Stability analysis of electromagnetic waves in negative refraction index materials

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1. Introduction

Left-handed (or *negative refraction index*) materials (LHM) are artificial meta-materials, which are characterized by a negative value of *both* the *dielectric permittivity* ϵ and *magnetic permeability* μ , as well as a negative value of the refraction index $n = -\sqrt{\epsilon\mu}$.

Inspired by the seminal work of Veselago in 1968 [1], a number of pioneering theoretical studies [2] suggested how these peculiar properties could be realized in purpose-designed and built materials, and experiments subsequently confirmed those predictions [3].

A number of applications (e.g. in optics [4]) were suggested to exploit the singular physical properties of LHM (beam re-focusing, inversion of Snell's law and of the Doppler shift effect, backward Cerenkov radiation, etc).

The theory of electromagnetic (EM) wave propagation in linear LHM was recently extended to account for *nonlinear* (i.e. field amplitude dependent) material properties [5, 6]. *Ab initio* calculations of the nonlinear dielectric and magnetic properties of split-ring resonator (SRR) lattice structures showed that magnetic nonlinearity, in principle, *dominates* in LH composite materials [5]. Taking this fact into account, the dynamics of the electric and magnetic field envelope of an EM wave propagating in a LH medium was recently related to the nonlinear amplitude modulation formalism by Lazarides and Tsironis [7], who showed that modulated EM wave propagation is governed by a pair of coupled Nonlinear Schrödinger-type equations (CNLS).

In this study, we present an investigation of the nonlinear stability of EM waves in a negative refractive index composite medium.

2. Nonlinear description of EM wave propagation in LHM

The dielectric and magnetic behaviors of negative index materials/LHM are characterized by both frequency *dispersion* and *nonlinearity* [5, 6]. Let us briefly review the existing theories modeling these mechanisms, in order to set the theoretical background of the stability analysis that will follow [7].

2.1. Nonlinear LHM properties

The dielectric and magnetic response of a nonlinear material is formally characterized by an electric flux density $\mathbf{D}$ and a magnetic induction $\mathbf{B}$, which depend on the electric and magnetic field intensities $\mathbf{E}$ and $\mathbf{H}$ as:

$$\mathbf{D} = \epsilon_{eff}\mathbf{E} = \epsilon\mathbf{E} + \mathbf{P}'$$

and

$$\mathbf{B} = \mu_{eff}\mathbf{H} = \mu\mathbf{H} + \mathbf{M}',$$

where ϵ and μ denote the medium (linear) *dielectric permittivity* and *magnetic permeability*, respectively, while $\mathbf{P}' = \epsilon_{NL}\mathbf{E}$ and $\mathbf{M}' = \mu_{NL}\mathbf{H}$ express the *nonlinear* contributions to the medium polarization and magnetization[a].

The *dielectric response* of LHM (SRR lattices, here) is given by the nonlinear and dispersive expression [5] (neglecting losses)

$$\begin{aligned}\epsilon_{eff} &= \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) + \epsilon_{NL}(|E|^2)\\ &\equiv \epsilon + \epsilon_{NL}(|E|^2)\\ &\equiv \epsilon_D(|E|^2) - \frac{\omega_p^2}{\omega^2},\end{aligned} \tag{1}$$

where the effective cutoff ("plasma") frequency ω_p is related to the geometrical features of the SRR lattice [2, 5]; ω is the propagating mode frequency; ϵ_D is related to the (nonlinear) dielectric response. The possibility for negative permittivity arises from the frequency dependence, for $\omega < \omega_p$.

The *magnetic response* of (SRR lattice-based) LHM reads [2, 5, 6]

$$\begin{aligned}\mu_{eff} &= \mu_0\left(1 + \frac{F\omega^2}{\omega_{0,NL}^2(|H|^2) - \omega^2}\right)\\ &= \mu_0\left(1 + \frac{F\omega^2}{\omega_0^2 - \omega^2}\right) + \mu_{NL}(|H|^2)\\ &\equiv \mu + \mu_{NL}(|H|^2),\end{aligned} \tag{2}$$

where the (linear) resonant SRR frequency ω_0 and the factor F (= the single ring to unit cell area ratio; $0 < F \ll 1$, ideally) are related to the intrinsic lattice structure [2] (losses neglected).

