Symposium on New Trends in Nonlinear Physics (17/09/2005), Abdus Salam ICTP, Trieste

Dynamics of coupled Bose-Einstein condensates

Brief overview of an exotic physical issue

$+$ *recent research work*

Ioannis Kourakis

R.U.B. Ruhr-Universität Bochum, Bochum, Germany

Email: `ioannis@tp4.rub.de`

`www.tp4.rub.de/∼ioannis`

in collaboration with P K Shukla, L Stenflo & M Marklund

`www.tp4.rub.de/∼ioannis/conf/2005-ICTP2-oral.pdf`

Outline

1. Bose-Einstein Condensates (BECs): *preliminaries*.

2. BEC modelling: the *Gross-Pitaevskii Equation* (GPE).

   Focus: *Scattering length* – role and sign control via Feshbach resonance.

3. Excitations in single BECs: a brief overview.

4. Coupled BECs (cBECs): notions and ideas.

5. Stability of cBECs.

6. Conclusions.

1. Intro.: Preliminaries on Bose-Einstein Condensates (BECs).

- Louis de Broglie hypothesis on *wave - particle duality* (1924, Nobel 1929):

  The wave nature of matter is manifested via the *de Broglie wave length* λ_{dB}

$$\lambda_{dB} = h/mv$$

  (h: Planck's constant; m: mass; v: velocity, e.g. $v_{th} \sim T^{1/2}/m^{1/2}$).

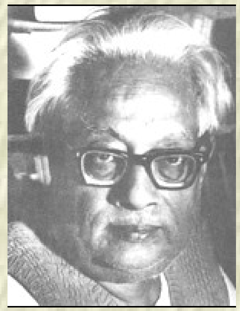
  - For *macroscopic particles* (dimension L), in room T, $\lambda_{dB} \approx 10^{-9}L$.
  - For *atoms* (radius r), in room temperature, $\lambda_{dB} \approx r$ (or smaller).
  - What should one expect for atoms ($\sim$ wavepackets) at *very low* T?

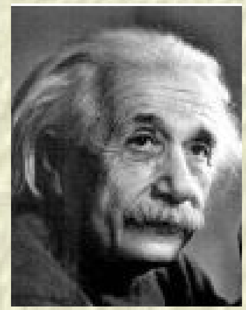
- *Boson condensation* occurs (in sophisticated experiments) at ultra low T ($< T_c \sim nK$), where $\lambda_{dB} \approx < r > \equiv n^{-3}$ or larger (n: particle density);

  $\rightarrow$ wave packets are *superposed* & individual particles *indistinguishable*!

Bose - Einstein condensation: a historical review

❑ 1924: Theoretical prediction by *Satyendra Nath Bose* and *Albert Einstein*;

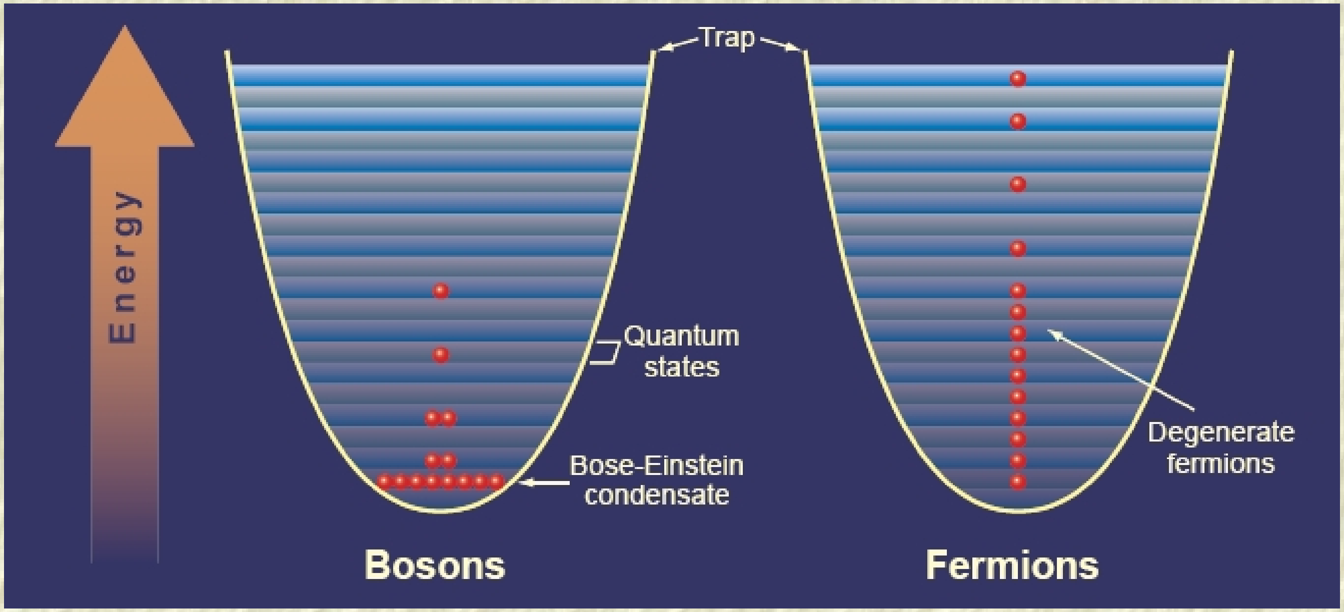




- ☛ Statistical approach to the photon spectrum of black-body radiation
- ☛ Publication (with A.E.) in *Zeitschrift für Physik*, and subsequent generalization to particle ensembles (by A.E.).
- ☛ B.E. statistics permit the occupation of the ground state by a large number of particles at ultra low (but finite) temperature.
- ☛ A.E.: *“From a certain temperature on, the molecules condense without attractive forces, that is, they accumulate at zero velocity. The theory is pretty, but is there also some truth to it?”*

Bose - Einstein condensation: a historical review *(cont.)*

- 1924: Theoretical prediction by *Satyendra Bose* and *Albert Einstein*.
- 1995: *Experimental confirmation* by E. Cornell & C. Wieman (Boulder, CO), W. Ketterle (Cambridge, MA) and R. Hulet (Rice Univ., Houston, TX): the first BECs are formed in atom gases (^{87}Rb, ^{23}Na, ^{7}Li).



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- 2001: *Nobel Prize* (E. Cornell, C. Wieman and W. Ketterle); the work by Hulet et al. was *"inconclusive"* (*Phys. World, Nov. 2001*) ...

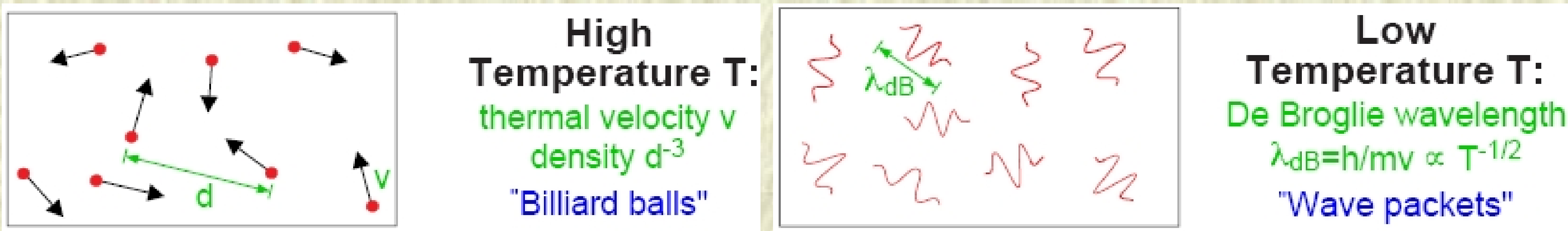
- Today:

  More than $\sim$40 experimental groups, working on various *alcali* gases (Rb, Na, Li, K, H, Cs, Cr),

  and

  more than $\sim$500 theoretical articles on BECs every year!

BEC experiments: concept and realization – prerequisites



- Experimentally BEC realization: *“freezing”* (parts of) boson gases;
- Below T_c , all of the atoms in the *boson* gas occupy the same quantum state (ground energy level);
- A collective wave is formed: atomic wave functions oscillate *in phase*;

Theoretical prerequisites to BEC experiments.

- Phase coherence is *only due to quantum BE statistics*, unlike in superfluidity or superconductivity (Cooper pairing, BCS etc.) where *interactions play the major role*.

- For an energy state density $\sim E^{1/2}$, the atoms N_0 in a BEC formed at temperature T below the critical temperature T_c are given by

$$N_0 = N \left[1 - \left(T/T_c\right)^3\right].$$

- *Magnetic trap*-imposed length scale a_0 vs. Gaussian width $(T > T_c)$

$$a_0 = [\hbar/(m\omega_{osc})]^{1/2} \qquad < \qquad R_G = [kT/(\hbar\omega_{osc})]^{1/2}\, a_0$$

($\omega_{osc} = (\omega_x\omega_y\omega_z)^{1/3}$: characteristic trapping potential frequency) unlike e.g. superfluid He (where inhomogeneity scale is fixed by interatomic spacing).

[F. Dalfovo *et al.*, RMP **71**, 463 (1999); A.J. Leggett, RMP **73**, 307 (2001)].

Boson gas density vs. T: theory meets experiment

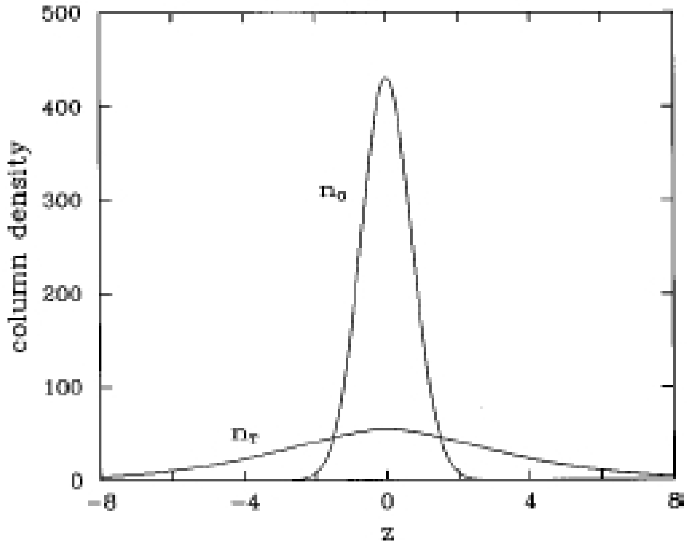


FIG. 4. Column density for 5000 noninteracting bosons in a spherical trap at temperature $T=0.9T_c^0$. The central peak is the condensate, superimposed on the broader thermal distribution. Distance and density are in units of a_{ho} and a_{ho}^{-2}, respectively. The density is normalized to the number of atoms. The same curves can be identified with the momentum distribution of the condensed and noncondensed particles, provided the abscissa and the ordinate are replaced with p_z, in units of a_{ho}^{-1}, and the momentum distribution, in units of a_{ho}^2, respectively.

