Autumn College on Plasma Physics

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Collective processes in dusty plasma crystals

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`www.tp4.rub.de/∼ioannis/conf/200509-ICTP1-oral.pdf`

Outline

1. Introduction
   - ☛ *Dusty Plasma (DP)*: a rapid overview of notions and ideas;
   - ☛ Prerequisites: *Linear waves* in 1d dust crystals;
   - ☛ *Nonlinearity* in 1d DP crystals: Origin and modeling.

2. Nonlinear effects on *transverse* dust-lattice waves (*TDLW*s): amplitude modulation, transverse envelope structures.

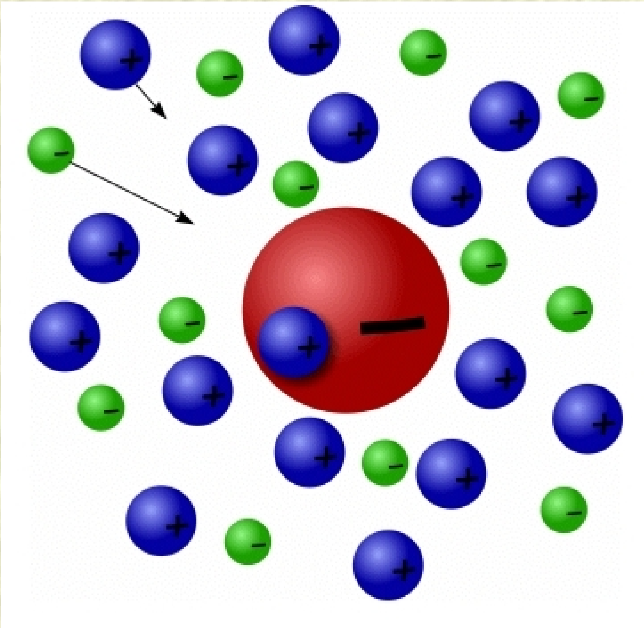
3. Nonlinear effects on *longitudinal* dust-lattice waves (*LDLW*s): modulation, longitudinal envelope excitations.

4. *Longitudinal localized excitations* : relation to soliton theories.

5. 1d *Discrete Breathers* (*Intrinsic Localized Modes*).

6. Conclusions.

1. Intro.: DP – Dusty Plasmas (or *Complex Plasmas*): definition and characteristics



❑ Ingredients:

– *electrons* e^- (charge $-e$, mass m_e),
– *ions* i^+ (charge $+Z_i e$, mass m_i), *and*
– charged micro-particles $\equiv$ *dust grains* d (most often d^-):

charge $Q = \pm Z_d e \sim \pm(10^3 - 10^4)\, e$,
mass $M \sim 10^9\, m_p \sim 10^{13}\, m_e$,
radius $r \;\sim\; 10^{-2}\, \mu m$ up to $10^2\, \mu$m.

Origin: Where does the dust come from?

- *Space:* cosmic debris (silicates, graphite, amorphous carbon), comet dust, man-made pollution (Shuttle exhaust, satellite remnants), ...
  → *talks by M Rosenberg & P K Shukla*

- *Atmosphere:* extraterrestrial dust (meteorites): $\geq 2 \cdot 10^4$ tons a year (!)(*), atmospheric pollution, chemical aerosols, ...

- *Fusion reactors:* plasma-surface interaction, carbonaceous particulates resulting from wall erosion-created debris (graphite, CFCs: Carbon Fiber Composites, ...)
  → *talk by S I Krasheninnikov*

- *Laboratory:* (man-injected) melamine–formaldehyde particulates (**) injected in *rf* or *dc* discharges; 3d ($=$ multiple 2d layers) or 1d (by appropriate experimental setting) crystals.
  → *talks by G E Morfill & V E Fortov*

Sources: [P. K. Shukla & A. Mamun 2002], (*) [DeAngelis 1992], (**) [G. E. Morfill *et al.* 1998]

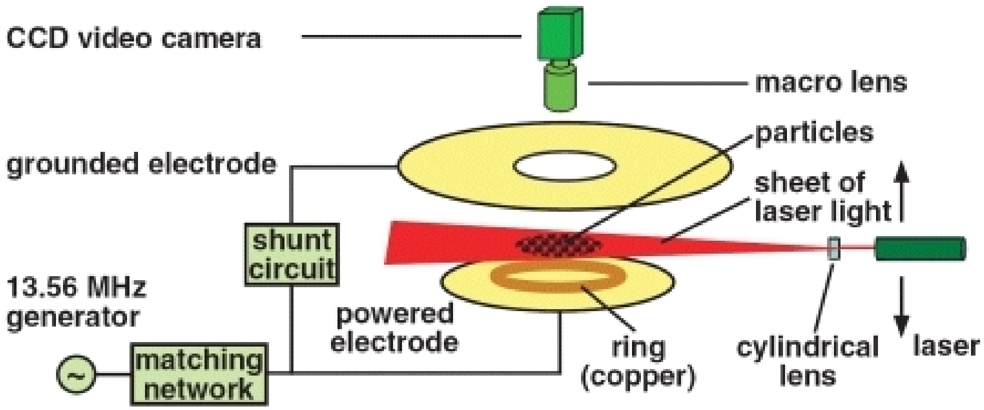
Some unique *"heretic"* features of *the Physics of* Dusty Plasmas:

- ❑ Complex plasmas are *overall charge neutral*; most e^- reside on d^-;
- ❑ The microparticles can be *dynamically dominant*: new (dust) modes;
- ❑ Studies in *slow motion* are possible; *low Q/M ratio*;
- ❑ The dust mesoparticles can be *visualised* individually and studied at the kinetic level (using digital image processing): kinetic theory *"à la carte"!*
- ❑ Dust charge ($Q \neq \mathrm{const.}$) is now a dynamical variable;
- ❑ Complex plasmas can be *strongly coupled*, may exist in *"liquid"* and *"crystalline"* ($\Gamma > 170$ [IKEZI 1986]) states; high coupling parameter Γ:

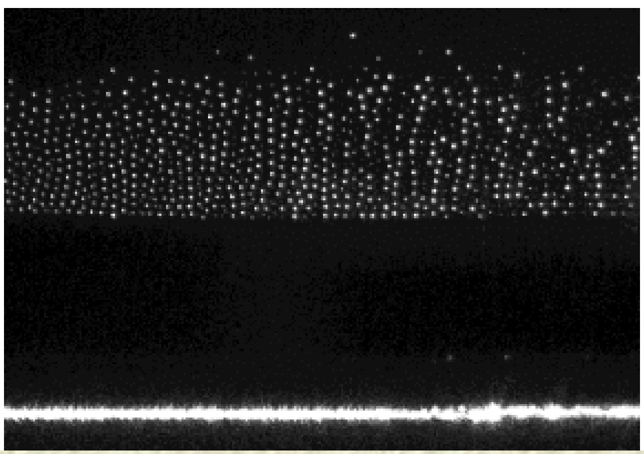
$$\Gamma_{eff} = \frac{< E_{potential} >}{< E_{kinetic} >} \sim \frac{Q^2}{r\,T} e^{-r/\lambda_D}$$

(r: inter-particle distance, T: temperature, λ_D: Debye length).

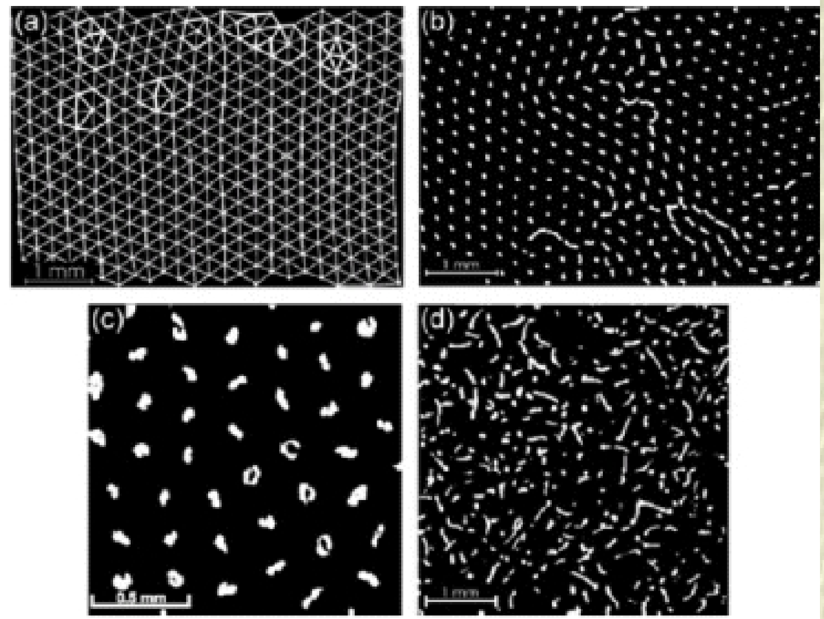
Dust laboratory experiments on Earth & in Space (ISS):



side view



top view



G E Morfill et al, Plasma Phys. Control. Fusion **44** *(2002) B263*

Focusing on 1d DP crystals:

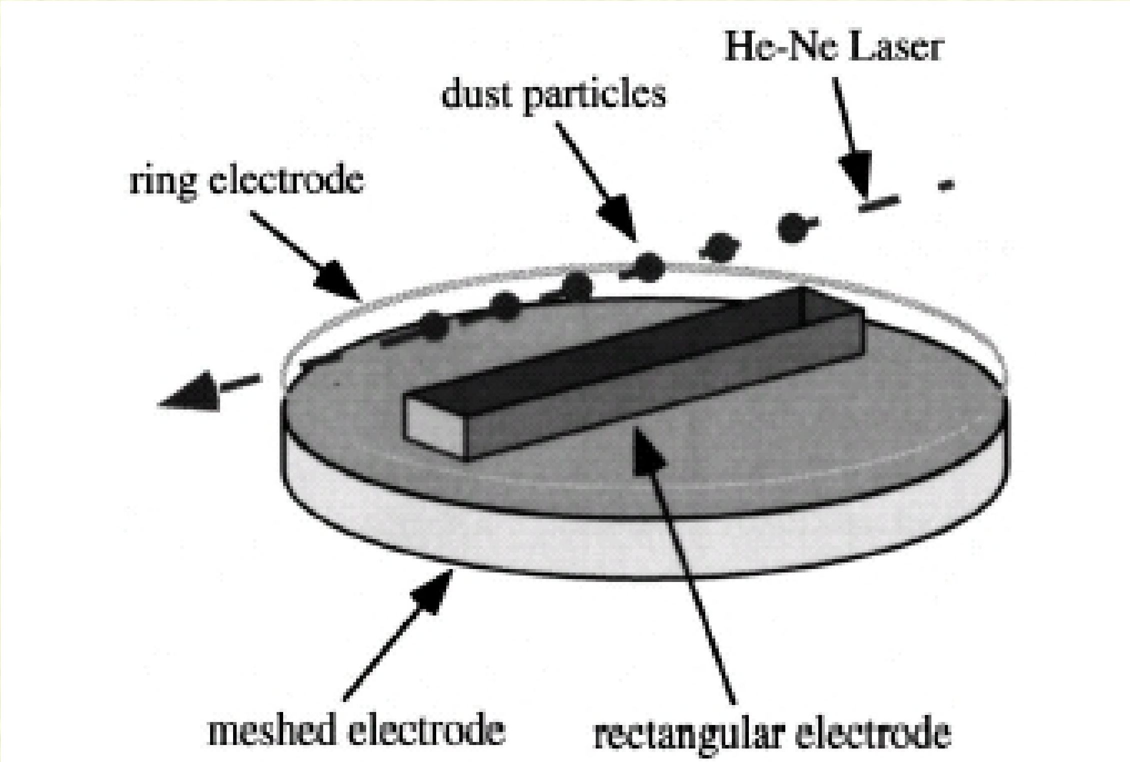


FIG. 10. Experimental configuration for forming a linear dust chain above a long rectangular box on a negatively biased mesh electrode.

[Figure from: S. Takamura *et al.*, Phys. Plasmas **8**, 1886 (2001).]

Focusing on 1d DP crystals: known linear modes.

❏ *Longitudinal Dust Lattice (LDL)* mode:

- *Horizontal* oscillations ($\sim \hat{x}$): cf. *phonons* in atomic chains;
- *Acoustic* mode: $\omega(k=0)=0$;
- *Restoring force* provided by electrostatic interactions.

❏ *Transverse Dust Lattice (TDL)* mode:

- *Vertical* oscillations ($\sim \hat{z}$);
- *Optical* mode:

$$\omega(k=0)=\omega_g \neq 0$$

  (center of mass motion);
- *Single* grain vibrations (propagating $\sim \hat{x}$ for $k \neq 0$): *Restoring force* provided by the *sheath electric potential* (*and* interactions).

❏ Transverse ($\sim \hat{y}$, in-plane, optical) d.o.f. *suppressed*.

Model *Hamiltonian*:

$$H = \sum_n \frac{1}{2} M \left(\frac{d\mathbf{r}_n}{dt} \right)^2 + \sum_{m \neq n} U_{int}(r_{nm}) + \Phi_{ext}(\mathbf{r}_n)$$

where:

– *Kinetic Energy* (1st term);

– $U_{int}(r_{nm})$ is the (binary) *interaction potential energy*;

– $\Phi_{ext}(\mathbf{r}_n)$ accounts for *'external' force fields*:
may account for *confinement potentials* and/or *sheath electric* forces, i.e.

$$F_{sheath}(z) = -\partial\Phi/\partial z \, .$$

Q.: Nonlinearity: *Origin: where from ? Consequence(s) ?*

Nonlinearity: Where does it come from?

- (i) *Interactions* between grains: Intrinsically *anharmonic*!
  - Electrostatic (e.g. Debye), long-range, screened ($r_0/\lambda_D \approx 1$); typically:

$$U_{Debye}(r) = \frac{q^2}{r} \exp\left(-r/\lambda_D\right).$$

  - Expanding $U_{pot}(r_{nm})$ near equilibrium:

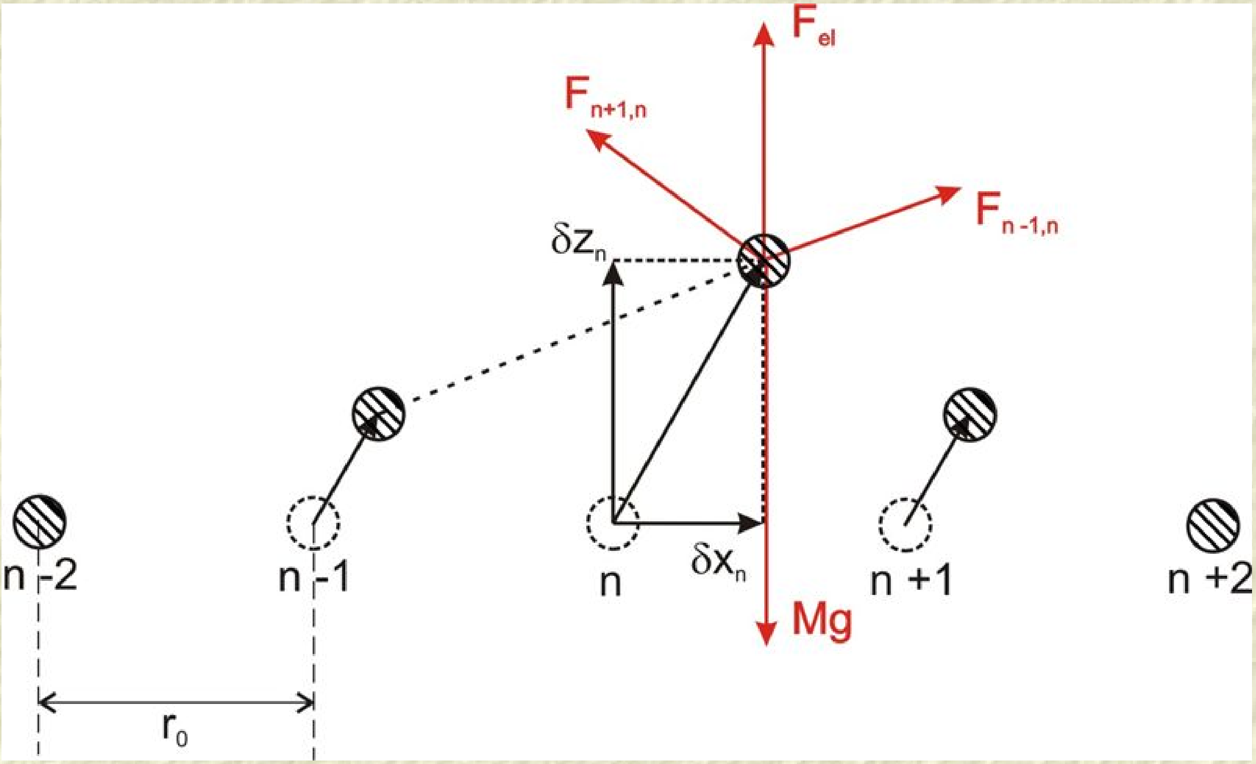
$$\Delta x_n = x_n - x_{n-m} = m r_0, \qquad \Delta z_n = z_n - z_{n-m} = 0$$

one obtains:

$$\begin{aligned} U_{nm}(r) \quad \approx \quad & \frac{1}{2} M \omega_{L,0}^2 (\Delta x_n)^2 + \frac{1}{2} M \omega_{T,0}^2 (\Delta z_n)^2 \\ & + \frac{1}{3} u_{30} (\Delta x_n)^3 + \frac{1}{4} u_{40} (\Delta x_n)^4 + ... + \frac{1}{4} u_{04} (\Delta z_n)^4 + ... \\ & + \frac{1}{2} u_{12} (\Delta x_n)(\Delta z_n)^2 + \frac{1}{4} u_{22} (\Delta x_n)^2 (\Delta z_n)^2 + ... \end{aligned}$$

Nonlinearity: Where from? *(continued ...)*

❑ (ii) Mode *coupling* also induces non linearity:
anisotropic motion, *not* confined along one of the main axes ($\sim \hat{x}$, $\hat{z}$).