The (linear) permeability μ attains *negative* values for $\omega_0 < \omega < \omega_0/\sqrt{1-F}$, while the (complete) effective permeability μ_{eff} yields an enriched behavior. Note that $\mu \to \mu_0$, for $\omega \to 0$.

The nonlinear frequency $\omega_{0,NL} \equiv X\omega_0$ is related to $\omega \equiv \Omega\omega_0$ and ω_0, via a complex expression [5].

For small $|E|$, one may assume a Kerr-type behavior [5, 6]

$$\epsilon_{eff} \approx \epsilon + \alpha|E|^2 .$$

α positive/negative denotes a focusing/defocusing behavior.

[a] Our notation incorporates the vacuum dielectric permittivity ϵ_0 and magnetic permeability $\mu_0 = 1/c^2\epsilon_0$ into ϵ_{eff} and μ_{eff} (contrary to a widely used notation, according to which $\mathbf{B} = \mu_{eff}\mu_0\mathbf{H}$ and $\mathbf{D} = \epsilon_{eff}\mu_0\mathbf{E}$).

One then has [5, 6, 7]

$$|H|^2 = \alpha\, A^2\, E_c^4\, \frac{(1-X^2)\,(X^2-\Omega^2)^2}{\Omega^2\, X^6}\,, \tag{3}$$

where E_c determines the characteristic dielectric nonlinearity scale (viz. $\alpha = \pm E_c^{-2}$) and the quantity A is related to the physical features of the material unit elements [5, 6]. Note that relation (3) suggests a multi-valued dependence of X and μ_{eff}, on $|H|^2$.

To simplify the description, one may consider a "Kerr-like" magnetic dependence

$$\mu_{eff} \approx \mu + \beta|H|^2$$

where β is related to intrinsic material properties [7].

Although, given the complexity of (3), it is not trivial to obtain an analytical expression for the phenomenological nonlinearity parameter β, this assumption seems to be justified for sufficiently low values of the magnetic field intensity H; also cf. [5].

We recall that the dispersive character of the medium is hidden in the frequency dependence of both ϵ and μ, and that, in fact, left-handed behavior is restricted within a certain range of frequency values. Thus, this formulation formally applies to both right-handed and left-handed behaving frequency ranges of composite materials. Nevertheless, rigorously speaking, this description refers to low magnetic fields, as explained above.

We note that the above relations are compatible with the causality requirements $d[\epsilon(\omega)\,\omega]/d\omega > 1$ and $d[\mu(\omega)\,\omega]/d\omega > 1$ [2].

2.2. EM wave modulation

Let us consider an EM plane wave propagating in a left-handed medium. The wave consists of an electric and a magnetic field(s) of intensities $\mathbf{E}$ and $\mathbf{H}$, respectively, representing transverse propagating oscillations in perpendicular directions.

$\mathbf{E} \times \mathbf{H}$ determines the (Poynting) direction of energy flow, which coincides (is opposed to) the propagation direction, say along z, in right-handed (RH) [left-handed (LH), respectively] media [2].

The field vector magnitudes are $E(z,t) = \mathcal{E}(z,t)\,\exp[i(kz-\omega t)]$ and $H(z,t) = \mathcal{H}(z,t)\,\exp[i(kz-\omega t)]$, where ω, $k = 2\pi/\lambda$ and λ here denote the cyclic frequency, the wavenumber and the wavelength, respectively.

EM wave propagation is governed by Maxwell's laws. The nonlinear modulation of the EM field amplitudes was shown to be governed [7] by the *coupled Nonlinear Schrödinger (CNLS) equations* :

$$i\,\frac{\partial\mathcal{E}}{\partial T} + P\,\frac{\partial^2\mathcal{E}}{\partial X^2} + Q_{11}\,|\mathcal{E}|^2\,\mathcal{E} + Q_{12}\,|\mathcal{H}|^2\,\mathcal{E} = 0 \tag{4}$$

$$i\,\frac{\partial\mathcal{H}}{\partial T} + P\,\frac{\partial^2\mathcal{H}}{\partial X^2} + Q_{22}\,|\mathcal{H}|^2\,\mathcal{H} + Q_{21}\,|\mathcal{E}|^2\,\mathcal{H} = 0\,. \tag{5}$$

The (slow) position and time variables are defined as $X = \delta(x - v_g t)$ and $T = \delta^2 t$, where $\delta \ll 1$ is a small real parameter.