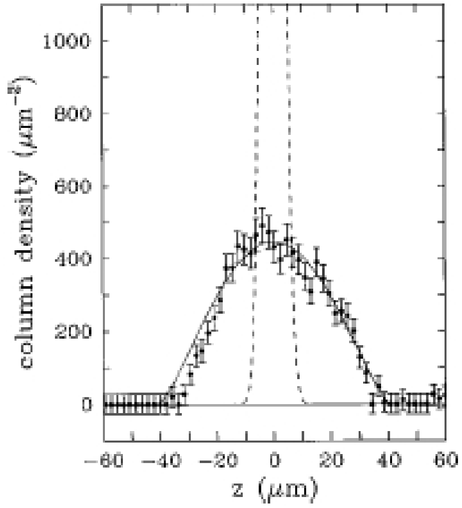


FIG. 3. Density distribution of 80 000 sodium atoms in the trap of Hau *et al.* (1998) as a function of the axial coordinate. The experimental points correspond to the measured optical density, which is proportional to the column density of the atom cloud along the path of the light beam. The data agree well with the prediction of mean-field theory for interacting atoms (solid line) discussed in Sec. III. Conversely, a noninteracting gas in the same trap would have a much sharper Gaussian distribution (dashed line). The same normalization is used for the three density profiles. The central peak of the Gaussian is found at about 5500 μm^{-2}. The figure points out the role of atom-atom interaction in reducing the central density and enlarging the size of the cloud.

[F. Dalfovo *et al.*, RMP **71**, 463 (1999); A.J. Leggett, RMP **73**, 307 (2001)].

The first (Boulder) BEC experiment (1995): concept & procedure

[Anderson *et al.*, Science, **269**, 5221 (1995)]

- A ^{87}Rb source in vacuum (10^{-6} mbar) is heated, so that atoms are "evaporated" at $v \sim 100$ cm/s;
- Part of the atom flux is isolated and slowed down by a laser beam [1], to a velocity as low as $v \sim$ cm/s (i.e. $T \sim \mu$K);
- Further cooling is achieved by magnetic trapping, down to 170 nK;
- A 10 μm -sized BEC is thus formed, consisting of ∼2000 Rb atoms.
- The BEC was sustained for more than 15 seconds.

[1] Laser cooling: cf. Nobel Prize 1997 awarded to Steven Chu, Claude Cohen-Tannoudji and William Phillips.

First BEC experiment *(continued)*

[Anderson *et al.*, Science, **269**, 5221 (1995)]

REPORTS

Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor

M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman,* E. A. Cornell

A Bose-Einstein condensate was produced in a vapor of rubidium-87 atoms that was confined by magnetic fields and evaporatively cooled. The condensate fraction first appeared near a temperature of 170 nanokelvin and a number density of 2.5×10^{12} per cubic centimeter and could be preserved for more than 15 seconds. Three primary signatures of Bose-Einstein condensation were seen. (i) On top of a broad thermal velocity distribution, a narrow peak appeared that was centered at zero velocity. (ii) The fraction of the atoms that were in this low-velocity peak increased abruptly as the sample temperature was lowered. (iii) The peak exhibited a nonthermal, anisotropic velocity distribution expected of the minimum-energy quantum state of the magnetic trap in contrast to the isotropic, thermal velocity distribution observed in the broad uncondensed fraction.

tons were found to limit the achievable temperatures (*8*) and densities (*9*), so that the resulting value for ρ_{ps} was 10^5 to 10^6 times too low for BEC. We began to pursue BEC in an alkali vapor by using a hybrid approach to overcome these limitations (*10, 11*). This hybrid approach involves loading a laser-cooled and trapped sample into a magnetic trap where it is subsequently cooled by evaporation. This approach is particularly well suited to heavy alkali atoms because they are readily cooled and trapped with laser light, and the elastic scattering cross sections are very large (*12*), which facilitates evaporative cooling.

There are three other attractive features of alkali atoms for BEC. (i) By exciting the easily accessible resonance lines, one can use light scattering to sensitively character-

The Boulder experiment (rubidium gas, ^{87}Rb) by Wiemann, Cornell *et al.*

[Anderson *et al.*, Science, **269**, 5221 (1995)]

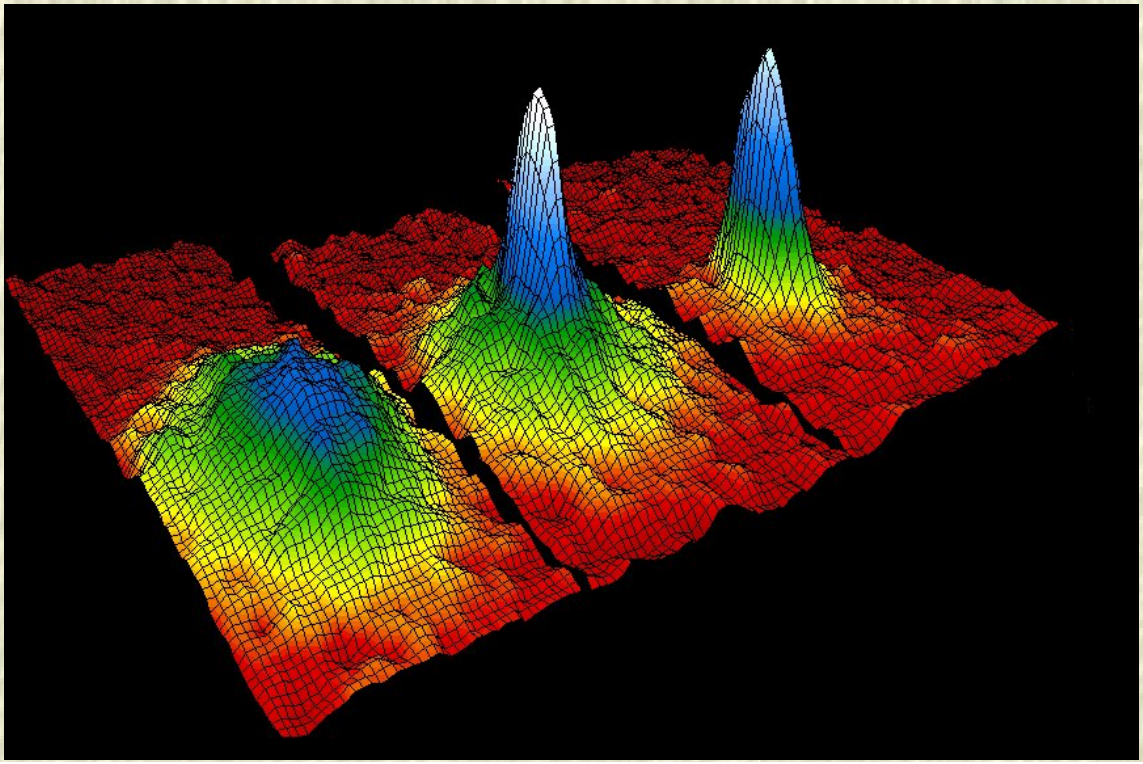


FIG. 1. (Color) Images of the velocity distribution of rubidium atoms in the experiment by Anderson *et al.* (1995), taken by means of the expansion method. The left frame corresponds to a gas at a temperature just above condensation; the center frame, just after the appearance of the condensate; the right frame, after further evaporation leaves a sample of nearly pure condensate. The field of view is 200 μm×270 μm, and corresponds to the distance the atoms have moved in about 1/20 s. The color corresponds to the number of atoms at each velocity, with red being the fewest and white being the most. From Cornell (1996).

The MIT experiment (sodium gas, ^{23}Na) by Ketterle *et al.*

[Davis *et al.*, PRL, **75**, 3969 (1995)] ; fig. from: Nobel lecture [Ketterle *et al.*, RMP, **74, 1131 (2002)]** ; *cf. movie 1*.

