Linear and nonlinear dynamics of a dust crystal with positive and negative dust

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1. Introduction

One of the most astonishing novel characteristics of dusty (complex) plasmas is the occurrence of strongly coupled dust configurations, such as the spontaneous formation of crystalline-like periodic arrangements, in the sheath region (above the negative electrode) in gas discharge experiments. Crystal formation and dynamics have been studied in numerous experiments, in which 'dust' particles were essentially created by injecting artificial (e.g. formaldehyde) micro-spheres, which subsequently acquire a fixed (negative, usually) charge via inherent dynamic charging mechanisms.
More recent experimental studies have been devoted to studies of alternating charge sign (positive-negative) dust configurations [1].

2. Formulation – a model dust bi-crystal

We shall consider a *one dimensional* horizontal chain (assumed infinite, for simplicity) consisting of negative and positive dust grains, located at equidistant sites (lattice constant r_0). Odd (even) sites, i.e. at $x = (2n+1)r_0$ ($x = 2nr_0$; $n \in \mathcal{N}$), are occupied by negative (positive) charge dust grains, of charge $-Q_1$ ($+Q_2$) and mass M_1 (M_2, respectively); we assume that $M_1 > M_2$, with no loss of generality. Vertical force equilibrium is ensured by (a balance between) gravity and electric/magnetic forces; only longitudinal displacement $\delta x_n = x_n - nr_0$ (where $n \in \mathcal{N}$) is permitted in this simplified model.
The electrostatic binary interaction force $F(r)$ exerted on two grains situated at a distance r is derived from a potential function $U(r)$, viz. $F(r) = -\partial U(r)/\partial x$. Considering the (attractive) interaction between first neighbors only, i.e. $r_{n,n+1} = x_{n+1} - x_n = r_0 + \delta x_{n+1} - \delta x_n$, we may Taylor expand $U(r)$ around r_0, to account for grain displacements. We formally have:

$$F(r) \approx -U''(r_0)(r-r_0) - \frac{1}{2}U'''(r_0)(r-r_0)^2 - \frac{1}{6}U''''(r_0)(r-r_0)^3$$

(the prime denotes differentiation). In the following, we shall set:

$$U''(r_0) = G\,, \quad U'''(r_0)/2 = GA\,, \quad U''''(r_0)/6 = GB\,.$$

The description of our dust crystal dynamics is thus effectively reduced to a problem of longitudinal atom motion in a diatomic chain, characterized by an anharmonic coupling 'spring' potential. Our dust bi-layer may therefore be analyzed by making use of standard analytical tools from solid state physics [2, 3].
If one considers a Debye-type interaction potential (energy)

$$U_D(r) = Q_1 Q_2\, e^{-r/\lambda_D}/r \equiv (Q_1 Q_2/\lambda_D)\, e^{-\kappa}/\kappa\,,$$

for which:

$$U_D'(r_0) = -(Q^2/\lambda_D^2)\, e^{-\kappa}\,(1+\kappa)/\kappa^2\,,$$

$$U_D''(r_0) = +(2Q^2/\lambda_D^3)\, e^{-\kappa}\,(1+\kappa+\kappa^2/2)/\kappa^3\,,$$

$$U_D'''(r_0) = -(6Q^2/\lambda_D^4)\, e^{-\kappa}\,(1+\kappa+\kappa^2/2+\kappa^3/6)/\kappa^4\,,$$

and

$$U_D''''(r_0) = +(24Q^2/\lambda_D^5)\, e^{-\kappa}\,(1+\kappa+\kappa^2/2+\kappa^3/6+\kappa^4/24)/\kappa^5\,,$$

where the lattice parameter is $\kappa = r_0/\lambda_D$.