[cf. A. Ivlev *et al.*, PRE **68**, 066402 (2003); I. Kourakis & P. K. Shukla, Phys. Scr. (2004)]

Nonlinearity: Where from? *(continued ...)*

- (iii) *Sheath* environment: *anharmonic* vertical potential:

$$\Phi(z) \approx \Phi(z_0) + \frac{1}{2} M \omega_g^2 (\delta z_n)^2 + \frac{1}{3} M \alpha \, (\delta z_n)^3 + \frac{1}{4} M \beta \, (\delta z_n)^4 + \ldots$$

cf. experiments [Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)];

$\delta z_n = z_n - z_{(0)}$; α, β, ω_g are defined via $E(z)$, $[B(z)]^\dagger$ and $Q(z)$;

(in fact, functions of n and P) [† V. Yaroshenko *et al.*, NJP 2003; PRE 2004]

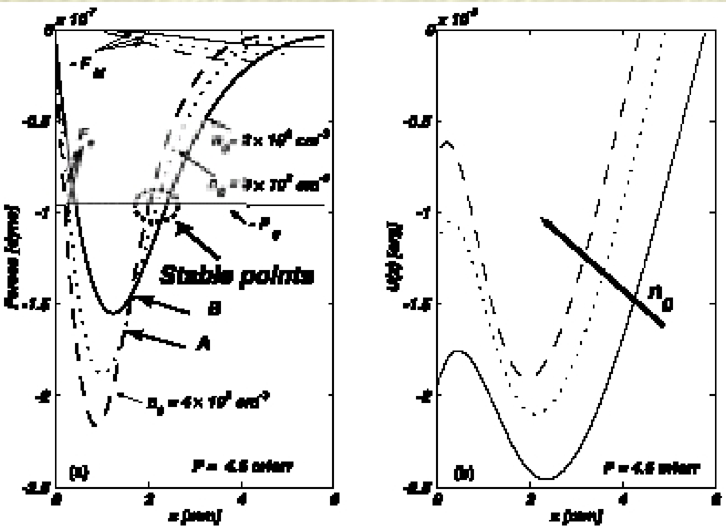


Figure 3: (a) Forces and (b) trapping potential profiles $U(z)$ as function of distance from the electrode for: $n_0 = 2\times10^8 cm^{-3}$ (solid line), $n_0 = 3\times10^8 cm^{-3}$ (dashed line), $n_0 = 4\times10^8 cm^{-3}$ (dotted line). The parameters are: P = 4.6 mtorr, $T_e = 1$ eV, $T_i = T_n = 0.05$ eV, $R = 2.5$ μm, $\rho_d = 1.5$ $g\,cm^{-3}$, $\phi_w = 6$ V.

Source: Sorasio *et al.* (2002).

2. *Transverse* oscillations

The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys (*):

$$\frac{d^2(\delta z_n)}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 \, (\, \delta z_{n+1} + \, \delta z_{n-1} - 2 \, \delta z_n) + \omega_g^2 \, \delta z_n = 0 \qquad (1)$$

- TDL eigenfrequency:

$$\omega_{T,0} = \left[-qU'(r_0)/(Mr_0) \right]^{1/2} = \omega_{DL}^2 \, \exp(-\kappa) \, (1+\kappa)/\kappa^3$$

(*for Debye interactions*); $\kappa = r_0/\lambda_D$ is the *lattice parameter*;

- $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$ is the characteristic DL frequency scale;
- λ_D is the *Debye length*.

(*) [Vladimirov, Shevchenko and Cramer, PRE 1997]

Transverse **oscillations** ***(linear)***

The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys

$$\frac{d^2(\delta z_n)}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 \, (\delta z_{n+1} + \delta z_{n-1} - 2\,\delta z_n) + \omega_g^2 \, \delta z_n = 0 \qquad (2)$$

- Neglect dissipation, i.e. set $\nu = 0$ in the following;

- *Continuum* analogue: $\delta z_n(t) \to u(x, t)$, where

$$\frac{\partial^2 u}{\partial t^2} + c_T^2 \frac{\partial^2 u}{\partial x^2} + \omega_g^2 \, u = 0$$

  where $c_T = \omega_{T,0} \, r_0$ is the *transverse "sound" velocity*.

Transverse* oscillations *(linear, "undamped")

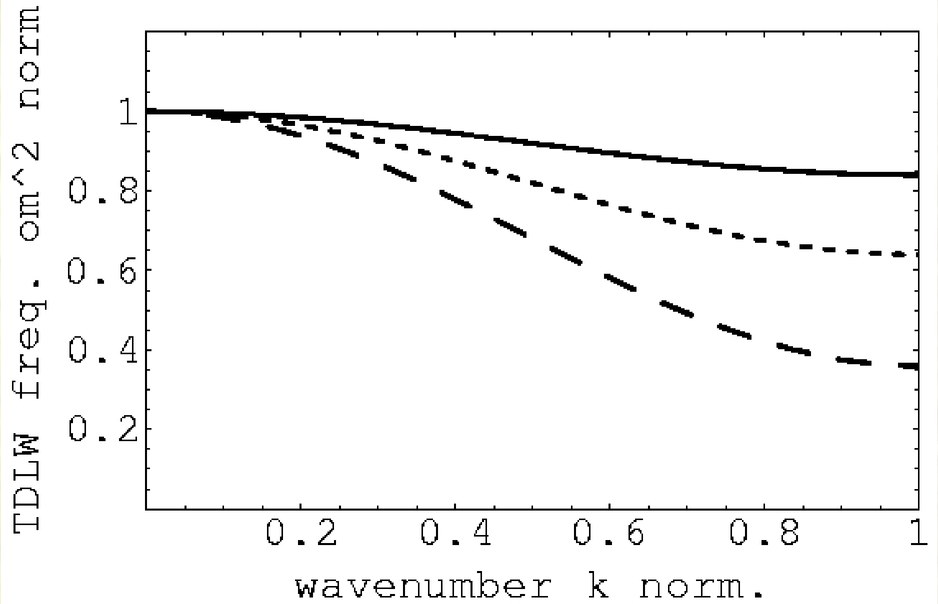
The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys

$$\frac{d^2(\delta z_n)}{dt^2} + \omega_{T,0}^2 \, (\, \delta z_{n+1} + \, \delta z_{n-1} - 2\, \delta z_n) + \omega_g^2 \, \delta z_n = 0 \qquad (3)$$

Optical dispersion relation

(*backward* wave, $v_g < 0$) †:

$$\omega^2 \; = \omega_g^2 - 4\omega_{T,0}^2 \, \sin^2\big(kr_0/2\big)$$



† Cf. experiments: T. Misawa *et al.*, *PRL* **86**, 1219 (2001); B. Liu *et al.*, *PRL* **91**, 255003 (2003).

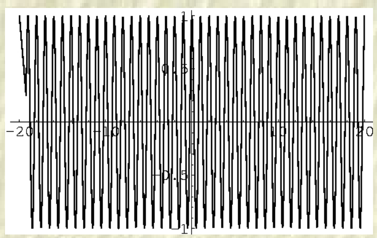
What if *nonlinearity* is taken into account?

$$\frac{d^2 \delta z_n}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 \left(\delta z_{n+1} + \delta z_{n-1} - 2\, \delta z_n \right) + \omega_g^2\, \delta z_n + \alpha\, (\delta z_n)^2 + \beta\, (\delta z_n)^3 = 0 \,. \qquad (4)$$

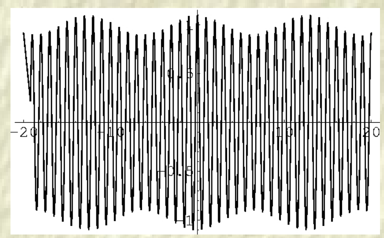
What if *nonlinearity* is taken into account?