1140 Wolfgang Ketterle: When atoms behave as waves

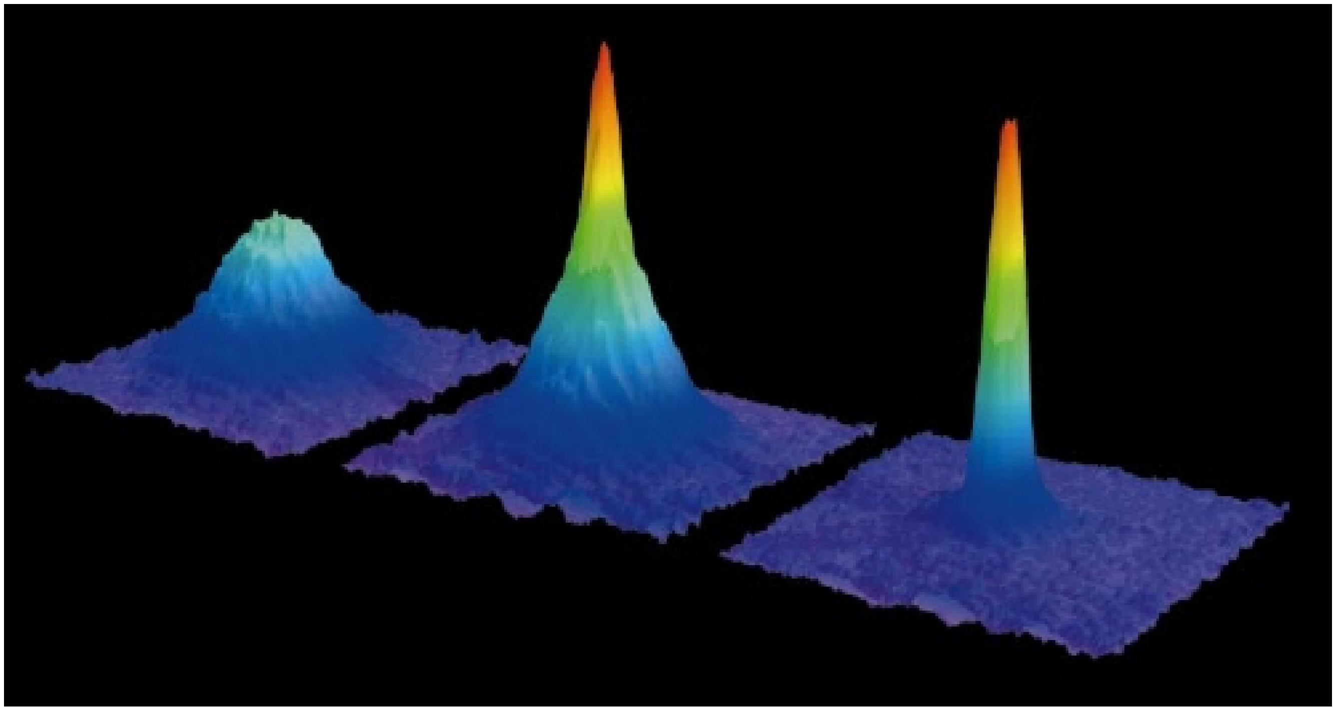


FIG. 7. Observation of Bose-Einstein condensation by absorption imaging. Shown is absorption vs two spatial dimensions. The Bose-Einstein condensate is characterized by its slow expansion observed after 6 ms time of flight. The left picture shows an expanding cloud cooled to just above the transition point; middle: just after the condensate appeared; right: after further evaporative cooling has left an almost pure condensate. The total number of atoms at the phase transition is about 7×10^5, the temperature at the transition point is 2 μK [Color].

2. BEC modelling: the *Gross-Pitaevskii Equation* (GPE)

Mean Field Theory for BE condensation → Gross-Pitaevskii Eq. (GPE):

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r}) + g|\psi|^2\right]\psi \qquad (1)$$

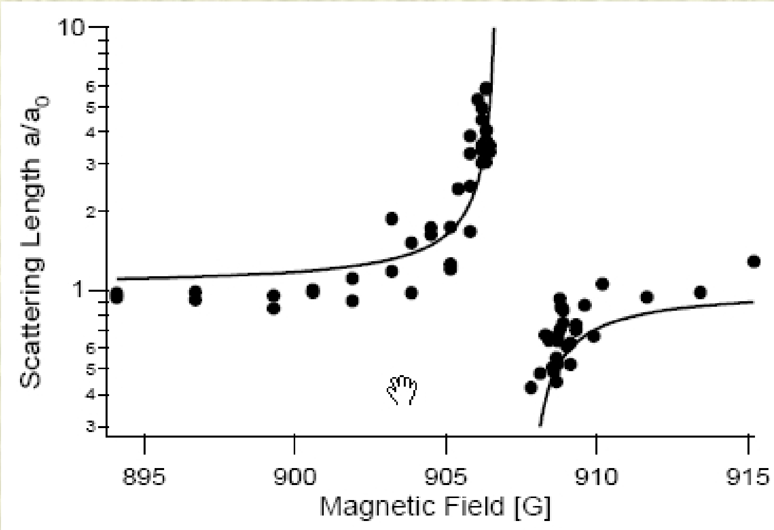
which takes into account:

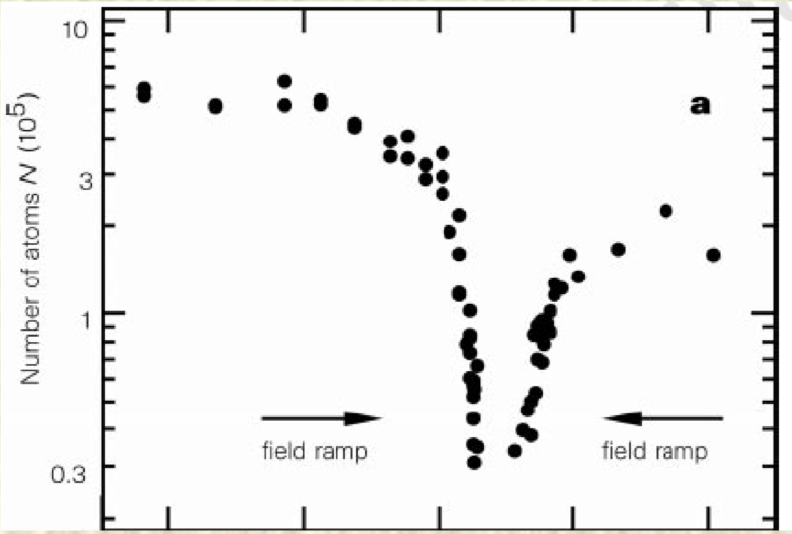
- $-\frac{\hbar^2}{2m}\nabla^2 \equiv -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$: a kinetic term;
- $V(\mathbf{r})$: the trapping potential, e.g. $V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$;
- $g|\psi|^2$: interatomic interactions, with $g = 4\pi\hbar^2 a/m$; these may be either
  - *repulsive*: $g > 0$ ($a > 0$: *positive* scattering length); BEC *stability*
    or
  - *attractive*: $g < 0$ ($a < 0$: *negative* scattering length); BEC *collapse*.

Scattering length: tuning via *Feshbach* resonance

The strength of atomic interactions varies strongly near a *Feshbach resonance*, at some specific value of an external magnetic field.

Thus,
the scattering length may be controlled via external magnetic fields
(experiments on ^{23}Na gases) [Inouye *et al.* (MIT + Ketterle), Nature **392**, 153 (1998)]:





Scattering length: sign inversion ....

2002: a *negative scattering length* is realized experimentally ($a < 0$), ...

($\rightarrow g < 0$: repulsive $\rightarrow$ attractive interaction; focusing nonlinearity)

[Strecker *et al.* (Rice U. + Hulet), Nature **417**, 150 (2002)]

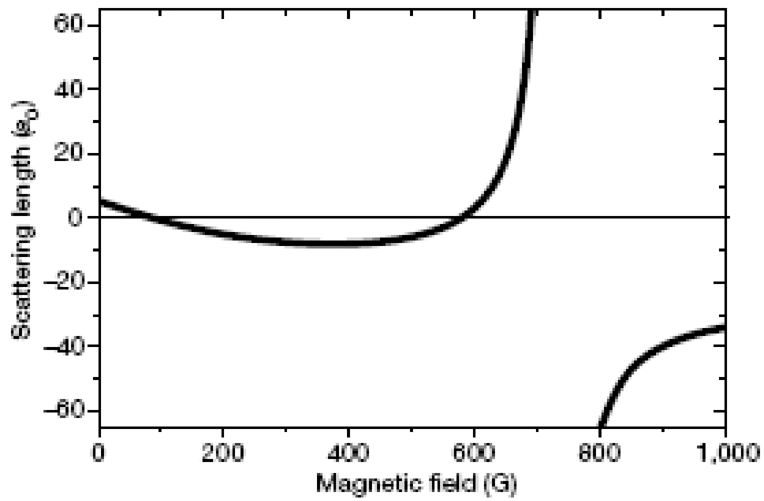


Figure 1 Feshbach resonance. Calculation of the scattering length versus magnetic field for atoms in the (1, 1) state of ^{7}Li using the coupled channels method[18]. The field axis has been scaled here by a factor of 0.91, to agree with the measured resonance position of 725 G shown in Fig. 2. The scattering length is given in units of the Bohr radius, a_0.

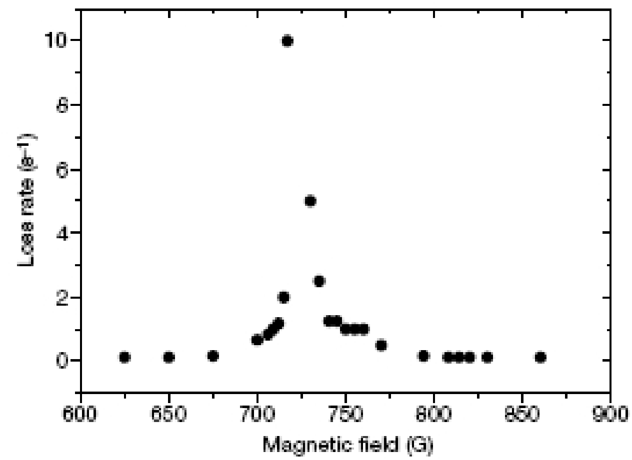


Figure 2 Measured rate of inelastic collisional loss of atoms near the Feshbach resonance. The temperature is ~1 μK, which is above the transition temperature for Bose–Einstein condensation. The initial peak density is estimated to be ~6×10^{12} cm^{-3}. The rate of loss is given by the time for the number of trapped atoms to fall to e^{-1} of the initial number. The magnetic field is determined spectroscopically by measuring the frequency of the (2, 2) → (1, 1) transition to within an uncertainty of 0.1 G.

Scattering length: sign inversion ....