3. Equations of motion

Denoting the odd (even) grain displacement, within the $n-$th pair, by $\delta z_{2n+1} = z_n$ ($\delta z_{2n} = w_n$), the resulting equations of motion read:

$$\begin{aligned} M_1 \frac{d^2 z_n}{dt^2} &= G(w_n - 2z_n + w_{n-1}) \\ &+ GA\left[(w_n - z_n)^2 - (z_n - w_{n-1})^2\right] \\ &+ GB\left[(w_n - z_n)^3 - (z_n - w_{n-1})^3\right] \\ M_2 \frac{d^2 w_n}{dt^2} &= G(z_{n+1} - 2w_n + z_n) \\ &+ GA\left[(z_{n+1} - w_n)^2 - (w_n - z_n)^2\right] \\ &+ GB\left[(z_{n+1} - w_n)^3 - (w_n - z_n)^3\right]. \end{aligned} \qquad (1)$$

4. Linear vibrations

Assuming a plane wave (*phonon*) solution in the form: $z = Z \exp i[(2n+1)kr_0 - \omega t]$+c.c. (for the heavy negative grains) and $w = W \exp i(2nkr_0 - \omega t)$+c.c (for the lighter positive grains), one finds that the frequency ω is related to the wavenumber k via the *dispersion relation*

$$\left(\frac{\omega^2}{2G} - \frac{1}{M_1}\right)\left(\frac{\omega^2}{2G} - \frac{1}{M_2}\right) = \frac{1}{M_1 M_2}\cos^2 kr_0\,;$$

the exact solution for the frequency reads

$$\omega_\pm^2 = \frac{G}{\mu}\left(1 \pm \sqrt{1 - \frac{4\mu^2}{M_1 M_2}\sin^2 kr_0}\right), \qquad (2)$$

where we have defined the *reduced mass* $\mu = M_1 M_2/(M_1 + M_2)$.

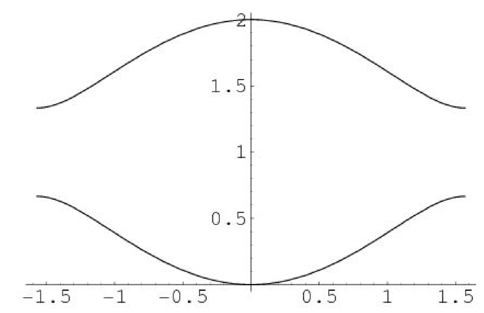

Fig. The dust bi-layer dispersion relation $\omega_\pm$ (normalized by G/μ) is depicted vs. the reduced wavenumber kr_0, for $M1 = 2M_2 = 1$ and $G = 1$ (indicative arbitrary values).

This relation defines a two-fold dispersion curve.

The *lower branch* ω_- defines an *acoustic mode*; at low k, it satisfies: $\omega_- \approx (\frac{2G}{M_1+M_2})^{1/2} kr_0 \equiv c_0 k$, and thus both the group velocity $v_{gr,-} = \omega_-'(k)$ and the phase velocity $v_{ph,-} = \omega_-/k$ tend to the (constant) *'sound velocity'* c_0 for low k.

The *upper branch* ω_+ defines an *optic mode*; at low k, it satisfies: $\omega_- \approx (\frac{2G}{M_1+M_2})^{1/2}$ =constant, and thus $v_{gr,+} = \omega_+'(k) = 0$ and $v_{ph,+} \to \infty$ for long wavelengths $\lambda = 2\pi/k$.

The *frequency band* scanned by the two modes are $\omega_- \in [0, \sqrt{2G/M_1}]$ and $\omega_+ \in [\sqrt{2G/M_2}, \sqrt{2G/\mu}]$.

We note the appearance of a *forbidden frequency range* between $\omega_-(k = \pm\pi/2r_0) = \omega_{-,max} = \sqrt{2G/M_1}$ and $\omega_+(k = \pm\pi/2r_0) = \omega_{+,min} = \sqrt{2G/M_2}$.

Furthermore, we point out that the optic mode ω_+ is characterized by an *inverse* dispersion, thus $v_{gr,+} = \omega_+'(k) \le 0$ everywhere in the first Brillouin zone (1BZ) $[0, \pi/2r_0]$.

The amplitude eigenmodes – i.e. the solutions of the linearized system of Eqs. (1), for Z and W – satisfy: $W/Z = (2G - M_2\omega_\pm^2)/(2G\cos kr_0)$. Therefore, in-phase (out-of-phase) motion is prescribed for long wavelength acoustic (optic) vibrations, since $W/Z \to 1$ ($W/Z \to -M_1/M_2$, respectively) for $k \to 0$.

