbf{B}$ force, where $\mathbf{J}$ is the plasma current and $\mathbf{B}$ is the total magnetic field in the plasma. Of interest here is also the existence of a new cut-off frequency for circularly polarized EM ion-cyclotron Alfvén waves and for magnetosonic waves in a dusty plasma [4, 5].

In this report, we present the linear dispersion properties of intermediate-frequency ($\omega_{p,d}$, $\omega_{c,d} < \omega < \omega_{c,e}$), long wavelength ($\lambda > \rho_{L,i/e}$, $\omega_{p,e}/c$) electromagnetic waves in a multi-component warm dusty magnetoplasma whose constituents are electrons, ions, and immobile charged dust macroparticles. At this range, the massive dust may be considered to be immobile (i.e. the dust density n_d is constant, and the dust velocity $\mathbf{u}_d$ vanishes), and electron inertia may be neglected. Here, $\omega_{p,\alpha} = (4\pi n_\alpha q_\alpha^2/m_\alpha)^{1/2}$, $\omega_{c,\alpha} = q_\alpha B_0/(m_\alpha c)$ and $\rho_{L,\alpha} = v_{th,\alpha}/\omega_{c,\alpha}$ respectively denote the plasma frequency, cyclotron frequency and Larmor radius (where $v_{th,\alpha} = (k_B T_\alpha/m_\alpha)^{1/2}$ is the thermal velocity, and T_α is the temperature) associated with species $\alpha = e, i, d$).

2. The model

A three-component, fully ionized dusty plasma is considered, composed of electrons, ions, and immobile charged dust particulates, with masses m_e, m_i and m_d, and charges $q_e = -e$, $q_i = +Z_i e$ and $q_d = -Z_d e$, where e is the magnitude of the electron charge, Z_i is the ion charge state, and Z_d is the number of electrons residing on a dust grain. Both the mass and charged of the heavy dust particles are assumed to be constant. The plasma is immersed in a homogeneous magnetic field $\mathbf{B_0} = B_0\,\hat{\mathbf{z}}$ along the z axis (B_0 =constant). We adopt the MHD system of equations for the electrons and ions. The electron and ion number densities $n_{e,i}$ and velocities $\mathbf{u}_{e,i}$ are governed by the continuity and momentum equations

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\,\mathbf{u}_e) = 0\,, \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\,\mathbf{u}_i) = 0\,, \tag{2}$$

$$0 = -e(\mathbf{E} + \frac{1}{c}\mathbf{u_e}\times\mathbf{B}) - \frac{1}{n_e}\nabla P_e\,, \tag{3}$$

and

$$\frac{\partial \mathbf{u_i}}{\partial t} + \mathbf{u_i}\cdot\nabla\,\mathbf{u_i} = \frac{Z_i\,e}{m_i}\left(\mathbf{E} + \frac{1}{c}\mathbf{u_i}\times\mathbf{B}\right) - \frac{1}{m_i n_i}\nabla P_i\,. \tag{4}$$

The pressure(s) $P_{e/i}$ is (are) assumed to obey $P_{e/i} \sim n_{e/i}^{\gamma_{e/i}}$, where γ is the adiabatic index (i.e. $\gamma = 3$ for adiabatic compression, $\gamma = 5/3$ in three dimensions, and $\gamma = 1$ for isothermal compression), thus $\nabla P_{e/i} = \gamma_{e/i} T_{e/i}\nabla n_{e/i}$. Also, $\mathbf{E}$ is the wave electric field and $\mathbf{B}$ is the sum of the static and wave magnetic fields, viz. $\mathbf{B} = \mathbf{B_0}+\mathbf{b}$. The system is closed with the Maxwell equations. Ampère's and Faraday's laws read

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} \equiv \frac{4\pi}{c}\sum_\alpha q_\alpha\,n_\alpha\,\mathbf{u}_\alpha = \frac{4\pi e}{c}(Z_i\,n_i\mathbf{u_i} - n_e\mathbf{u_e}) \tag{5}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,. \tag{6}$$

The displacement current was neglected in the Eq. (5), since low phase speed ($\omega/k \ll c$) EM waves are considered here. At equilibrium, the overall neutrality condition is

$$n_{e,0} - Z_i\,n_{i,0} + Z_d\,n_d = 0\,, \tag{7}$$

the subscript 0 denotes the unperturbed densities.