→ formation of *bright* soliton trains! [Strecker *et al.* (Rice U. + Hulet), Nature **417**, 150 (2002)]

Formation and propagation of matter-wave soliton trains

Kevin E. Strecker*, Guthrie B. Partridge*, Andrew G. Truscott*†
& Randall G. Hulet*

* *Department of Physics and Astronomy and Rice Quantum Institute, Rice University, Houston, Texas 77251, USA*

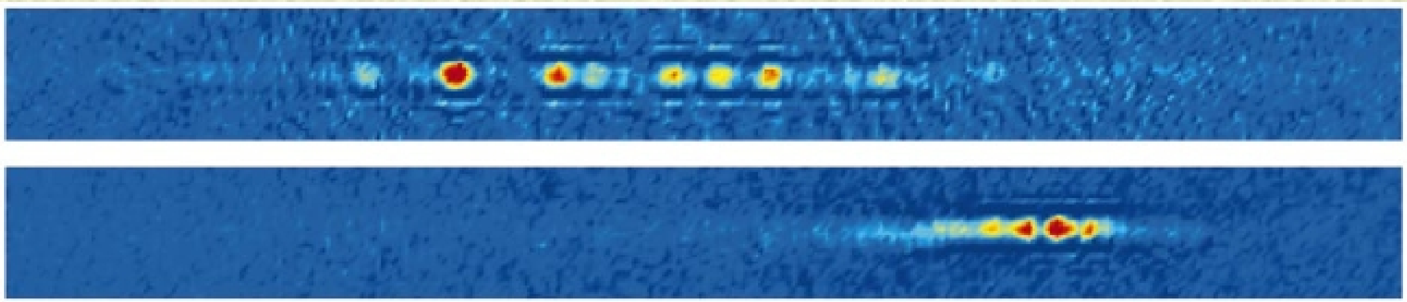


Figure 4 Repulsive interactions between solitons. The three images show a soliton train near the two turning points and near the centre of oscillation. The spacing between solitons is compressed at the turning points, and spread out at the centre of the oscillation. A simple model based on strong, short-range, repulsive forces between nearest-neighbour solitons indicates that the separation between solitons oscillates at approximately twice the trap frequency, in agreement with observations. The number of solitons varies from image to image because of shot to shot experimental variations, and because of a very slow loss of soliton signal with time. As the axial length of a soliton is expected to vary as $1/N$ (ref. 11), solitons with small numbers of atoms produce particularly weak absorption signals, scaling as N^2. Trains with missing solitons are frequently observed, but it is not clear whether this is because of a slow loss of atoms, or because of sudden loss of an individual soliton.

Scattering length: sign inversion *bis* ....: *Paris - Texas: 1-0*

2002 (*one month earlier!*): a similar experiment is carried out in Paris

[Khaykovich *et al.* (ENS, Paris + Salomon), Science **296**, 1290 (2002)]

Formation of a Matter-Wave Bright Soliton

L. Khaykovich,[1] F. Schreck,[1] G. Ferrari,[1,2] T. Bourdel,[1]
J. Cubizolles,[1] L. D. Carr,[1] Y. Castin,[1] C. Salomon[1]*

We report the production of matter-wave solitons in an ultracold lithium-7 gas. The effective interaction between atoms in a Bose-Einstein condensate is tuned with a Feshbach resonance from repulsive to attractive before release in a one-dimensional optical waveguide. Propagation of the soliton without dispersion over a macroscopic distance of 1.1 millimeter is observed. A simple theoretical model explains the stability region of the soliton. These matter-wave solitons open possibilities for future applications in coherent atom optics, atom interferometry, and atom transport.

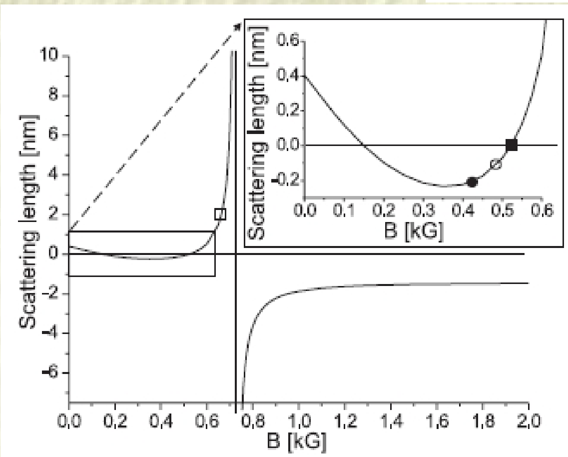
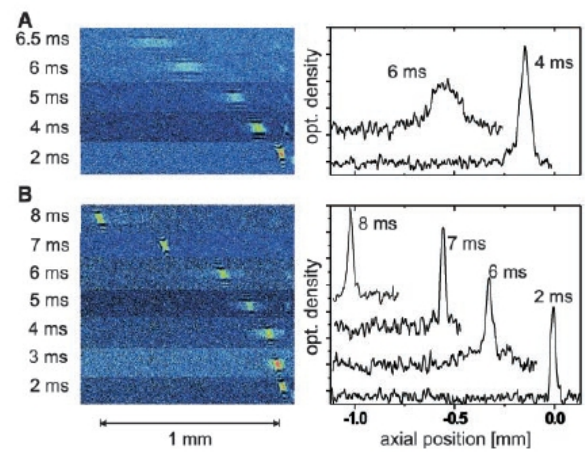


Fig. 2. Predicted magnetic field dependence of the scattering length a for ^{7}Li in state $|F = 1, m_F = 1\rangle$ (11). (**Inset**) Expanded view of the 0 to 0.6 kG interval with the various values of a used to study soliton formation. (□) Initial BEC; (■) ideal BEC gas; (○) attractive gas; (●) soliton.

Fig. 3. Absorption images at variable delays after switching off the vertical trapping beam. Propagation of an ideal BEC gas (**A**) and of a soliton (**B**) in the horizontal 1D waveguide in the presence of an expulsive potential. Propagation without dispersion over 1.1 mm is a clear signature of a soliton. Corresponding axial profiles are integrated over the vertical direction.



3. (Linear and) nonlinear excitation in BECs

- Scattering length a tuning has boosted nonlinear analysis of BECs.
- A plethora of excitations have been predicted (and realized in appropriate designed experiments).

These include ...

(→ next slide)

Propagating *linear oscillations* (waves)

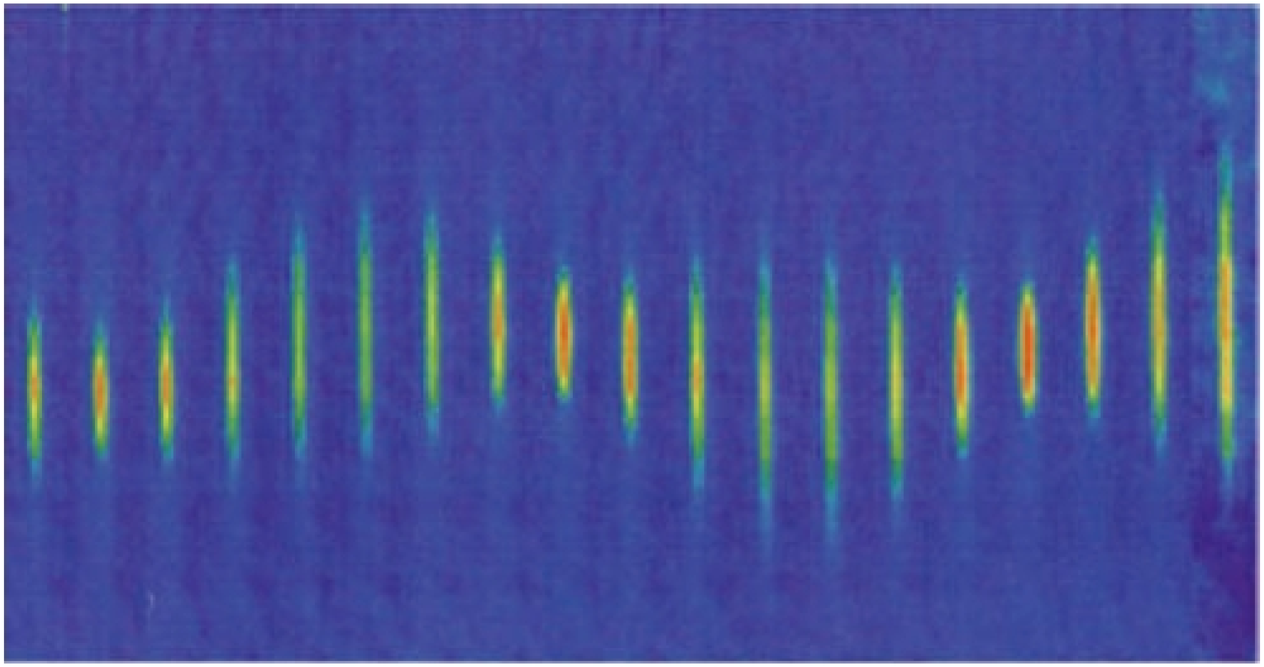


FIG. 2. (Color) Collective excitations of a Bose-Einstein condensate. Shown are *in situ* repeated phase-contrast images taken of a "pure" condensate. The excitations were produced by modulating the magnetic fields which confine the condensate, and then letting the condensate evolve freely. Both the center-of-mass and the shape oscillations are visible, and the ratio of their oscillation frequencies can be accurately measured. The field of view in the vertical direction is about 620 μm, corresponding to a condensate width of the order of 200–300 μm. The time step is 5 ms per frame. From Stamper-Kurn and Ketterle (1998).

reprinted from [F. Dalfovo *et al.*, RMP **71**, 463 (1999)]

Bright soliton trains (for attractive interactions)