The field envelopes $\mathcal{E}$ and $\mathcal{H}$ move at the group velocity, related to the wavevector $\mathbf{k}$ as $\mathbf{v_g} = c^2\mathbf{k}/\omega$ [i.e. $v_g = \omega'(k) = c^2 k/\omega$].

The (common) group velocity dispersion coefficient is

$$P = \omega''(k)/2 = (c^2 - \omega'^2)/2\omega > 0\,;$$

($v_g < c$ is prescribed by causality in both RHM and LHM [2c]).

The frequency ω is related to the wavenumber k via a dispersion relation (related to the perplex expression for $\epsilon_{eff}(\omega)$ [2c]), which to lowest order reads $\omega = k/\sqrt{\epsilon\mu} \equiv ck$.

The nonlinearity coefficients are related to the nonlinearity ("Kerr") parameters α and β (both assumed $\sim \delta^2$ here):

$$Q_{11} = Q_{21} = \omega c^2 \alpha\mu/2 \equiv Q_1$$

and

$$Q_{22} = Q_{12} = \omega c^2 \beta\epsilon/2 \equiv Q_2\,.$$

Note the peculiar symmetry of the nonlinear part of Eqs. (4) and (5) (contrary to the 'usual' case in nonlinear optics, where one would have $Q_{11} = Q_{22}$ and $Q_{12} = Q_{21}$, instead).

3. Coupled plane wave solutions

Setting $\mathcal{E}(X,T) = \rho_1 \exp(i\theta_1)$ and $\mathcal{H}(X,T) = \rho_2 \exp(i\theta_2)$, where $\rho_{1,2}$ and $\theta_{1,2}$ are real functions of $\{X,T\}$, and substituting in Eqs. (4) and (5), one obtains

$$\rho_{i,T} + P\,(2\,\rho_{i,X}\,\theta_{i,X} + \rho_i\,\theta_{i,XX}) = 0\,, \tag{6}$$

and

$$\theta_{i,T} = P\,[\rho_{i,XX}/\rho_i - (\theta_{i,X})^2] + Q_1\rho_1^2 + Q_2\rho_2^2\,, \tag{7}$$

($i = 1,2$); the subscripts denote partial differentiation, viz. $f_X \equiv \partial f/\partial X$ and so forth. Taking $\rho_{1,2}$ = constant, we obtain a set of coupled monochromatic envelope (*Stokes*) wave solutions in the form

$$\{\mathcal{E}(X,T),\, \mathcal{H}(X,T)\} = \{\mathcal{E}_0,\, \mathcal{H}_0\}\, e^{i(Q_1|\mathcal{E}_0|^2 + Q_2|\mathcal{H}_0|^2)\,T}\,. \tag{8}$$

These solutions represent two co-propagating modulated field envelopes, oscillating (slowly) at a frequency $\Omega = (Q_1|\mathcal{E}_0|^2 + Q_2|\mathcal{H}_0|^2)$ (which depends on the constant field amplitudes $\mathcal{E}_0$ and $\mathcal{H}_0$). Note the common phase ΩT, due to the symmetry of the CNLSEs.

4. Modulational stability analysis and criterion

In order to study the stability of solution (8), we set $\mathcal{E}_0 \to \mathcal{E}_0 + \xi\mathcal{E}_1(X,T)$ and $\mathcal{H}_0 \to \mathcal{H}_0 + \xi\mathcal{H}_1(X,T)$, where the small ($\xi \ll 1$) perturbations $\mathcal{E}_1$ and $\mathcal{H}_1$ are *complex* functions of $\{X,T\}$.