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n$$
$$+\alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \qquad (5)$$

* *Intermezzo:* The mechanism of *wave amplitude modulation*:
The *amplitude* of a harmonic wave may vary in space and time:



$\longrightarrow$



What if *nonlinearity* is taken into account?

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n$$
$$+\alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \qquad (6)$$

* *Intermezzo:* The mechanism of *wave amplitude modulation*:
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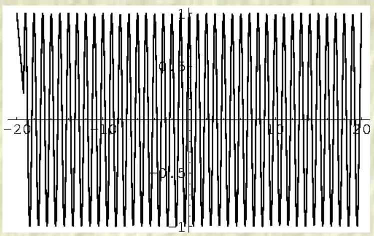
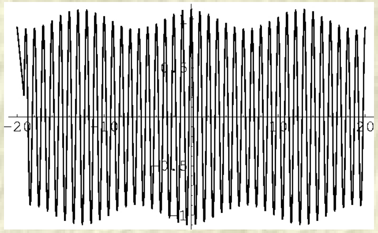
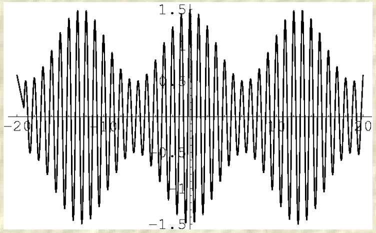


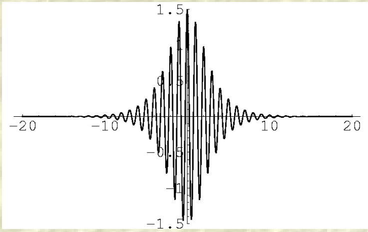
$\longrightarrow$



This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* or formation of *envelope solitons*:



$\longrightarrow$?



Large amplitude oscillations - envelope structures

A reductive perturbation (multiple scale) technique, viz.

$$t \to \{t_0, t_1 = \epsilon t, t_2 = \epsilon^2 t, ...\},\ x \to \{x_0, x_1 = \epsilon x, x_2 = \epsilon^2 x, ...\}$$

yields ($\epsilon \ll 1$; damping omitted):

$$\delta z_n \approx \epsilon\,(A\,e^{i\phi_n} + \text{c.c.}) \,+\, \epsilon^2\,\alpha\left[-\frac{2|A|^2}{\omega_g^2} + \left(\frac{A^2}{3\omega_g^2}\,e^{2i\phi_n} + \text{c.c.}\right)\right] + ...$$

Here,

- $\phi_n = nkr_0 - \omega t$ is the (fast) TDLW carrier phase;
- the amplitude $A(X, T)$ depends on the (*slow*) variables

$$\{X,\,T\} = \{\epsilon(x - v_g t),\,\epsilon^2 t\}\,.$$

Transverse oscillations - the envelope evolution equation

The amplitude $A(X, T)$ obeys the *nonlinear Schrödinger equation* (NLSE):

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \qquad (7)$$

where

- The *dispersion coefficient* ($\rightarrow$ see dispersion relation)

$$P = \frac{1}{2}\,\frac{d^2\omega_T(k)}{dk^2} = ...$$

  is negative/positive for low/high values of k.

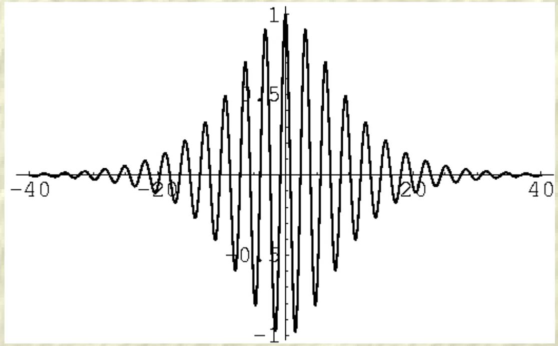
- The *nonlinearity coefficient* is $Q = \left[10\alpha^2/(3\omega_g^2) - 3\,\beta\right]/2\omega$.

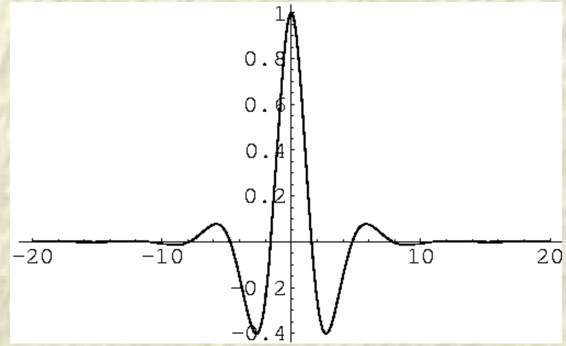
- Cf.: known properties of the NLS Eq.: Cf. previous talks.

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 2322 (2004); also *PoP*, **11**, 3665 (2004).]

Modulational stability analysis & envelope structures

❑ $PQ > 0$: *Modulational instability* of the carrier, *bright solitons*:



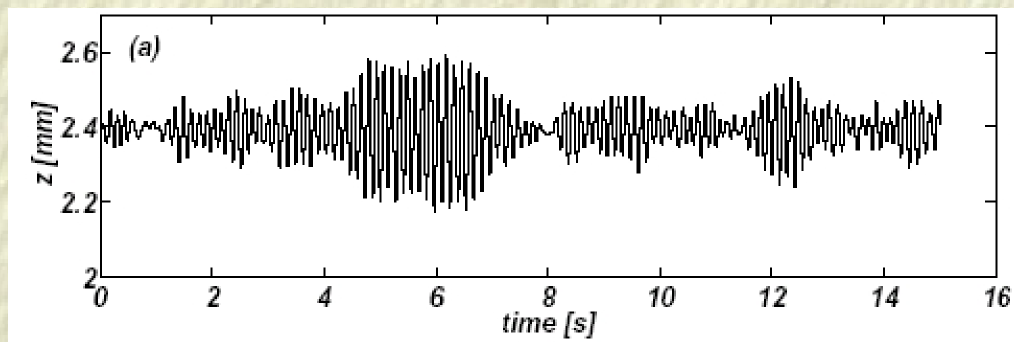


→ *TDLWs*: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

Rem.: $Q > 0$ for *all* known experimental values of α, β.

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)]

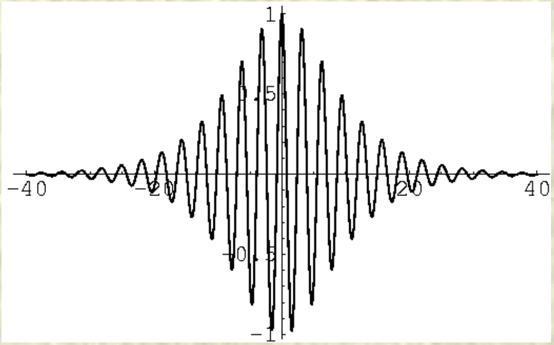
Figure 9: Dust grain oscillations induced by a 1% fluctuation in plasma density. The simulation parameters are: $P = 0.9\ mtorr$, $n_0 = 0.8 \times 10^8\ cm^{-3}$, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$, $\varsigma_t = 0.06$, $\varsigma_p = 1\% n_0$

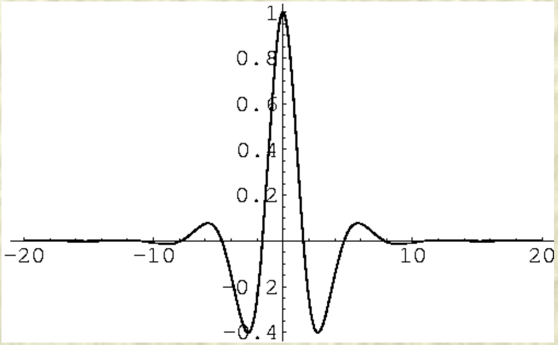


Source: G. Sorasio *et al.* (2002).