3. Reduced system of equations

Letting $n_i \approx n_{i,0} + n_1$ and $\mathbf{u_i} = \mathbf{0} + \mathbf{v}$, where $n_1 \ll n_{i,0}$ is a small perturbation in the density, a reduced system of evolution equations can be obtained from Eqs. (1) to (6) (see in [6] for details). These are the ion continuity equation

$$\partial n_1/\partial t + n_{i,0}\nabla\cdot\mathbf{v} = 0\,, \tag{8}$$

the ion momentum equation

$$\frac{\partial\mathbf{v}}{\partial t} = -\Omega_R\,(\mathbf{v}\times\hat{\mathbf{z}}) + \frac{\alpha B_0}{4\pi n_{i,0} m_i}(\nabla\times\mathbf{b})\times\hat{\mathbf{z}} - \frac{c_s^2}{n_{i,0}}\nabla n_1\,, \tag{9}$$

and the magnetic field evolution equation

$$\frac{\partial\mathbf{b}}{\partial t} = \alpha\,B_0\,\nabla\times(\mathbf{v}\times\hat{\mathbf{z}}) - \frac{\alpha c\,B_0}{4\pi Z_i e n_{i,0}}\nabla\times\left[(\nabla\times\mathbf{b})\times\hat{\mathbf{z}}\right]\,, \tag{10}$$

where we have used the quasi-neutrality condition – here $Z_d > 0 (Z_d < 0)$ for negative (positive) dust charge – and defined the quantities $\Omega_R = Z_d n_d \omega_{ci}/n_{e,0}$ and $\alpha = Z_i n_{i,0}/n_{e,0}$, where $\omega_{ci} = Z_i e B_0/m_i c$ is the ion gyrofrequency. The modified ion sound speed is $c_s = \left[\left(\gamma_i n_{i,0} T_i + \gamma_e Z_i^2 n_{i,0}^2 T_e/n_{e,0}\right)/m_i n_{i,0}\right]^{1/2}$. The latter equations form a closed system which describes the evolution of small ion density, ion velocity and magnetic field perturbations in our dust Hall-MHD plasma model. In a dust-free (e-i) plasma, where $\alpha = 1$, the ion rotation frequency Ω_R vanishes and Eq. (9) describes ion acceleration by the $\mathbf{J}\times\mathbf{B_0}$ and $\nabla(P_{e1}+P_{i1})$ forces, while Eq. (10) then simply depicts the evolution of the wave magnetic field in the presence of a non-solenoidal electric field $\mathbf{E} = -\mathbf{v}_e\times\mathbf{B}_0$. In a dusty Hall-MHD plasma with negatively charged dust grains, where $\alpha > 1$, an enhanced charge separation appears due to the wave electric field. The resulting enhanced electron fluid velocity produces a new Lorentz centripetal force [the first term in the right-hand side of Eq. (9)], which in combination with the $\mathbf{J}\times\mathbf{B_0}$ and pressure gradient forces produces ion rotation around the negatively charged static dust grains. The rotational force acting on the ions is then responsible for a non-trivial coupling between various wave modes in dusty plasmas. Furthermore, due to $\alpha > 1$, we obtain an increased Alfvén wave phase speed and ion skin depth.

Let us now consider small amplitude propagating electromagnetic waves around the equilibrium state $\{n_{i,0}, \mathbf{0}, \mathbf{B_0}\}$ (where the perturbations are $\{n_1, \mathbf{v}, \mathbf{b}\}$). Following Ref. [9], we thus linearize our governing equations to derive a fairly general dispersion relation for the wave propagation in our uniform dusty magnetoplasma. By letting $\partial/\partial t \to -i\omega$ and $\nabla \to i\mathbf{k}$, where ω and $\mathbf{k} = \mathbf{k}_\perp + \hat{\mathbf{z}}k_z$ denote the wave frequency and the wavevector, respectively, we then obtain from Eqs. (8) - (10)

$$\omega n_1 - n_{i,0}\,\mathbf{k}\cdot\mathbf{v} = 0\,, \tag{11}$$

$$\omega\mathbf{v} = -i\Omega_R(\mathbf{v}\times\hat{\mathbf{z}}) - \frac{\alpha B_0}{4\pi m_i n_{i,0}}(k_z\mathbf{b} - b_z\mathbf{k}) + (\mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\frac{c_s^2}{\omega}\,, \tag{12}$$

and

$$\omega\mathbf{b} = -\alpha\,B_0\,[k_z\mathbf{v} - (\mathbf{k}\cdot\mathbf{v})\hat{\mathbf{z}}] + i\frac{\alpha c B_0}{4\pi e n_{i,0} Z_i}k_z\,(\mathbf{k}\times\mathbf{b})\,.$$