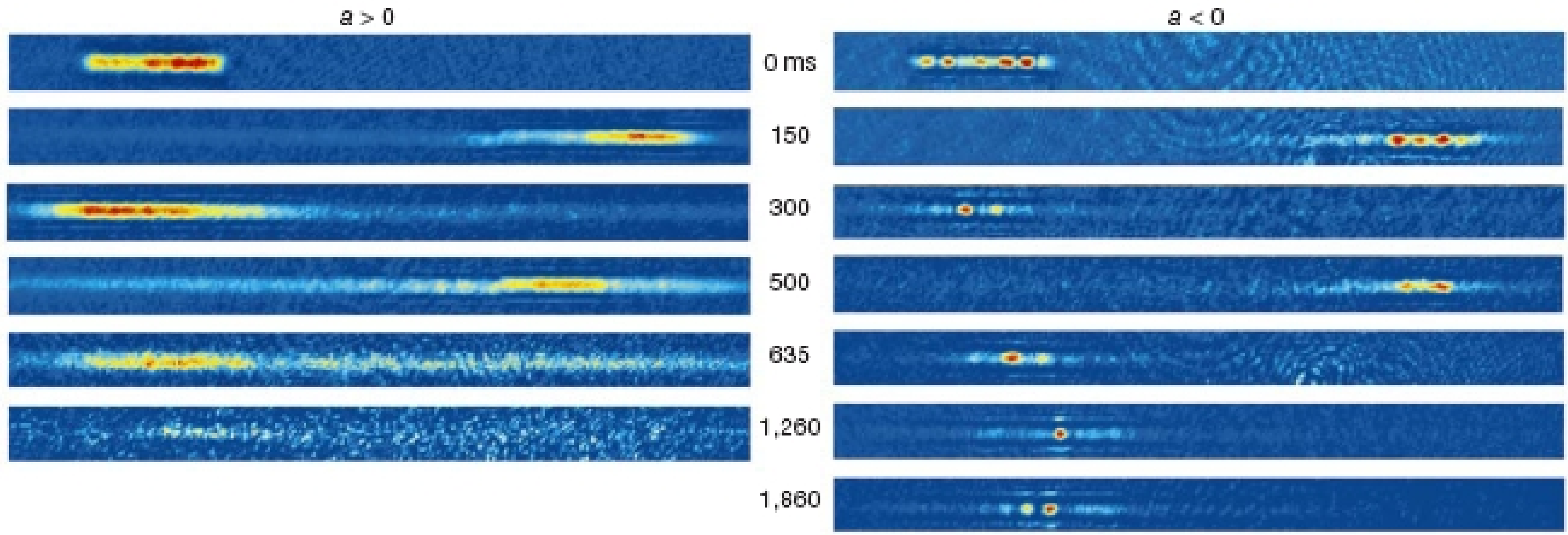


Figure 3 Comparison of the propagation of repulsive condensates with atomic solitons. The images are obtained using destructive absorption imaging, with a probe laser detuned 27 MHz from resonance. The magnetic field is reduced to the desired value before switching off the end caps (see text). The times given are the intervals between turning off the end caps and probing (the end caps are on for the $t = 0$ images). The axial dimension of each image frame corresponds to 1.28 mm at the plane of the atoms. The amplitude of oscillation is $\sim 370\,\mu$m and the period is 310 ms. The $a > 0$ data correspond to 630 G, for which $a \approx 10a_0$, and the initial condensate number is $\sim 3 \times 10^5$. The $a < 0$ data correspond to 547 G, for which $a \approx -3a_0$. The largest soliton signals correspond to $\sim$5,000 atoms per soliton, although significant image distortion limits the precision of number measurement. The spatial resolution of $\sim 10\,\mu$m is significantly greater than the expected transverse dimension $l_r \approx 1.5\,\mu$m.

reprinted from [Strecker *et al.* (Rice U. + Hulet), Nature **417**, 150 (2002)]

Stable *Dark solitons (for repulsive interactions) ...*

i.e. void regions versus a finite density background:

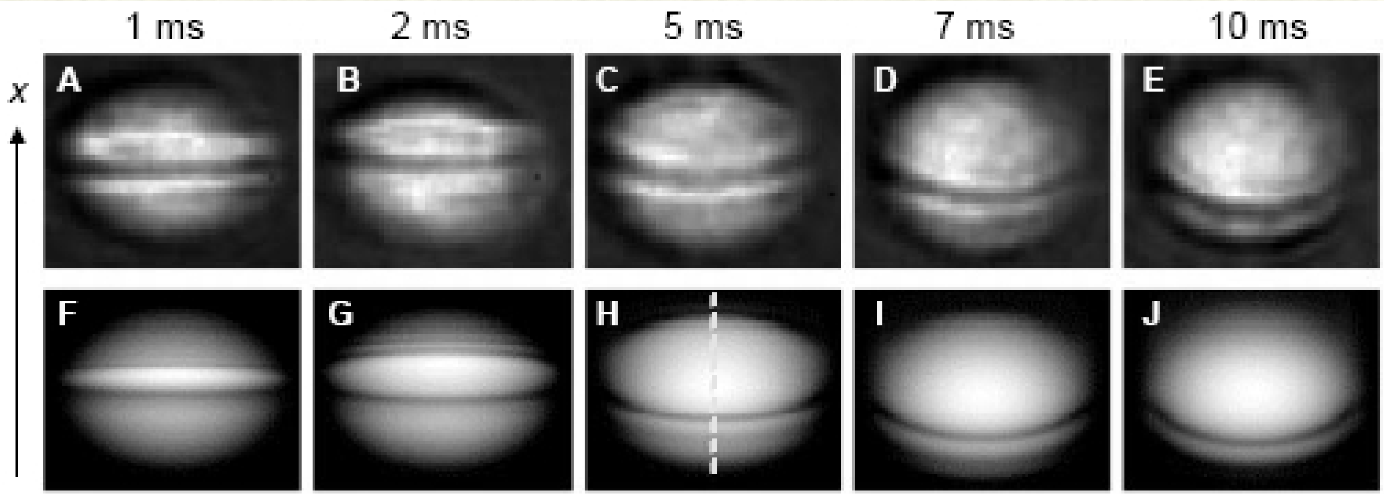


Fig. 3. Experimental (A to E) and theoretical (F to J) images of the integrated BEC density for various times after we imprinted a phase step of ~1.5π on the top half of the condensate with a 1-μs pulse. The measured number of atoms in the condensate was 1.7 ($\pm$0.3) $\times$ 10^6, and this value was used in the calculations. A positive density disturbance moved rapidly in the +x direction, and a dark soliton moved oppositely at significantly less than the speed of sound. Because the imaging pulse (*27*) is destructive, each image shows a different BEC. The width of each frame is 70 μm.

reprinted from [Denschlag *et al.*, Science **287**, 97 (2000)]

Stable *Dark soliton ensembles (interacting)*

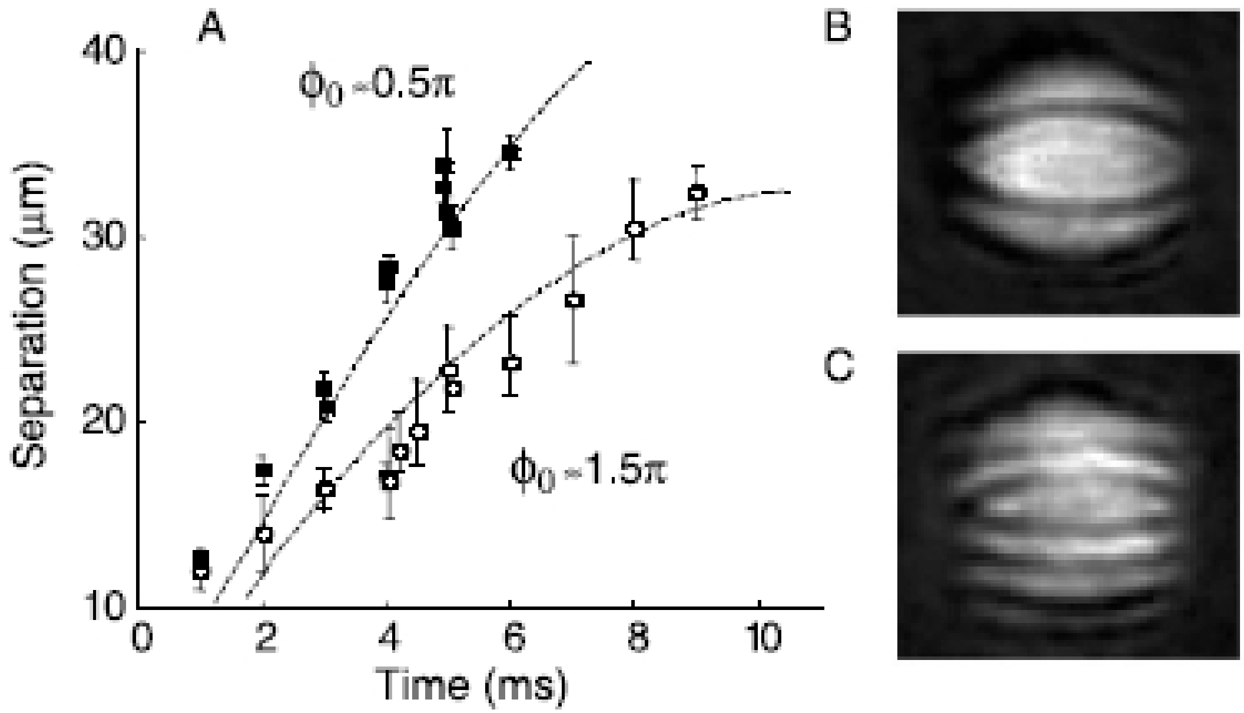


Fig. 5. (A) Plot of separation versus time for two oppositely propagating solitons after a phase imprint in the form of a stripe. For a small phase imprint ($\phi_0 \approx 0.5\pi$, squares), the solitons move at almost the local speed of sound. For a larger phase imprint ($\phi_0 \approx 1.5\pi$, circles), they are much slower. The dashed lines are from numerical simulations, from which we extract speeds for the corresponding solitons of 2.56 mm/s ($\phi_0 = 0.5\pi$) and 1.75 mm/s ($\phi_0 = 1.5\pi$) at 4 ms. (B) The condensate 6 ms after a stripe phase imprint of $\phi_0 \approx 1.5\pi$. (C) For a larger phase imprint of $\phi_0 \approx 2\pi$ many solitons appeared.

reprinted from [Denschlag *et al.*, Science **287**, 97 (2000)]

Stable *Dark solitons decaying into vortices*

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Watching Dark Solitons Decay into Vortex Rings in a Bose-Einstein Condensate

B. P. Anderson,[1,2] P. C. Haljan,[1] C. A. Regal,[3] D. L. Feder,[4] L. A. Collins,[5] C. W. Clark,[4] and E. A. Cornell[1,2]
[1]*JILA, National Institute of Standards and Technology and Department of Physics, University of Colorado, Boulder, Colorado 80309-0440*
[2]*Quantum Physics Division, National Institute of Standards and Technology, Boulder, Colorado 80305*
[3]*Physics Department, Lawrence University, P.O. Box 599, Appleton, Wisconsin 54912*
[4]*Electron and Optical Physics Division, National Institute of Standards and Technology, Gaithersburg, Maryland 20899-8410*
[5]*Theoretical Division, Mail Stop B212, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
(Received 11 December 2000)

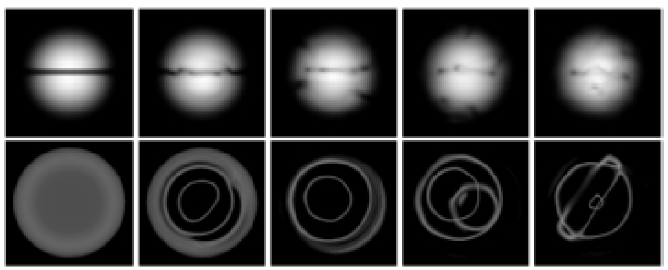


FIG. 1. Results of numerical simulations showing the decay of a black soliton in a BEC. The simulation corresponds to 3×10^5 ^{87}Rb atoms in a spherically symmetric trap with frequency 7.8 Hz. Successive frames are shown at 50-ms intervals, with the first frame at 100 ms after the start of the simulation. The first row shows the density profile of the condensate, integrated down an axis parallel to the soliton plane. The low-density regions within the cloud are also rendered (second row), with views perpendicular to the soliton plane.