Modulational stability analysis & envelope structures

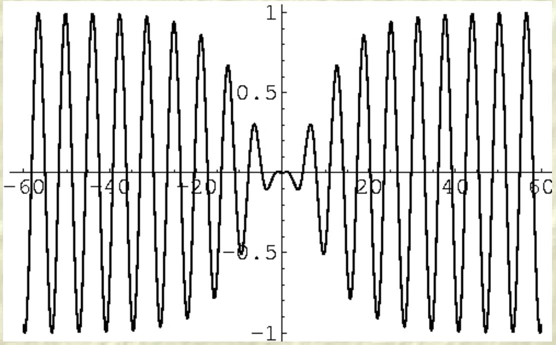
- $PQ > 0$: *Modulational instability* of the carrier, *bright solitons*:

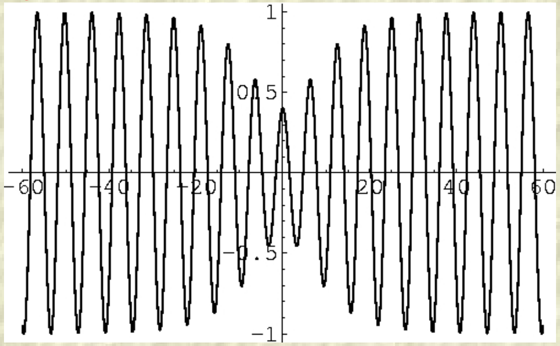




→ *TDLW*s: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

- $PQ < 0$: Carrier wave is *stable*, *dark/grey solitons*:





→ *TDLW*s: possible for *long* wavelengths i.e. $k < k_{cr}$.

Rem.: $Q > 0$ for *all* known experimental values of α, β

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)] *(end of TDL-Part 2.)*.

3. *Longitudinal* excitations

The (*linearized*) equation of *longitudinal* ($\sim \hat{x}$) motion reads (*):

$$\frac{d^2(\delta x_n)}{dt^2} + \nu \frac{d(\delta x_n)}{dt} = \omega_{0,L}^2 \left(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n\right)$$

– $\delta x_n = x_n - n r_0$: longitudinal dust grain displacements

– *Acoustic* dispersion relation:

$$\omega^2 = 4\omega_{L,0}^2 \sin^2\!\left(k r_0/2\right) \equiv \omega_L^2(k)$$

– LDL eigenfrequency: $\omega_{0,L}^2 = U''(r_0)/M = 2\,\omega_{DL}^2 \exp(-\kappa)\,(1+\kappa+\kappa^2/2)/\kappa^3$ $(*)$

$^{(*)}$ *for Debye interactions*; Rem.: $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$.

– Neglect damping in the following, viz. $\nu \to 0$.

(*) [Melandsø PoP 1996, Farokhi *et al*, PLA 1999]

Longitudinal **excitations (linear, "undamped")**

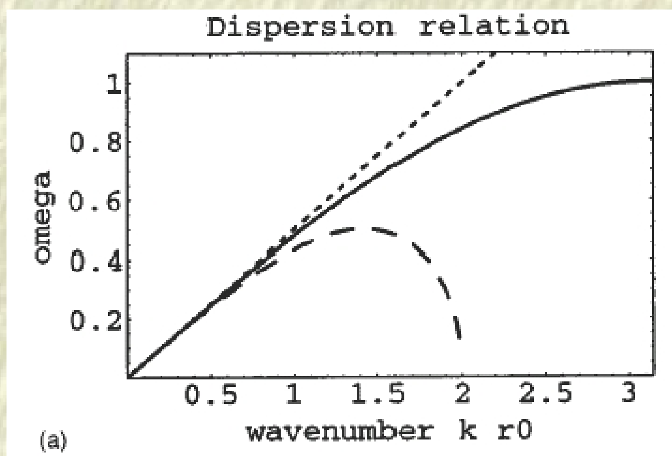
The (*linearized*) equation of longitudinal motion reads:

$$\frac{d^2(\delta x_n)}{dt^2} = \omega_{0,L}^2 \, (\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n)$$

or, in the continuum approximation:

$$\frac{\partial^2(\delta x_n)}{\partial t^2} - c_L^2 \, \frac{\partial^2(\delta x_n)}{\partial t^2} = 0$$

$(c_L = \omega_{0,L} r_0)$



***Longitudinal* excitations (*nonlinear*).**

The *nonlinear* equation of longitudinal motion reads:

$$\frac{d^2(\delta x_n)}{dt^2} = \omega_{0,L}^2 (\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n) \\ -a_{20} \left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2 \right] \\ + a_{30} \left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3 \right] \quad (8)$$

– $\delta x_n = x_n - n r_0$: longitudinal dust grain displacements

– Cf. *Fermi-Pasta-Ulam (FPU) problem*: anharmonic spring chain model:

$$U_{int}(r) \approx \frac{1}{2} M \omega_{0,L}^2 r^2 - \frac{1}{3} M a_{20} r^3 + \frac{1}{4} M a_{30} r^4 .$$

Longitudinal **envelope structures.**

The reductive perturbation technique (cf. above) now yields:

$$\delta x_n \approx \epsilon \left[u_0^{(1)} + \left(u_1^{(1)} e^{i\phi_n} + \text{c.c.} \right) \right] + \epsilon^2 \left(u_2^{(2)} e^{2i\phi_n} + \text{c.c.} \right) + \ldots ,$$

[Harmonic generation; Cf. experiments: K. Avinash PoP 2004].

Longitudinal **envelope structures.**

The reductive perturbation technique (cf. above) now yields:

$$\delta x_n \approx \epsilon \left[u_0^{(1)} + (u_1^{(1)}\, e^{i\phi_n} + \text{c.c.})\right] + \epsilon^2\, (u_2^{(2)}\, e^{2i\phi_n} + \text{c.c.}) + \ldots ,$$

where the amplitudes obey the coupled equations:

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L\, \frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\, |u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\, u_1^{(1)} \frac{\partial u_0^{(1)}}{\partial X} = 0 \,,$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - c_L^2}\, \frac{\partial}{\partial X}\, |u_1^{(1)}|^2 \equiv R(k) \frac{\partial}{\partial X}\, |u_1^{(1)}|^2$$

– $Q_0 = -\frac{k^2}{2\omega}\left(q_0\, k^2 + \frac{2p_0^2}{c_L^2 r_0^2}\right)$; $\quad v_{g,L} = {\omega_L}'(k)$; $\quad \{X, T\}$: *slow* variables;

– $p_0 = -U'''(r_0) r_0^3/M \equiv 2a_{20} r_0^3$, $\quad q_0 = U''''(r_0) r_0^4/(2M) \equiv 3a_{30} r_0^4$.

– $R(k) > 0$, since $\forall\, k \quad v_{g,L} < \omega_{L,0}\, r_0 \equiv c_L$ *(subsonic LDLW envelopes)*.

Asymmetric *longitudinal* envelope structures.

– The system of Eqs. for $u_1^{(1)}$, $u_0^{(1)}$ may be combined into a closed (*NLSE*) equation (for $A = u_1^{(1)}$, here);

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0$$

– $P = P_L = \omega_L''(k)/2 < 0$;

– $Q > 0$ (< 0) prescribes *stability* (instability) at *low* (high) k.

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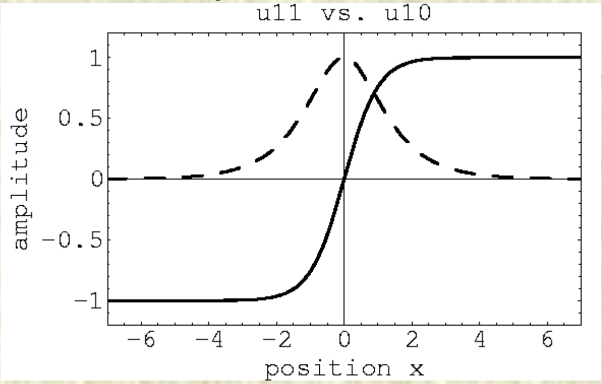
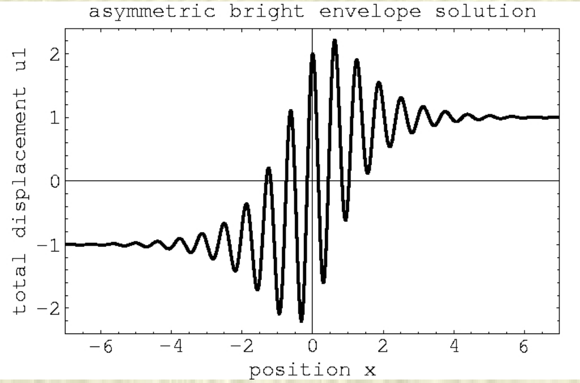
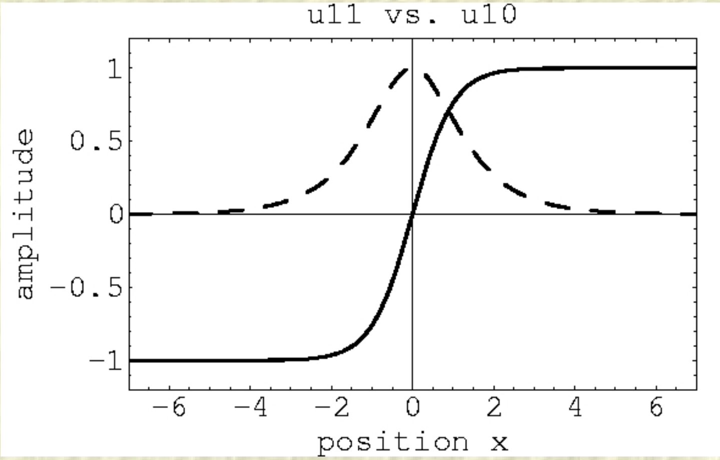
– Envelope excitations are now *asymmetric*:

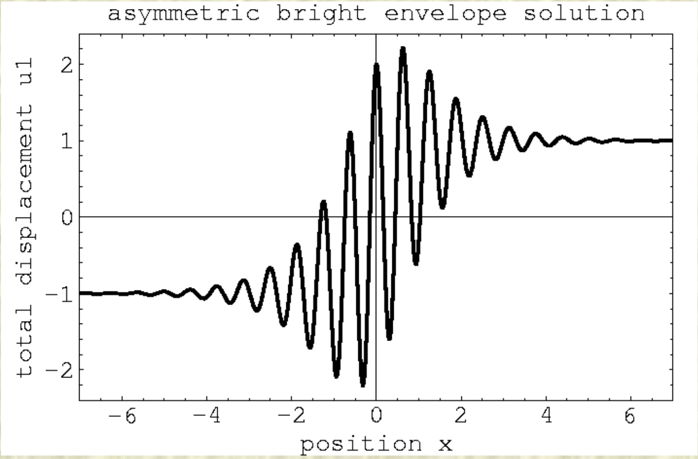




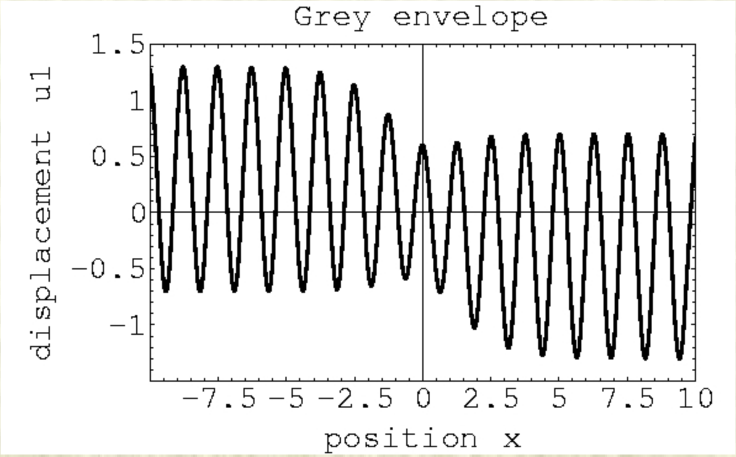
(at high k, i.e. $PQ > 0$).

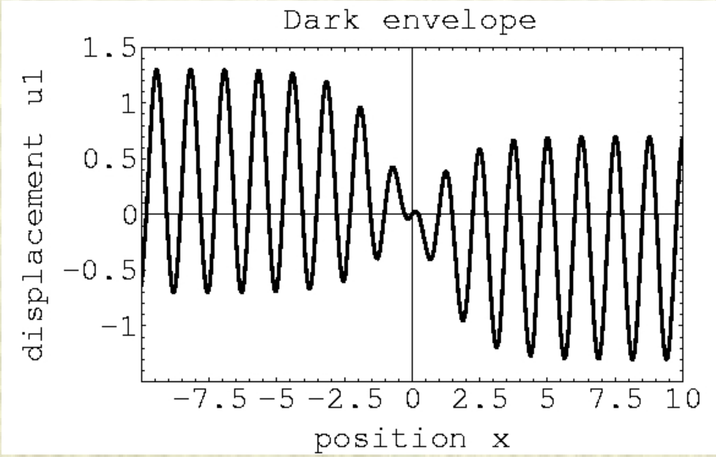
Asymmetric *longitudinal* envelope structures.





(at high k)





(at low k)

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 1384 (2004).] *(end of L-Part 3)*.

4. *Longitudinal* soliton formalism.

Q.: *A link to soliton theories: the Korteweg-deVries Equation.*

– *Continuum approximation*, viz. $\delta x_n(t) \to u(x,t)$.

– *"Standard"* description: keeping *lowest order* nonlinearity,

$$\ddot{u} + \nu\, \dot{u} - c_L^2\, u_{xx} - \frac{c_L^2}{12}\, r_0^2\, u_{xxxx} \; = \; -\, p_0\, u_x\, u_{xx}$$

$c_L = \omega_{L,0}\, r_0$; $\quad \omega_{L,0}$ and p_0 were defined above.

– Let us neglect damping ($\nu \to 0$), once more.

– For *near-sonic propagation* (i.e. $v \approx c_L$), slow profile evolution in time τ and defining the *relative displacement* $w = u_\zeta$, one obtains the KdV equation:

$$w_\tau \; - \; a\, w\, w_\zeta + b\, w_{\zeta\zeta\zeta} \; = \; 0$$

(for $\nu = 0$); $\;\zeta = x - vt$; $\;a = p_0/(2c_L) > 0$; $\;b = c_L r_0^2/24 > 0$.

– This KdV Equation yields soliton solutions, ... (→ *next page*)

The KdV description

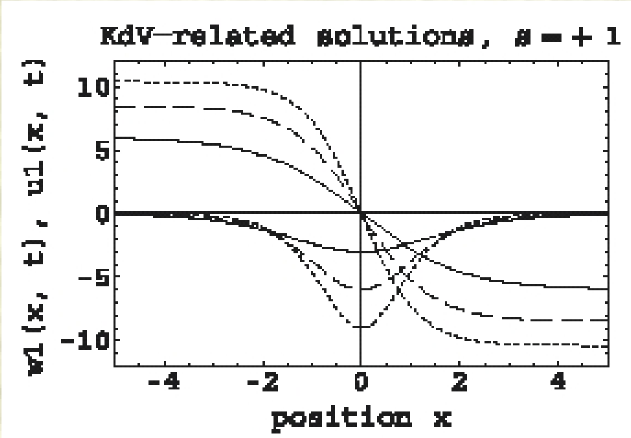
The Korteweg-deVries (KdV) Equation

$$w_\tau - a\, w\, w_\zeta + b\, w_{\zeta\zeta\zeta} = 0$$

yields *compressive* (*only*, here) solutions, in the form (here):

$$w_1(\zeta, \tau) = -w_{1,m} sech^2 \left[(\zeta - v\tau - \zeta_0)/L_0 \right]$$

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.



The KdV description

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$$w_\tau - a\, w\, w_\zeta + b\, w_{\zeta\zeta\zeta} = 0$$

yields *compressive* (*only*, here) solutions, in the form (here):

$$w_1(\zeta, \tau) = -w_{1,m} sech^2 \Big[(\zeta - v\tau - \zeta_0)/L_0 \Big]$$

– Pulse amplitude: $w_{1,m} = 3v/a = 6vv_0/|p_0|$;

– Pulse width: $L_0 = (4b/v)^{1/2} = [2v_1^2 r_0^2/(vv_0)]^{1/2}$;

– Note that: $w_{1,m}L_0^2 = \mathrm{constant}$ (cf. experiments)†.

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.
– This is the standard treatment of dust-lattice solitons today ... †

† F. Melandsø 1996; S. Zhdanov *et al.* 2002; K. Avinash *et al.* 2003; V. Fortov *et al.* 2004.