Using the constraint $\nabla\cdot\mathbf{B} = 0$, i. e. $\mathbf{k}\cdot\mathbf{b} = 0$, the inner product of $\mathbf{k}$ with Eq. (12) gives

$$\mathbf{k}\cdot\mathbf{v} = \frac{\omega}{\omega^2 - k^2 c_s^2}\left[\frac{\alpha B_0}{4\pi m_i n_{i,0}}k^2\,b_z - i\Omega_R\,(\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}}\right]\,, \tag{13}$$

which, combined with Eqs. (12) and (13), yields

$$\omega b_z = \frac{(\omega^2 - k_z^2 c_s^2)k^2 V_A^2}{\omega(\omega^2 - k^2 c_s^2)}\,b_z + i\frac{\alpha c B_0}{4\pi Z_i e n_{i,0}}k_z j_z - i\alpha B_0\Omega_R\frac{\omega^2 - k_z^2 c_s^2}{\omega(\omega^2 - k^2 c_s^2)}(\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}}\,, \tag{14}$$

where we defined $j_z = (\mathbf{k}\times\mathbf{b})\cdot\hat{z}$, as well as the (dust-modified) Alfvén speed $V_A = \alpha\,B_0/\sqrt{4\pi m_i n_{i,0}} \equiv \alpha\,v_A$, where $\alpha = Z_i n_{i,0}/n_{e,0}$.

From Eqs. (12) and (13), one also obtains

$$\omega j_z = -\alpha\,B_0\,k_z\,(\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}} - i\frac{\alpha c B_0}{4\pi Z_i e n_{i,0}}k^2 k_z\,b_z\,, \tag{15}$$

and

$$\omega\left[1 - \frac{\Omega_R^2(\omega^2 - k_z^2 c_s^2)}{\omega^2(\omega^2 - k^2 c_s^2)}\right](\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}} = -\frac{\alpha B_0}{4\pi m_i n_{i,0}}k_z\,j_z + i\frac{\omega\Omega_R(\omega^2 - k_z^2 c_s^2)}{\omega^2(\omega^2 - k^2 c_s^2)}\frac{\alpha\,B_0}{4\pi m_i n_{i,0}}k^2\,b_z\,. \tag{16}$$

Equations (14) - (16) form a closed system in terms of b_z, j_z and $(\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}}$. The latter two equations may now be solved, and the solutions may then be inserted into Eq. (14).

Combining Eqs. (14)-(16) we obtain

$$j_z = -i\left[\omega^2 - \frac{\Omega_R^2(\omega^2 - k_z^2 c_s^2)}{(\omega^2 - k^2 c_s^2)} - k_z^2 V_A^2\right]^{-1} \left\{\left[1 - \frac{\Omega_R^2(\omega^2 - k_z^2 c_s^2)}{\omega^2(\omega^2 - k^2 c_s^2)}\right]\frac{1}{\alpha\omega_{ci}} + \frac{\Omega_R(\omega^2 - k_z^2 c_s^2)}{\omega^2(\omega^2 - k^2 c_s^2)}\right\}k_z k^2 V_A^2\omega b_z \tag{17}$$

and

$$(\mathbf{k}\times\mathbf{v})\cdot\hat{\mathbf{z}} = i\left[\omega^2 - \frac{\Omega_R^2(\omega^2 - k_z^2 c_s^2)}{(\omega^2 - k^2 c_s^2)} - k_z^2 V_A^2\right]^{-1} \left[\frac{\Omega_R(\omega^2 - k_z^2 c_s^2)}{(\omega^2 - k^2 c_s^2)} + \frac{k_z^2 V_A^2}{\alpha\omega_{ci}}\right]\frac{k^2\,V_A}{\sqrt{4\pi m_i n_{i,0}}}\,b_z. \tag{18}$$

Substituting into Eq. (14), we obtain the new dispersion relation

$$\left(\omega^2 - k_z^2 V_A^2\right)\left[\omega^2(\omega^2 - k^2 c_s^2) - (\omega^2 - k_z^2 c_s^2)k^2 V_A^2\right] = \omega^2\left(\omega^2 - k_z^2 c_s^2\right)\Omega_R^2 + \frac{k_z^2 k^2 V_A^4}{\alpha^2\omega_{ci}^2}\left[\omega^2(\omega^2 - k^2 c_s^2) - \Omega_R^2(\omega^2 - k_z^2 c_s^2)\right] + \frac{2k_z^2 k^2 V_A^4 \Omega_R}{\alpha\omega_{ci}}(\omega^2 - k_z^2 c_s^2)\,, \tag{19}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2 \equiv k_\perp^2 + k_z^2$. We have defined the (dust-modified) Alfvén speed $V_A = \alpha\,B_0/\sqrt{4\pi m_i n_{i,0}} \equiv \alpha\,v_A$. Note the effect of the dust expressed via Ω_R and α. The terms in the second line are due to the Hall-term, and thus disappear in the *ideal* MHD limit (the first term in the right-hand side would then be the sole modification due to the stationary dust). The same terms also disappear in the purely perpendicular propagation limit.