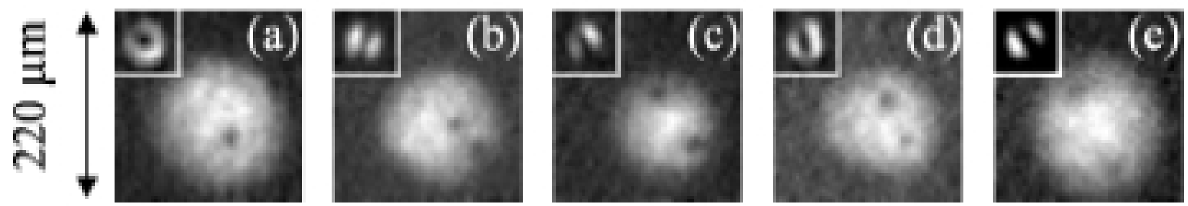


FIG. 2. Typical images of expanded condensates, and their initial states (insets) before the $|2\rangle$ atoms were removed. (a) A vortex. (b)–(e) The decay products of solitons. Image (e) was taken with a hold time of 500 ms; all other images were taken with a hold time of 0 ms, in addition to the 100 ms $|2\rangle$ removal and the 56-ms expansion [22].

Vortices & vortex lattices

Observation of Vortex Lattices in Bose-Einstein Condensates

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle

Quantized vortices play a key role in superfluidity and superconductivity. We have observed the formation of highly ordered vortex lattices in a rotating Bose-condensed gas. These triangular lattices contained over 100 vortices with lifetimes of several seconds. Individual vortices persisted up to 40 seconds. The lattices could be generated over a wide range of rotation frequencies and trap geometries, shedding light on the formation process. Our observation of dislocations, irregular structure, and dynamics indicates that gaseous Bose-Einstein condensates may be a model system for the study of vortex matter.

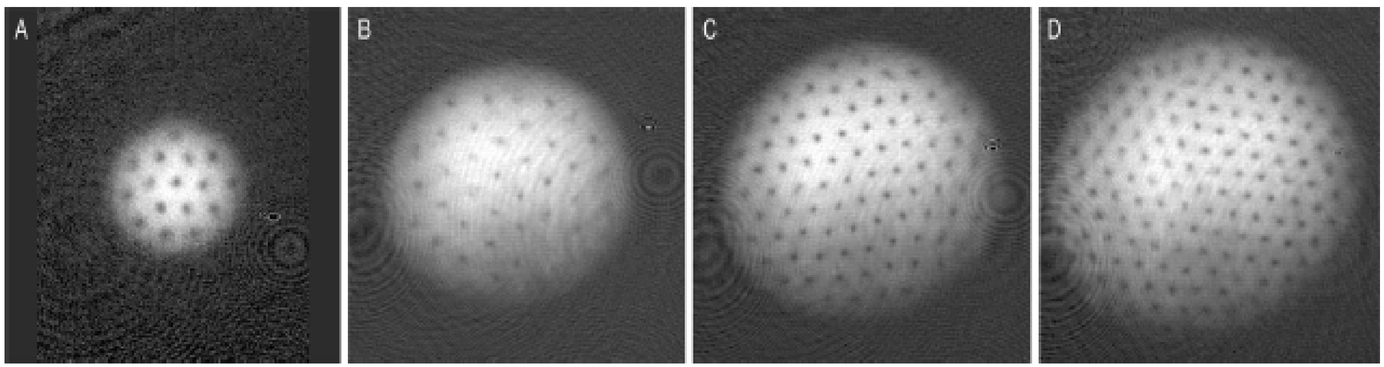


Fig. 1. Observation of vortex lattices. The examples shown contain approximately (A) 16, (B) 32, (C) 80, and (D) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.

reprinted from [Abo-Shaeer *et al.*, Nature **292**, 476 (2001)]

Vortices & vortex lattices (continued...)

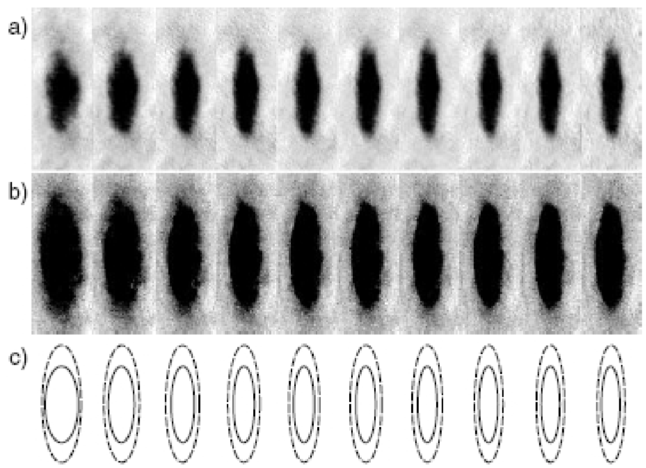
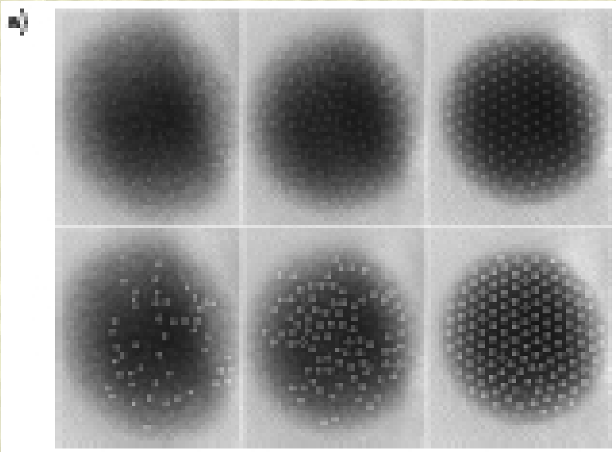


FIG. 1. (a) Spin down of a rotating condensate in a static magnetic trap. The first phase contrast image is taken 100 ms after turning off the drive, with each subsequent image spaced by 100 ms. The rotating vortex lattice caused a radial swelling of the condensate, reducing the aspect ratio. As vortices leave the system the aspect ratio approaches its static value of 4.2. The field of view is 25 by 75 μm. (b) Observation of a rotating thermal cloud. The parameters are identical to (a), but the probe light detuning is closer to resonance, enhancing the sensitivity to the more dilute thermal cloud. The phase shift of the dense condensate exceeds 2π and is displayed as saturated (black) in the image. The apparent loss in atom number is due to Rayleigh scattering of the probe light. (c) The inner and outer contours represent the aspect ratio of the condensate and thermal cloud, respectively, as obtained from two-dimensional fits to phase-contrast images.



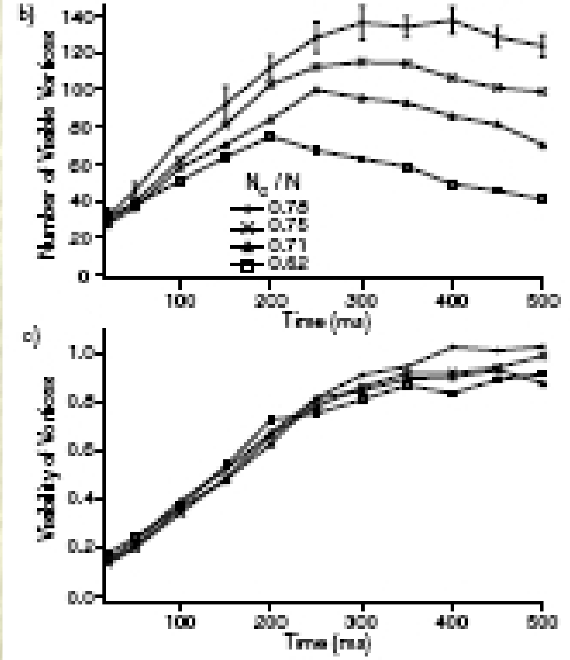
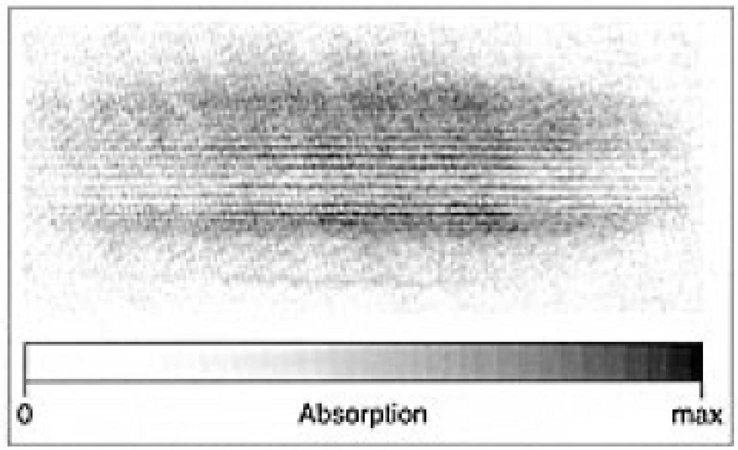