5. Continuum approximation

Assuming a long excitation extension $L \gg r_0$, one may substitute the discrete space variables $z_n(t)$ and $w_n(t)$ with continuous ones, say $z(x,t)$ and $w(x,t)$, by Taylor expanding, i.e.

$$z_{n\pm1} \approx z \pm 2r_0 z_x + 2r_0^2 z_{xx} \pm \frac{4}{3}r_0^3 z_{xxx} + \frac{2}{3}r_0^4 z_{xxxx} + \mathcal{O}[(2r_0/L)^5]$$

(and the analogous expression for $w_n \to w$), where the subscript denotes differentiation, e.g. $z_x = \partial z/\partial x$ and so forth. Inserting into the discrete equations of motion (1), one thus obtains two coupled partial derivative equations (PDEs). For analytical manipulation purposes, the second one may be neglected by employing the *'Büttner ansatz'* [3]:

$$w \approx \sigma[z + b_1 r_0 z_x + \frac{b_2}{2}r_0^2 z_{xx} + \frac{b_3}{6}r_0^3 z_{xxx} + \frac{b_4}{24}r_0^4 z_{xxxx} + b_0 f(z)] + \mathcal{O}(\epsilon^5)\,,$$

where σ is set equal to 1 ($-M_1/M_2$) for the acoustic (optic) mode, and the parameters b_j and the function $f(z)$ are appropriately adjusted for compatibility. One thus remains with one PDE, in terms of $z(x,t)$, while $w(x,t)$ is defined accordingly.
In the following, we shall present some recent results regarding the acoustic mode. The remaining results will be exposed in a detailed article, in preparation.

6. Nonlinear analysis: the Boussinesq equation for the accoustic mode

The compatibility among the eqs. of motion is ensured by choosing [3]: $\sigma = b_1 = 1$, $b_2 = 2\mu/M_2$, $b_3 = 6\mu(2M_1 - M_2)/(3M_1 M_2)$, $b_4 = 24\mu[1/(3M_2) - b_2^2/(4M_1)]$, and $b_0 = 0$ (for first-neighbor only interactions); see that an ordinary Taylor expansion (viz. $b_j = 1$) is recovered in the limit $M_1 = M_2$.
The system of Eqs. (1) now yield to the nonlinear PDE

$$z_{tt} - c_0^2 z_{xx} = p_0 z_x z_{xx} + q_0 z_x^2 z_{xx} + h_0 z_{xxxx}\,, \qquad (3)$$

or (in an equivalent manner) the Generalized BOUSSINESQ equation

$$u_{tt} - c_0^2 u_{xx} = p(u^2)_{xx} + q(u^3)_{xx} + h_0 u_{xxxx}\,, \qquad (4)$$

where we have set:

$$u = z_x\,, \quad p = p_0/2 = GAb_2/M_1\,, \quad q = q_0/3 = GBb_2/M_1\,,$$

and

$$h_0 = \frac{2G}{M_1}r_0^4\left(\frac{b_4}{24} - \frac{b_3}{6} + \frac{b_2}{2} - \frac{1}{3}\right)$$

(the sound velocity c_0 was defined above).
The ordinary (modified, respectively) BOUSSINESQ equation is recovered from Eq. (4), upon setting $q = 0$ ($p = 0$), or $B = 0$ ($A = 0$), i.e. by neglecting quartic (cubic) interaction potential contributions.

The GBq Eq. (4) yields two distinct pulse soliton solutions (whose exact form is omitted here, for brevity); these lead (since $u = z_x$) to the *kink* (shock-like) soliton:

$$z(x,t) = \pm 2\left(\frac{6h_0}{q_0}\right)^{1/2} \arctan\left[\frac{1}{P_1}\tanh\left(\frac{x - vt}{L_1} + x_0\right)\right], \qquad (5)$$

where

$$P_1 = \left\{\left[\sqrt{p_0^2 + 6(v^2 - c_0^2)q_0} \pm p_0\right]/\left[\sqrt{p_0^2 + 6(v^2 - c_0^2)q_0} \mp p_0\right]\right\}^{1/2};$$

x_0 and v are real constants, which determine the soliton center and (supersonic, since $v > c_0$) velocity, respectively;
the soliton width is expressed by $L_1 = 2\sqrt{h_0/(v^2 - c_0^2)}$.
Recall that $L_1 \gg r_0$ in order for the continuum theory to be valid.
The two solutions above correspond to a rarefactive and a compressive localized excitation, propagating in the dust bi-layer.