4. Dust-free two-component plasmas.

Let us now check the above results in the vanishing dust limit. With $\Omega_R = 0$, $\alpha = 1$ and $V_A = v_A$, we obtain

$$\left(\omega^2 - k_z^2 v_A^2\right)\left[\omega^4 - k^2(v_A^2 + c_s^2)\omega^2 + k_z^2 k^2 c_s^2 v_A^2\right] = (\omega^2 - k^2 c_s^2)\frac{\omega^2 k_z^2 k^2 v_A^4}{\omega_{ci}^2}\,. \tag{20}$$

This dispersion relation was derived in Ref. [7] and analyzed in Refs. [3, 8]; it incorporates the electromagnetic ion-cyclotron Alfvén modes, the fast and slow magnetosonic modes, the kinetic Alfvén waves, and long wavelength whistlers. For instance, in the limit $c_s = 0$ and $k_\perp = 0$, one obtains the magnetic field aligned dispersive electromagnetic ion-cyclotron-Alfvén wave, i.e. $\omega = k_z v_A\,(1 \pm \omega/\omega_{ci})^{1/2}$ (here $+/-$ corresponds to right-/left-hand circularly polarized waves). The whistler frequency $\omega = k_z k c^2\omega_{ce}/\omega_{pe}^2$ is recovered for $\omega \gg k v_A$ and $c_s = 0$. The kinetic Alfvén waves, $\omega \approx k_z v_A\left(1 + k_\perp^2 c_s^2/\omega_{ci}^2\right)^{1/2}$ are obtained in the limits $c_s \ll v_A$, $k_z c_s \ll \omega \ll \omega_{ci}, k_\perp v_A, k_\perp c_s$. For perpendicular wave propagation ($k^2 = k_x^2 + k_y^2 \equiv k_\perp^2$), one recovers the (fast) magnetosonic mode, $\omega = k_\perp(v_A^2 + c_s^2)^{1/2}$.

5. Dusty (three-component) plasmas.

5.1. Perpendicular propagation.

For perpendicular wave propagation, viz. $k_z = 0$ and $k = k_\perp \equiv (k_x^2 + k_y^2)^{1/2}$, one recovers from Eq. (19) the *modified magnetosonic* mode

$$\omega^2 = \Omega_R^2 + k_\perp^2(c_s^2 + V_A^2)\,.$$

Note the Rao frequency cutoff $\omega(k_\perp = 0) = \Omega_R$ [5].

5.2. Parallel propagation.

For wave propagation along the magnetic field direction (i.e. for $k = k_z$, one obtains from Eq. (19)

$$\left(\omega^2 - k_z^2 V_A^2\right)^2 = \omega^2\Omega_R^2 + \frac{k_z^4 V_A^4}{\alpha^2\omega_{ci}^2}(\omega^2 - \Omega_R^2) + 2k_z^4 V_A^4\frac{\Omega_R}{\alpha\omega_{ci}}\,, \tag{21}$$

which can be exactly rewritten in the simpler form

$$k_z^2 v_A^2 = \frac{\omega^2\omega_{ci}}{\omega_{ci} \pm \omega} \mp \frac{Z_d n_d}{Z_i n_{i,0}}\omega\omega_{ci}, \tag{22}$$

and which precisely agrees with the well known result [4] for the magnetic field aligned circularly polarized electromagnetic waves.

5.3. Cold dusty plasma.

In a cold dusty plasma ($c_s = 0$), the dispersion relation (19) reduces to

$$(\omega^2 - k_z^2 V_A^2)(\omega^2 - \omega_A^2) = \omega^2\omega_A^2 b + \Omega_R^2(\omega^2 - \omega_A^2 b) + 2\alpha\Omega_R\omega_{ci}\omega_A^2 b,$$

where $\omega_A = kV_A$ and $b = k_z^2 V_A^2/\alpha^2\omega_{ci}^2$. In the limits $\omega_A^2 b \ll \Omega_R^2, \omega^2\Omega_R/2\alpha\omega_{ci}, \omega^2$ and $k_z V_A\omega_A \ll \omega^2$, one obtains

$$\omega^2 = \Omega_R^2 + (k^2 + k_z^2)$$

whereas for $\omega_A^2 b \ll \Omega_R^2, \omega^2\Omega_R/2\alpha\omega_{ci}, \omega^2$ and $\omega^2 \ll k_z V_A\omega_A$,

$$\omega^2 = k_z^2 k^2 V_A^4/\Omega_R^2\,.$$

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