FIG. 4. Crystallization of the vortex lattice. (a) The top row shows three condensates that have equilibrated for 50, 150, and 300 ms, respectively, and have 48, 86, and 140 vortices recognized as visible by our algorithm. The bottom row shows the same condensates with the visible vortices circled. The field of view was 1.4 by 1.6 mm. (b) Growth of the number of visible vortices for several temperatures expressed by the condensate fraction (N_c/N). (c) Visibility of vortices derived from the data in (b), normalized by the number of vortices inferred from centrifugal distortion measurements.

reprinted from [Abo-Shaeer *et al.*, PRL **88**, 070409 (2002)]

4. Coupled BECs: History and modelling

❏ *Coupling among BECs* was considered and realized very early. *Interference patterns* were observed, motivating further investigation:



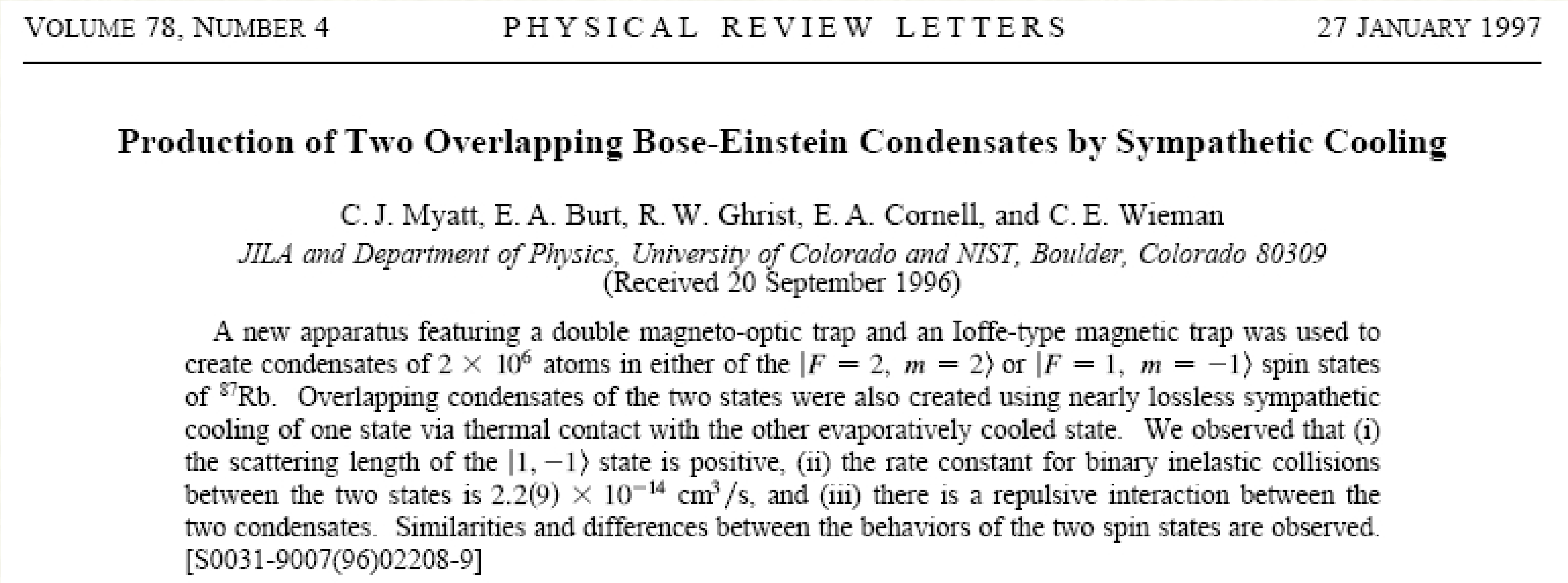
High resolution image (jpeg 1,61 mb)

Figure 2. Pattern of interference between two overlapping Bose-Einstein condensates of sodium atoms. The image was made in absorption. Matter-wave interferences have a periodicity of 15 micrometer. The recording shows that the atoms of the two condensates were fully co-ordinated.

4. Coupled BECs: History and modelling

- *Coupling among BECs* was considered and realized very early.

- A number of experiments were devoted to coupled boson gases, either of similar nature (*symmetric BEC pairs*) ... :



VOLUME 78, NUMBER 4 PHYSICAL REVIEW LETTERS 27 JANUARY 1997

Production of Two Overlapping Bose-Einstein Condensates by Sympathetic Cooling

C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman
JILA and Department of Physics, University of Colorado and NIST, Boulder, Colorado 80309
(Received 20 September 1996)

A new apparatus featuring a double magneto-optic trap and an Ioffe-type magnetic trap was used to create condensates of 2×10^6 atoms in either of the $|F = 2, m = 2\rangle$ or $|F = 1, m = -1\rangle$ spin states of ^{87}Rb. Overlapping condensates of the two states were also created using nearly lossless sympathetic cooling of one state via thermal contact with the other evaporatively cooled state. We observed that (i) the scattering length of the $|1, -1\rangle$ state is positive, (ii) the rate constant for binary inelastic collisions between the two states is $2.2(9) \times 10^{-14}$ cm^3/s, and (iii) there is a repulsive interaction between the two condensates. Similarities and differences between the behaviors of the two spin states are observed. [S0031-9007(96)02208-9]

4. Coupled BECs: History and modelling

- *Coupling among BECs* was considered and realized very early.

- A number of experiments were devoted to coupled boson gases, either of similar nature (*symmetric BEC pairs*)...

- ... or distinct ones (*asymmetric BEC pairs*), e.g. ^{87}Rb-^{23}Na, ^{87}Rb-^{85}Rb:

PHYSICAL REVIEW A, VOLUME 62, 043605

Atom loss and the formation of a molecular Bose-Einstein condensate by Feshbach resonance

V. A. Yurovsky and A. Ben-Reuven
School of Chemistry, Tel Aviv University, 69978 Tel Aviv, Israel

P. S. Julienne and C. J. Williams
Atomic Physics Division, Stop 8423, National Institute of Standards and Technology, Gaithersburg, Maryland 20889
(Received 25 May 2000; published 13 September 2000)

Dynamics of Component Separation in a Binary Mixture of Bose-Einstein Condensates

D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell*
JILA, National Institute of Standards and Technology and Department of Physics, University of Colorado, Boulder, Colorado 80309-0440
(Received 2 April 1998)

from: Hall *et al.*, PRL **81**, 1539 (1998)

4. Coupled BECs: History and modelling

- *Coupling among BECs* was considered and realized very early.

- A number of experiments were devoted to coupled boson gases, either of similar nature (*symmetric BEC pairs*) ...

- ... or distinct ones (*asymmetric BEC pairs*), e.g. ^{87}Rb-^{23}Na, ^{87}Rb-^{85}Rb, or ...

- ... or, more recently, *optical lattices* (beyond the dilute, weakly interacting boson gas limit), i.e. chains consisting of N BECs, realized via appropriate magnetic traps.

4. Coupled BEC (cBEC) modelling (at the bottom of the trapping well)

$\rightarrow$ coupled Gross-Pitaevskii Eqs. (cGPE):

$$i\hbar \frac{\partial \psi_1}{\partial t} + \frac{\hbar^2}{2m_1} \nabla^2 \psi_1 - g_{11} |\psi_1|^2 \psi_1 - g_{12} |\psi_2|^2 \psi_1 + \mu_1 \psi_1 = 0$$

$$i\hbar \frac{\partial \psi_2}{\partial t} + \frac{\hbar^2}{2m_2} \nabla^2 \psi_2 - g_{22} |\psi_2|^2 \psi_2 - g_{21} |\psi_1|^2 \psi_2 + \mu_2 \psi_2 = 0 \qquad (2)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and

- $g_{jj} = 4\pi\hbar^2 a_{jj}/m_j$ $(j = 1, 2)$ measure *intra-BEC interatomic interactions*: *repulsive* (*attractive*) for $g_{jj} > 0$ $(g_{jj} < 0)$, i.e. $a_{jj} > 0$ $(a_{jj} < 0)$;
- $g_{jj'} = 2\pi\hbar^2 a_{jj}/m_{jj'}$ $(j \neq j' = 1, 2)$ measure *inter-BEC interactions*, where $m_{jj'} = m_j m_{j'}/(m_j + m_{j'})$ is the *reduced mass*;
- μ_j are the chemical potentials, i.e. $\Phi_j = \psi_j \exp(-i\mu_j t/\hbar)$ $(\int d\mathbf{r}\psi_j^2 = N_{j,0})$.

5. Stability analysis of cBECs

Material from [I. Kourakis, P. K. Shukla, M. Marklund, and L. Stenflo, *Eur. Phys. J. B* **46**, 381 (2005)].

- Let us consider *harmonic cBEC vibrations* in the form

$$\psi_j = \psi_{j0} \exp[i\varphi_j(t)] \, ;$$

- Substituting in the cGPE Eqs., we obtain $\varphi_j(t) = \Omega_{j0} t$, with

$$\Omega_{j0} = -\frac{g_{jj}}{\hbar}\psi_{j0}^2 - \frac{g_{jj'}}{\hbar}\psi_{j'0}^2 + \mu_j \, , \qquad \text{for } j \text{ and } j'(\neq j) = 1 \text{ and } 2 \, .$$

- We now assume a small external perturbation ψ_{j1} by setting

$$\psi_j \to (\psi_{j0} + \epsilon \psi_{j1}) \exp[i\varphi_j(t)] \qquad\qquad (\epsilon \ll 1) \, ,$$

where $\psi_{j1} \sim \exp[i(\mathbf{k}\cdot\mathbf{r} - \Omega_k t)]$; the *perturbation* wavenumber $\mathbf{k}$ and frequency Ω_k will be determined by the cGPE Eqs.