7. KdV acoustic soliton theory.

By assuming near-sonic propagation, i.e. $v \approx c_0$, and a very slow time variation (viz. $u_{\tau\tau} \ll u_\tau, u_\xi$), one obtains from the GBq Eq. (4) the canonical form of the Generalized KORTEWEG - DE VRIES (GKdV) equation

$$u_\tau + 6uu_\xi + 6u^2 u_\xi + u_{\xi\xi\xi} = 0 \qquad (6)$$

[3], where we have defined: $\xi = p_0(x - c_0 t)/\sqrt{6h_0 q_0}$, $\tau = p_0^3 t/[2c_0(6q_0)^{3/2}h_0^{1/2}]$, and $u = z_\xi\sqrt{q_0/(6h_0)}$. The GKdV Eq. (6) yields two distinct exact soliton solutions, which may be inverted to u; one thus obtains two different *kink solitons* in the form:

$$z(x,t) = \pm 2\left(\frac{6h_0}{q_0}\right)^{1/2} \arctan\left[\frac{1}{P_2}\tanh\left(\frac{x - vt}{L_2} + x_2\right)\right], \qquad (7)$$

where

$$P_2 = \left\{\left[\sqrt{p_0^2 + 12c_0(v - c_0)q_0} \pm p_0\right]/\left[\sqrt{p_0^2 + 12c_0(v - c_0)q_0} \mp p_0\right]\right\}^{1/2};$$

x_2 and v are real constants, which determine the soliton center and (slightly supersonic) velocity, respectively;
the soliton width is expressed by $L_2 = 2\sqrt{h_0/[2c_0(v - c_0)}$.
Again, $L_2 \gg r_0$ is assumed, in order for the continuum theory to be valid.
These two solutions correspond to a *rarefactive* and *compressive* localized excitation. Notice that Eq. (7) are recovered from Eq. (5), by setting $v + c_0 \approx 2c_0$.

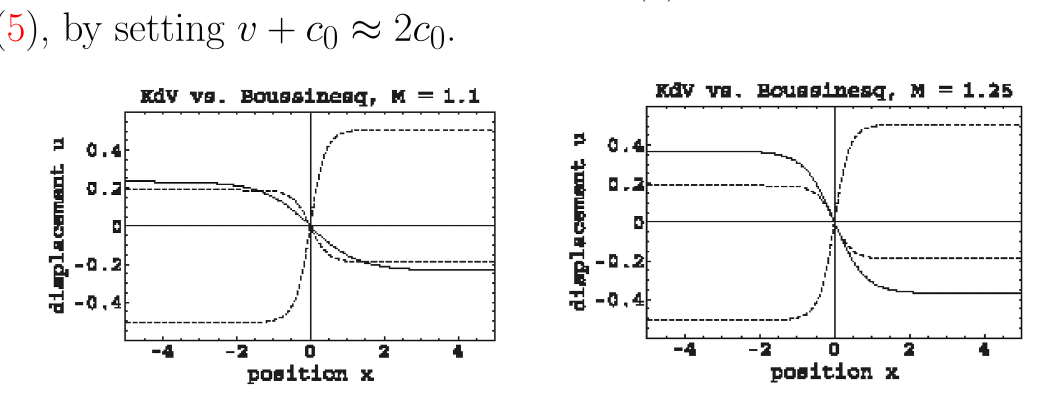

Fig. KdV vs. Boussinesq theory (displacement) solitons, for different Mach numbers $M = v/c_L$.

We conclude that the KdV (and associated) equation related theory adds no extra information to that obtained via the (less approximate) Boussinesq theory.

8. Amplitude modulational analysis of dust bi-lattice waves

A study of the amplitude modulation of the dust bi-layer described above is being carried out and results will be reported soon.
The standard multiple scale analysis provides a set of nonlinear Schrödinger equations, whose compatibility is ensured via an appropriate choice of the Büttner ansatz parameters (see above). Finding the exact expressions for these parameters is a tedious task, whose results are too lengthy to report here.
Let us point out that the (electrostatic) coupling 'potential possesses a strong cubic term, which leads to an asymmetry in the envelope excitations (cf. our neighboring poster for a 1d dust crystal).

Similar results have been obtained for one-dimensional dust monolayers [4]. These theoretical considerations will hopefully be confirmed by appropriate experiments.

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