Isolating terms in ξ, we obtain

$$i\mathcal{E}_{1,T} + P\,\mathcal{E}_{1,XX} + Q_1(\mathcal{E}_1 + \mathcal{E}_1^*)\mathcal{E}_0^2 + Q_2(\mathcal{H}_1 + \mathcal{H}_1^*)\mathcal{E}_0\mathcal{H}_0 = 0\,,$$

along with the analogous equation, upon $\mathcal{E}_1 \longleftrightarrow \mathcal{H}_1$, $Q_1 \longleftrightarrow Q_2$.

Separating real and imaginary parts, and assuming a perturbation wavenumber $\tilde{k}$ and frequency $\tilde{\omega}$, we obtain

$$\begin{aligned}&[-\tilde{\omega}^2 + P\tilde{k}^2(P\tilde{k}^2 - 2Q_1\mathcal{E}_0^2)]\,\tilde{a}_1 - 2PQ_2\mathcal{E}_0\mathcal{H}_0\tilde{k}^2\,\tilde{a}_2 = 0\,,\\ &-2PQ_1\mathcal{E}_0\mathcal{H}_0\tilde{k}^2\,\tilde{a}_1 + [-\tilde{\omega}^2 + P\tilde{k}^2(P\tilde{k}^2 - 2Q_2\mathcal{H}_0^2)]\,\tilde{a}_2 = 0\,.\end{aligned} \tag{9}$$

This system is tantamount to the eigenvalue problem

$$(\mathbf{M} - \tilde{\omega}^2\mathbf{I})\,\tilde{\mathbf{a}} = \mathbf{0}$$

where the elements of the vector $\tilde{\mathbf{a}} = (a_1, a_2)^T$ are the perturbation amplitudes, $\mathbf{I}$ is the unit matrix ($I_{ij} = \delta_{ij}$, for $i,j = 1,2$) and the elements of the matrix $\mathbf{M}$ are:

$$M_{11} = P\tilde{k}^2(P\tilde{k}^2 - 2Q_1\mathcal{E}_0^2)\,, \quad M_{22} = P\tilde{k}^2(P\tilde{k}^2 - 2Q_2\mathcal{H}_0^2)\,,$$

$$M_{12} = -2PQ_2\mathcal{E}_0\mathcal{H}_0\tilde{k}^2\,, \quad M_{21} = -2PQ_1\mathcal{E}_0\mathcal{H}_0\tilde{k}^2\,.$$

The eigenvalue existence condition $\mathrm{Det}(\mathbf{M} - \tilde{\omega}^2\mathbf{I}) = 0$, provides the bi-quadratic polynomial equation

$$\tilde{\omega}^4 - T\tilde{\omega}^2 + D = 0\,, \tag{10}$$

where

$$T \equiv \mathrm{Tr}\mathbf{M} = M_{11} + M_{22} = 2P^2\tilde{k}^2\,(\tilde{k}^2 - K)$$

and

$$D \equiv \mathrm{Det}\mathbf{M} = M_{11}M_{22} - M_{12}M_{21} = P^4\tilde{k}^6\,(\tilde{k}^2 - 2K)$$

denote the trace and the determinant, respectively, of matrix $\mathbf{M}$.

We have defined the quantity $K = \big(Q_1|\mathcal{E}_0|^2 + Q_2|\mathcal{H}_0|^2\big)/P$.

Since $T^2 - 4D = 4P^4\tilde{k}^4K^2 \geq 0$, two *real* solutions exist:

$$\tilde{\omega}_\pm^2 = \frac{1}{2}\,(T \pm \sqrt{T^2 - 4D})\,, \tag{11}$$

or, explicitly

$$\tilde{\omega}_+^2 = P^2\tilde{k}^4\,, \qquad \tilde{\omega}_-^2 = P^2\tilde{k}^2\,(\tilde{k}^2 - 2K)\,. \tag{12}$$

Imposing the reality of ω_- (for modulational stability) amounts to

$$\tilde{k}^2 - \frac{2}{P}\,\big(Q_1|\mathcal{E}_0|^2 + Q_2|\mathcal{H}_0|^2\big) > 0\,, \tag{13}$$

or

$$\tilde{k}^2 - \frac{\omega}{P}\left(\frac{\alpha}{\epsilon}\,|\mathcal{E}_0|^2 + \frac{\beta}{\mu}\,|\mathcal{H}_0|^2\right) \equiv \tilde{k}^2 - \frac{\omega}{P}K' > 0\,. \tag{14}$$

The EM stability profile thus depends on K' (rem.: $P > 0$ here).

In "ordinary" RH materials, one has $\mu, \epsilon > 0$, so (for $\beta = 0$, say, i.e. for a linear magnetic response) a modulational instability may or may not occur, depending on the focusing or de-focusing dielectric property of the medium (i.e. on the sign of α).

In LHM, both μ and ϵ are negative, while α and β depend on the medium's structure. Clearly, the EM wave will be *stable* if

$$K' = \frac{\alpha}{\epsilon}\,|\mathcal{E}_0|^2 + \frac{\beta}{\mu}\,|\mathcal{H}_0|^2 \leq 0 \tag{15}$$

If, on the other hand, $K' > 0$, the EM wave will be unstable to external perturbations with $\tilde{k} < \tilde{k}_{cr} \equiv \sqrt{2K} = \sqrt{\omega K'/P}$.

The growth rate $\sigma = i\sqrt{-\tilde{\omega}_-^2}$ of the instability then attains its maximum value $\sigma_{max} = PK = \omega K'/2$ at $\tilde{k} = \sqrt{K} = \tilde{k}_{cr}/\sqrt{2}$.

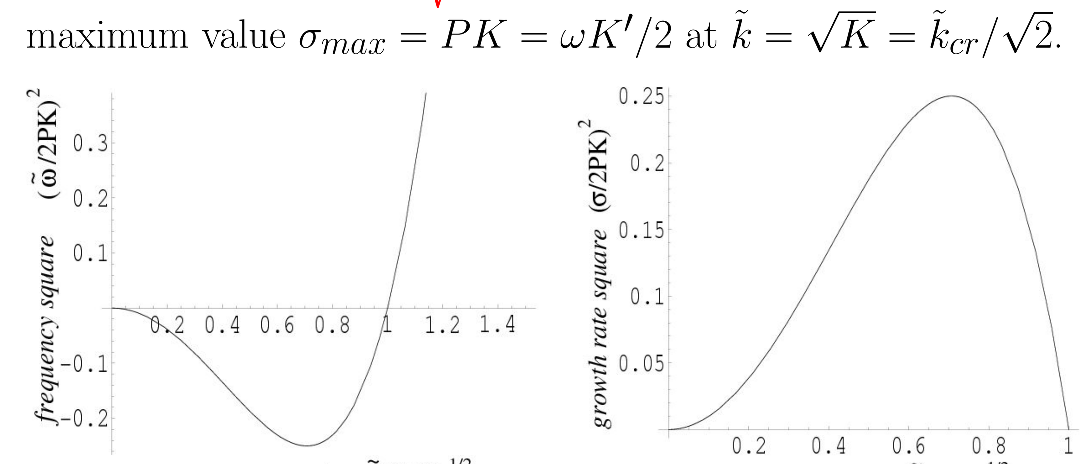

Figure 1. (a) The square of the perturbation frequency $\tilde{\omega}$ (scaled by $2KP = \omega K'$) is depicted vs. the perturbation wavenumber $\tilde{k}$ (scaled by $\sqrt{2K}$), as derived from the relation (12) for $K > 0$ (unstable case). (b) The square of the instability growth rate σ (scaled by $2KP = \omega K'$) is depicted vs. the perturbation wavenumber $\tilde{k}$ (scaled by $\sqrt{2K}$).

The final (physically transparent) stability criterion reads:

$$K' \approx \epsilon_{eff}/\epsilon + \mu_{eff}/\mu - 2 \leq 0\,. \tag{16}$$

The well known focusing/defocusing nonlinearity criterion, related to the Kerr property of optical media, is thus generalized to account for the intrinsically nonlinear properties of LHM.

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