Perturbation dispersion relation

An eigenvalue problem is obtained, viz. $\mathbf{Ma} = (\hbar\omega)^2\mathbf{a}$, leading to

$$\left(\Omega_k^2 - \Omega_1^2\right)\left(\Omega_k^2 - \Omega_2^2\right) = \Omega_c^4\,, \tag{3}$$

where:

- the *individual BEC terms* read: $M_{jj} = \frac{\hbar^2 k^2}{2m_j}\left(\frac{\hbar^2 k^2}{2m_j} + 2g_{jj}|\psi_{j0}|^2\right) \equiv \hbar^2\Omega_j^2$;
- the *BEC coupling terms* read: $M_{jj'} = -2\frac{\hbar^2 k^2}{2m_j} g_{jj'}|\psi_{j0}||\psi_{j'0}| \equiv \hbar^2\Omega_{jj'}^2$;
- the *coupling* is expressed via $\Omega_c^4 = \Omega_{12}^2\Omega_{21}^2 \equiv M_{12}M_{21}/\hbar^4$.
- INTERMEZZO: *"switching off"* the coupling:

$$\Omega_k^2 = M_{jj}/\hbar^2 = \frac{k^2}{2m_j}\left(\frac{\hbar^2 k^2}{2m_j} + 2g_{jj}|\psi_{j0}|^2\right) \qquad j = 1 \,\text{or}\, 2\,.$$

INTERMEZZO: *"switching off"* the coupling:

$$\Omega_k^2 = M_{jj}/\hbar^2 = \frac{k^2}{2m_j}\left(\frac{\hbar^2 k^2}{2m_j} + 2g_{jj}|\psi_{j0}|^2\right) \qquad j = \text{either } 1 \text{ or } 2\,.$$

- For $g_{jj} > 0$ (repulsive interactions): $rhs > 0$, hence *stability*;
- For $g_{jj} < 0$ (attractive interactions): $rhs < 0$ for

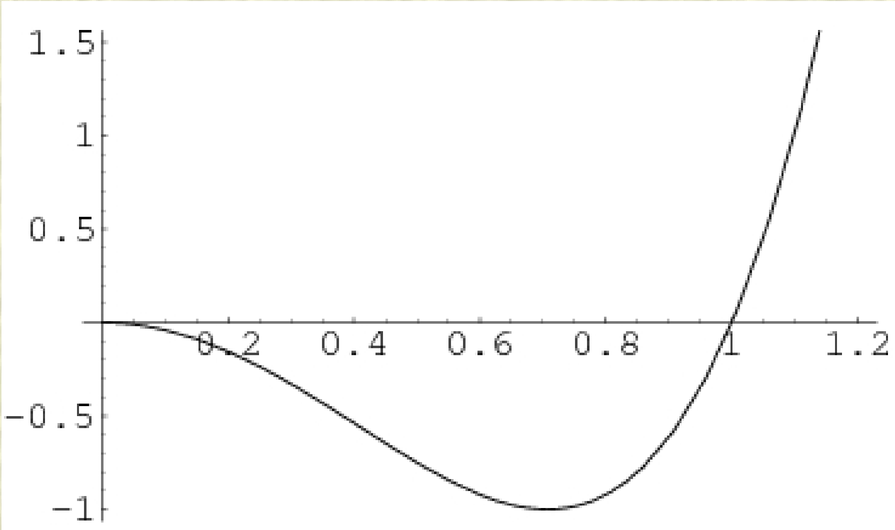
$$k < k_{cr} \equiv 2\,(m_j g_{jj})^{1/2}|\psi_{j0}|/\hbar\,,$$

hence *instability* for large perturbation wavelengths $\rightarrow$ ***collapse!***

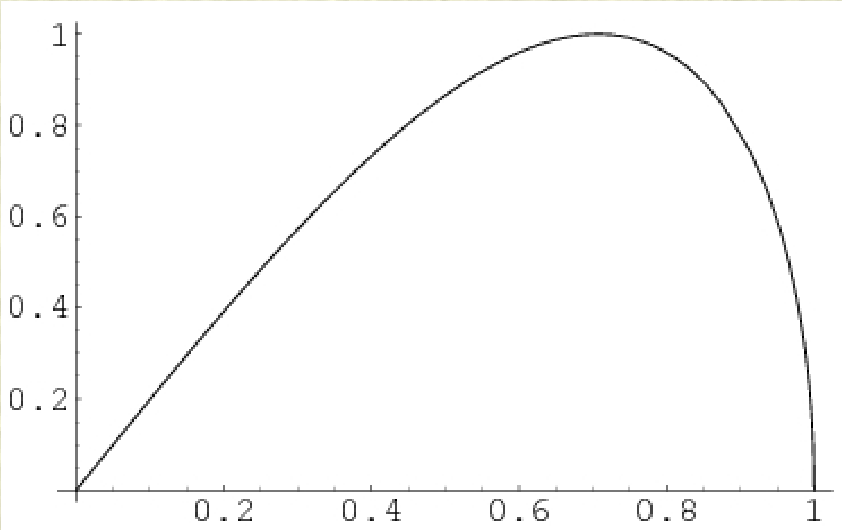
INTERMEZZO: ***"switching off"*** the coupling:

- *Max. instability growth rate* $\sigma_{max} = g_{jj}|\psi_{j0}|^2/\hbar \equiv \Omega_0$ occurs at $k = k_{cr}/\sqrt{2}$.

Ω_j^2/Ω_0^2 vs. k/k_{cr}



σ_j/Ω_0 vs. k/k_{cr}



- This result is reminiscent of (yet not identical to) *modulational (Benjamin-Feir) instability* in nonlinear dispersive media, e.g. in *optical media* featuring a *focusing* nonlinearity; also in *plasmas*, *water waves*, etc.

(Stability analysis of) The *coupled* BEC system ...

The cBEC *dispersion relation* is satisfied by the (*complex*, in general) roots of a 4th order (biquadratic) polynomial in Ω (8th order in k), viz. $p(\Omega, k) = 0$:

$$\Omega_k^2 = \frac{1}{2}(\Omega_1 + \Omega_2)^2 \pm \frac{1}{2}[(\Omega_1 - \Omega_2)^2 + 4\Omega_c^4]^{1/2}$$

The analysis involves an enlarged parameter space $\{k;\, g_{jj}, g_{jj'}\}$.

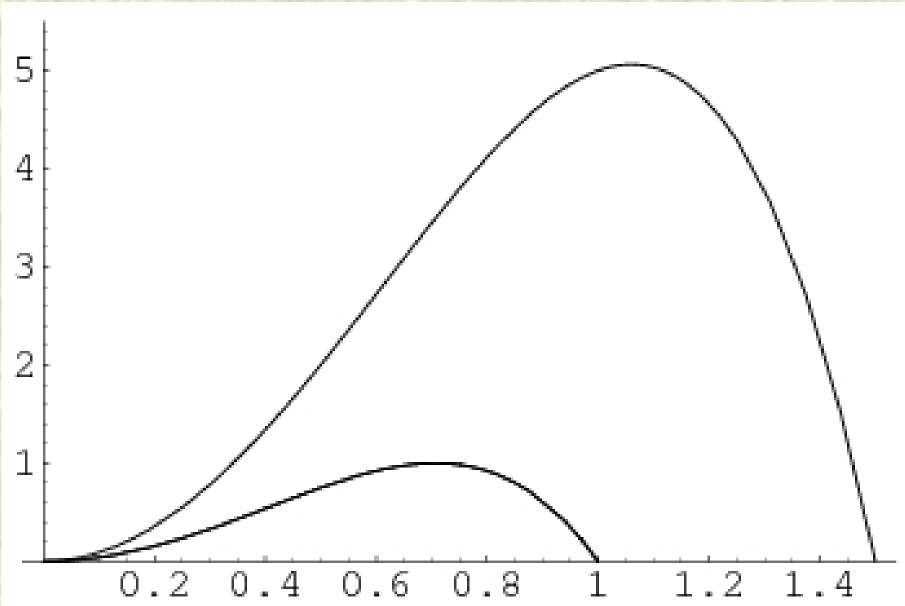
The (lengthy!) investigation of the conditions for *reality of* Ω (hence stability) for an *arbitrary* (*asymmetric*, in principle) pair of boson gases (cBEC) yields:

- (possibility for) an enlarged unstable wavenumber range;
- (possibility for) an increased growth rate (in case of cBEC *instability*);
- a *modified value of* Ω (in case of cBEC *stability*).

(Stability analysis of) The *coupled* BEC system ...

- An *enlarged unstable wavenumber range* is obtained, in addition to ...
- an *increased growth rate* (in case of cBEC *instability*);

$\sigma_j/\Omega_{0,1}$ & $\sigma_{12}/\Omega_{0,1}$ vs. $k/k_{1,cr}$



Stability criteria for coupled BECs

Stability is ensured if *all of* the following criteria are satisfied:

- Criterion 1:

$$g_{11} > 0 \qquad \text{and} \qquad g_{22} > 0\,.$$

$\rightarrow$ Instability if one (or more) BEC with attractive interactions is involved.

- Criterion 2:

$$g_{11}g_{22} - g_{12}g_{21} > 0\,;$$

$\rightarrow$ Generalization of previous criterion: $|g_{12}/g_{11}| < 1$, for *symmetric* cBECs.
$\rightarrow$ Previous criterion (obtained via energetic arguments) extended.
[Stamper-Kurn, 1999; Modugno, Roati, 2002; Svidzinsky, 2003].

- Criterion 3:

$$V_{12}V_{21} > 0\,.$$

($\rightarrow$ Automatically satisfied since $V_{12} = V_{21}$, in principle.)

Conclusions

- BECs provide an excellent test-bed for nonlinear theories, related e.g. to the dynamics of localized excitations, in various dimensionalities (related to trapping geometry).

- The actual realization of BECs requires sophisticated experimental techniques, but in turn offers the possibility for refined tuning of nonlinear parameters and design of purpose-built devices.

- On coupled BECs: one still needs to refine the theory, by taking into account the existence of an external magnetic trap, of variable size and geometry.

- Further work on coupled BECs is on the way and will be reported shortly.

Thank you for your attention !

I.K.

www.tp4.rub.de/∼ioannis