Characteristics of the KdV theory

The *Korteweg - deVries* theory, as applied in DP crystals:

– provides a *correct qualitative description* of *compressive* excitations observed in experiments;

– benefits from the KdV *"artillery"* of analytical know-how obtained throughout the years: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

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The *Korteweg - deVries* theory, as applied in DP crystals:

– provides a *correct qualitative description* of *compressive* excitations observed in experiments;

– benefits from the KdV *"artillery"* of analytical know-how obtained throughout the years: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

but possesses a few drawbacks:

– *approximate* derivation: (i) propagation velocity v near (longitudinal) sound velocity c_L, (ii) time evolution terms omitted *'by hand'*, (iii) higher order nonlinear contributions omitted;

– *only* accounts for *compressive* solitary excitations (for Debye interactions); nevertheless, the existence of *rarefactive* dust lattice excitations is, *in principle*, *not excluded*.

Longitudinal soliton formalism *(continued)*

Q.: *What if we also kept the next order in nonlinearity ?*

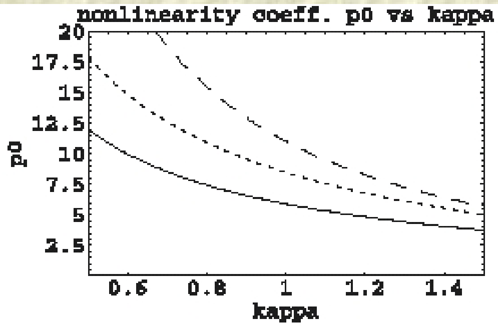
Longitudinal soliton formalism *(continued)*

Q.: *What if we also kept the next order in nonlinearity ?*

– *“Extended”* description: :

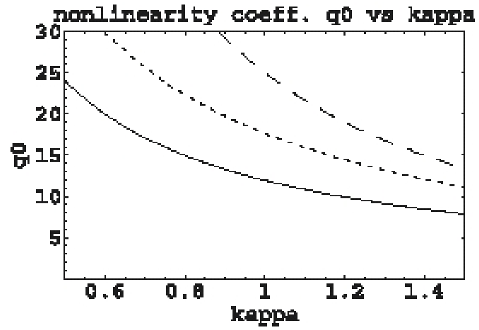
$$\ddot{u} - c_L^2\, u_{xx} - \frac{c_L^2}{12}\, r_0^2\, u_{xxxx} = -\, p_0\, u_x\, u_{xx} + q_0\, (u_x)^2\, u_{xx}$$

$c_L = \omega_{L,0}\, r_0$; $\omega_{L,0}$, $p_0 \sim -U'''(r)$ and $q_0 \sim U''''(r)$ (cf. above).



(b)

Fig. 4. (a) The nonlinearity coefficient p_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.



(b)

Fig. 5. (a) The nonlinearity coefficient q_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.

Rq.: q_0 is *not* negligible, compared to p_0! (instead, $q_0 \approx 2p_0$ practically, for $r_0 \approx \lambda_D$!)

Longitudinal soliton formalism *(continued)*

Q.: *What if we also kept the next order in nonlinearity ?*

– *"Extended"* description: :

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}$$

$c_L = \omega_{L,0}\,r_0$; $\omega_{L,0}$, p_0 and q_0 were defined above.

– For *near-sonic propagation* (i.e. $v \approx c_L$), and defining the *relative displacement* $w = u_\zeta$, one obtains the E-KdV equation:

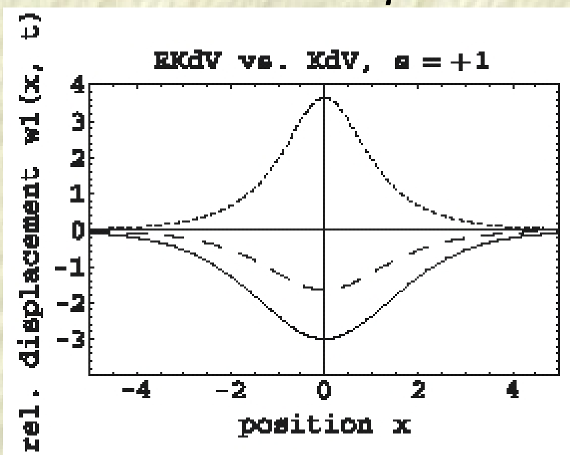
$$w_\tau - a\,w\,w_\zeta + \hat{a}\,w^2\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0 \qquad (9)$$

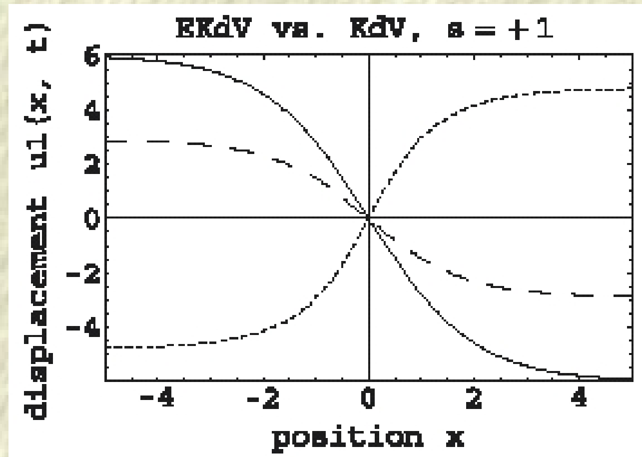
(for $\nu = 0$); $\zeta = x - vt$; $a = p_0/(2c_L) > 0$; $b = c_L r_0^2/24 > 0$;
$\hat{a} = q_0/(2c_L) > 0$.

Characteristics of the EKdV theory

The *extended Korteweg - deVries* Equation:

– accounts for *both compressive and rarefactive* excitations;





(*horizontal grain displacement* $u(x, t)$)

– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– is previously widely studied, in literature;

Still, ...

– It was derived under the *assumption*: $v \approx c_L$.

One more alternative: the Boussinesq theory

The *Generalized* *Boussinesq* (Bq) Equation (for $w = u_x$):

$$\ddot{w} - c_L^2 \, w_{xx} = \frac{c_L^2 r_0^2}{12} \, w_{xxxx} - \frac{p_0}{2} \, (w^2)_{xx} + \frac{q_0}{2} \, (w^3)_{xx}$$

– predicts *both compressive and rarefactive* excitations;

– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– has been widely studied in literature;

and, ...

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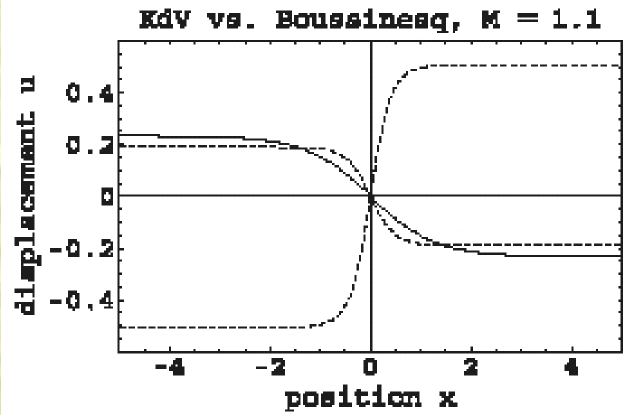
– predicts *both compressive and rarefactive* excitations;

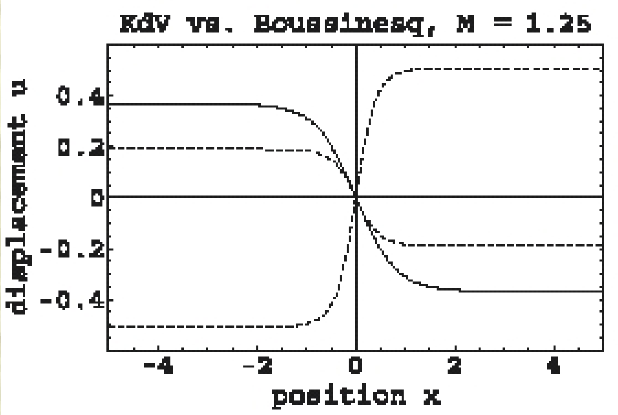
– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– has been widely studied in literature;

and, ...

– relaxes the velocity assumption, i.e. is valid $\forall \quad v > c_L$.





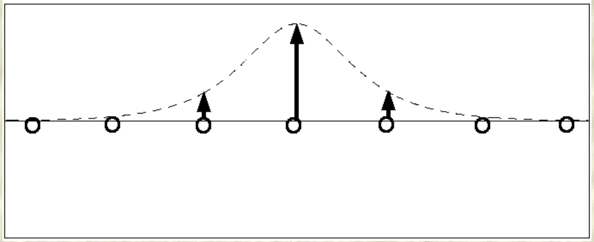
(end of L-Part 4.)

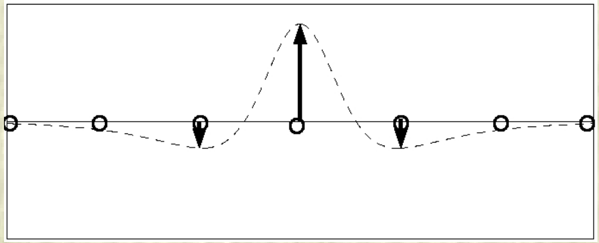
5. Transverse Discrete Breathers (DB)

- DBs are *highly discrete* oscillations (*Intrinsic Localized Modes, ILMs*);
- Looking for DB solutions in the *transverse* direction, viz.

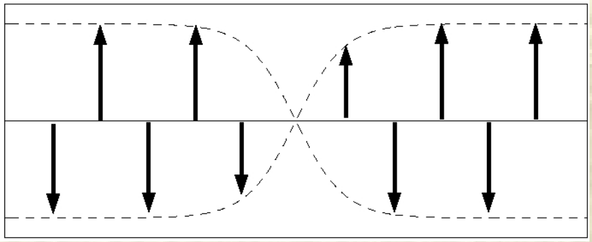
$$\frac{d^2 u_n}{dt^2} + \omega_{T,0}^2 \, (u_{n+1} + u_{n-1} - 2\, u_n) + \omega_g^2 \, \delta z_n + \alpha \, u^2 + \beta \, u^3 = 0$$

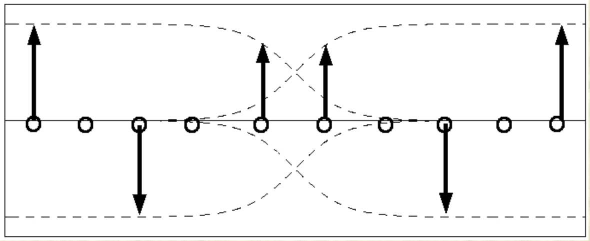
one obtains the *bright-type* DB solutions (localized pulses):





as well as the *dark-type* excitations (holes; *Kivshar dark* modes):





- Similar modes may be sought in the longitudinal direction.

Transverse Discrete Breathers (DB)

- Existence and stability criteria still need to be examined.

- It seems established that DBs exist if the *non-resonance criterion*:

$$n\,\omega_B \neq \omega_k \qquad \forall n \in \mathcal{N}$$

  is fulfilled, where:

  – ω_B is the *breather frequency*;

  – ω_k is the *linear ("phonon") frequency* (cf. dispersion relation).

- If ω_B (or its harmonics) enter(s) into resonance with the linear spectrum ω_k, discrete oscillations will decay into a "sea" of linear lattice waves.

- The DB existence condition is satisfied in *all* known lattice wave experiments.

6. Conclusions

We have seen that:

– *Energy localization* via *modulational instability*, leading to the formation of *envelope excitations*, is possible in both *transverse* and *longitudinal* directions;

– Solitary waves can be efficiently modeled by existing soliton theories (e.g. KdV, EKdV, MKdV; more accurately: Bq, EBq) ;

– Compressive *and* rarefactive excitations are predicted ;

– *Discrete Breather*-type localized modes may exist (need to study further);

– *Urge* (!) *for experimental confirmation* (technical constraints?) ;

– Future directions: include *dissipation* (dust-neutral friction, ion drag); *particle-wake effects*; *mode coupling* effects; ... *(Realism!)*

– Fertile soil for future studies: still *a lot to be done!...*

Thank you !!!

Ioannis Kourakis

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Acknowledgments: *V Basios* (U.L.B., Brussels), *T Bountis* (Patras, Greece), *V Koukouloyannis* (Thessaloniki, Greece).

Material from:
I. Kourakis & P. K. Shukla, *PoP*, **11**, 2322 (2004);
idem, *PoP*, **11**, 3665 (2004).
idem, *PoP*, **11**, 1384 (2004);
idem, *Eur. Phys. J. D*, **29**, 247 (2004).

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Appendix I: Solutions of the NLSE
Localized envelope excitations 1: *bright solitons*

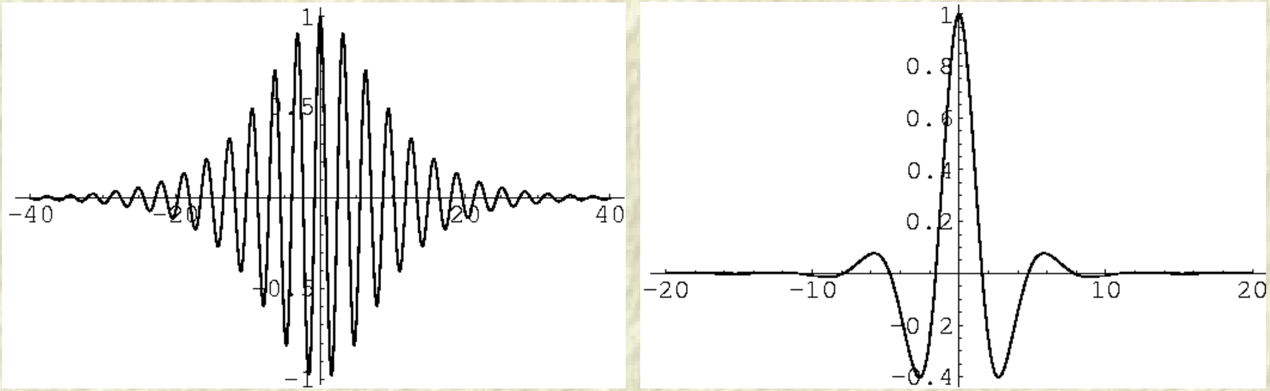
- The NLSE accepts various soliton solutions: $\psi = \rho\, e^{i\Theta}$;
  the *total* wavepacket is then: $u \approx \epsilon\, \rho\, \cos(kx - \omega t + \Theta)$ where the *amplitude* ρ and *phase correction* Θ depend on ζ, τ.
- Bright–type envelope soliton (pulse):

$$\rho = \rho_0\, sech\left(\frac{X - u_e\, T}{L}\right), \quad \Theta = \frac{1}{2P}\left[u_e\, X + (\Omega - \frac{1}{2}u_e^2)T\right]. \qquad (10)$$

$L = \sqrt{\frac{2P}{Q}\frac{1}{\rho_0}}$

$(X_0 = 0)$

$(\Theta_0 = 0)$

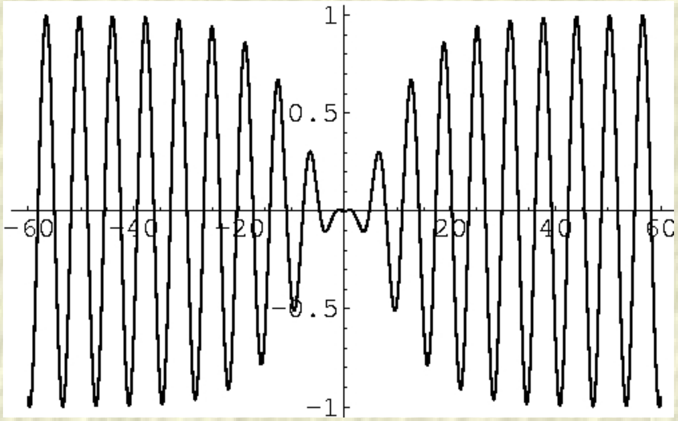


Localized envelope excitations 2: *dark/grey solitons*

❑ Dark–type envelope solution (*hole soliton*):

$$
\begin{aligned}
\rho &= \pm\rho_1 \left[1 - \operatorname{sech}^2\left(\frac{X - u_e T}{L'}\right)\right]^{1/2} = \pm\rho_1 \tanh\left(\frac{X - u_e T}{L'}\right), \\
\Theta &= \frac{1}{2P}\left[u_e X - \left(\frac{1}{2}u_e^2 - 2PQ\rho_1^2\right)\tau\right] \\
L' &= \sqrt{2\left|\frac{P}{Q}\right|\frac{1}{\rho_1}} \qquad (X_0 = 0) \qquad (11)
\end{aligned}
$$

This is a *propagating localized hole* (zero density void):



dark/grey solitons (continued...)

❑ Grey–type envelope solution (*void soliton*):

$$\rho = \pm\rho_2 \left[1 - d^2\, sech^2 \left(\frac{X - u_e T}{L''} \right) \right]^{1/2}$$

$$\Theta = \dots$$

$$L'' = \sqrt{2 \left| \frac{P}{Q} \right| \frac{1}{d\rho_2}} \qquad (12)$$

This is a propagating *(non zero-density)* void: