On some fundamental problems involved in the statistical-mechanical description of test-particle motion in a plasma

or

Derivation of a *Fokker-Planck* kinetic equation from first principles: Application in magnetized plasma [1,*]

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* I. Kourakis, ICTP, Trieste, 09.09.2005 (www.tp4.rub.de/∼ioannis/conf/200509-ICTP3-oral.pdf)

1 Introduction – theoretical framework

We are interested in the description of the dynamics of a large physical system of N particles $(j = 1, 2, ..., N)$, which interact:

- among themselves ($\equiv$ *collisions*)
- with an external force field.

Application:

Plasma = large ensemble of charged particles (e^-, i^+, ...)

Particular features:

- *long-range* electrostatic interactions;
- presence of EM fields, *Lorentz forces*.

1.1 Statistical Mechanics - Review of notions

* Probability density (*distribution function*) ρ_N, in *phase space* $\mathit{\Gamma}_N = \{\mathbf{x_j}, \mathbf{v_j}\}$.

* **Liouville Equation for N particles:**

$$\frac{\partial \rho_N}{\partial t} = L_N \, \rho_N \tag{1}$$

* General (*formal*) solution of the Liouville Equation:

$$\rho_N(t) = e^{L_N(t-t_0)} \, \rho_N(t_0) \tag{2}$$

* $e^{L_N(t-t_0)}$: *Time evolution operator* ("Propagator"):

its exact knowledge is tantamount to the knowledge of the complete problem of motion

(of N particles): impossible for $N = 10^{23}$ particles!!!

* **Kinetic evolution equation** (for 1 particle, d.f.: $\rho_1(\mathit{\Gamma}_1) = f$)

$$\frac{\partial f}{\partial t} = \mathcal{T}\{f\} \tag{3}$$

* $\mathcal{T}$: *Kinetic evolution operator* (to be determined for a given specific physical problem).

1.2 Kinetic equation (K.E.) - Collision term

General form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + m^{-1}\mathbf{F}\frac{\partial f}{\partial \mathbf{v}} = \mathcal{K}\{f\} \tag{4}$$

* $f = f(\mathbf{x}, \mathbf{v}; t)$.

* $\mathbf{F} = \mathbf{F}_{ext} + \mathbf{F}_{int}$: *external forces and mean-field forces* (Vlasov).

* The collision operator $\mathcal{K}$ *should take into account the existence of an external field.*

* $\mathbf{F}_{int}$ and $\mathcal{K}$ express the mutual interactions between particles.

Some *known collision terms* (to be used with caution in Plasma Physics) include:

- Boltzmann: Not applicable for long-range (e.g. Coulomb) interactions.

- Vlasov: Contains *no* collision term (hence no irreversibility, no H-Theorem).

- Landau: Contains a collision term, but takes into account *no* external force field.

- Fokker-Planck: Phenomenological description of stochastic processes:

NO rigorous link to microscopic dynamics in the presence of the field.

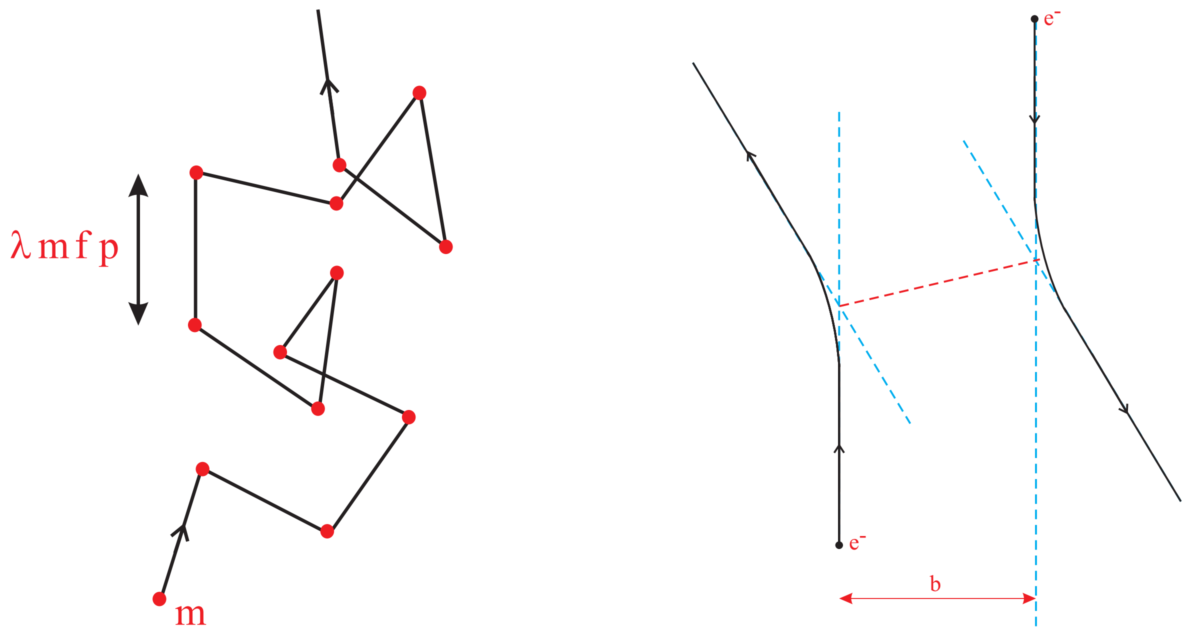

Figure 1: Inter-particle interaction – notice the difference between

(a) Point-like interactions between charge-neutral particles (sphere-model)

and

(b) *long-range* electrostatic interactions between *charged* particles.

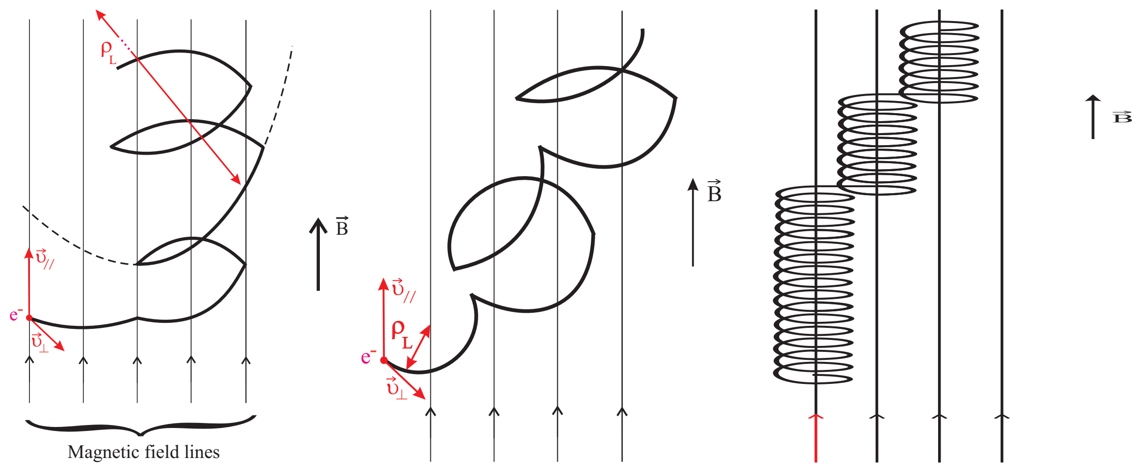

Figure 2: Heuristic representation of the trajectory of colliding charges, in the presence of a magnetic field. Compare the typical interaction space scale (e.g. Debye radius r_D) to the typical Larmor gyration scale (Larmor radius ρ_L) in three cases:

(a) $\rho_L \gg r_D$, (b) $\rho_L \approx r_D$ kai (g) $\rho_L \ll r_D$.

1.3 Macroscopic description

* Observable quantity (macroscopic) $A(\mathbf{x};t)$ = mean value of a:

$$A = \int d\mathbf{v}\, a\, f \;\equiv < a >_{\Gamma_v}$$

where a: a function of microscopic variables $\{\mathbf{x_j}, \mathbf{v_j}\}$,

e.g. density $n =< 1 >_{\Gamma_v}$, velocity $\mathbf{u} =< \mathbf{v} >_{\Gamma_v}$, and so forth.

The evolution of A in time obeys a relation in the form:

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t} \int d\mathbf{v}\, a f = \int d\mathbf{v}\, \frac{\partial a}{\partial t} f \;\simeq \int d\mathbf{v}\, a\, \frac{\partial f}{\partial t} = \int d\mathbf{v}\, a\, \mathcal{T} f = ...$$

→ *Fluid-dynamical description* of a Stat. Mechanical system

→ *Magnetohydrodynamic (MHD) Plasma Theory*

Ref. [R. Balescu, *Statistical Mechanics* (1975)] etc.

2 Model description – Test-particle formalism

Ingredients:

- a heat-bath (the *"reservoir"* R), in thermal equilibrium;

- a reference particle (the *test-particle* σ) ;

- an external field;

- Weak interaction between R kai σ .

Application 1: 3d plasma: N charged particles in a *homogeneous* & *static* magnetic field

$$\mathbf{B} = B\,\hat{z}\,.$$

Application 2: a chain of N coupled harmonic oscillators, in 1d.

"Application 3": Free motion (vanishing field limit).

3 Hamiltonian function – Equations of motion

- Hamiltonian:

$$H = H_R + H_\sigma + \lambda\, H_{int} \qquad (5)$$

- H_R: Hamiltonian of the reservoir (N particles)

$$H_R = \sum_{j=1}^{N} H_j + \sum_{j<n} \sum_{n=1}^{N} V_{jn} \qquad (6)$$

- H_j: 1 particle term ($j = 1, 2, ..., N$ *and* σ);

- H_{int}: interaction term (among the two subsystems):

$$H_{int} = \sum_{n=1}^{N} V_{\sigma n}$$

- $V_{ij} \equiv V(|\mathbf{x_i} - \mathbf{x_j}|) \quad (i, j = 1, 2, ..., N, \sigma)$;

- $\lambda \ll 1$ (*Weak interaction*).

- *Case study 1 (1d harmonic oscillators)*:

$$H_j = \frac{1}{2} m_j v_j^2 + \frac{1}{2} m_j \omega_j^2 x_j^2$$

- *Case study 2 (magnetized plasma)*:

$$H_j(\mathbf{x_j}, \mathbf{p_j}) = \frac{1}{2m_j} \left| \mathbf{p_j} - \frac{e_j}{c} \mathbf{A}(\mathbf{x_j}) \right|^2 \equiv \frac{1}{2} m_j v_j^2$$

where $\mathbf{A}(\mathbf{x_j})$ is the vector potential, i.e.

$$\mathbf{B}(\mathbf{x_j}) = \nabla \times \mathbf{A}(\mathbf{x_j})$$

[H. Goldstein, *Classical Mechanics*, 1980]etc.

- *Case study 3 (free motion, no field)*:

$$H_j = \frac{1}{2} m_j v_j^2 \, .$$

3.1 Equations of motion

$$\dot{\mathbf{x}} = \mathbf{v} \ ; \qquad \dot{\mathbf{v}} = \frac{1}{m}\left(\mathbf{F_0} + \lambda\,\mathbf{F_{int}}\right) \tag{7}$$

- $\mathbf{x} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z)$.

- $\mathbf{F_0}$: External force (due to the field)

  e.g. Lorentz force: $\mathbf{F_L} = \frac{e}{c}(\mathbf{v} \times \mathbf{B})$,

  e.g. restoring (spring) force: $\mathbf{F_0} = -m\omega_0^2 x^2$,

  $\mathbf{F_0} = \mathbf{0}$, for a free particle,

  and so forth ...

- $\mathbf{F_{int}}$: *interaction* force

$$\mathbf{F_{int}} = -\frac{\partial}{\partial \mathbf{x}} \sum V(|\mathbf{x} - \mathbf{x_j}|) \tag{8}$$

→ Collisions: *Random*, "*stochastic*" process !

3.2 Solution of the free (collisionless) problem of motion (for $\lambda = 0$)

Plasma:

$$\begin{aligned} \mathbf{v}^{(0)}(t) &= \mathbf{v} + \frac{1}{m}\int_0^t dt'\, \mathbf{F_0}(t') = \mathbf{R}(t)\,\mathbf{v} \\ \mathbf{x}^{(0)}(t) &= \mathbf{x} + \int_0^t dt'\, \mathbf{v}(t') = \mathbf{x} + \mathbf{N}(t)\,\mathbf{v} \end{aligned} \tag{9}$$

$$\mathbf{N}'^{\alpha}(t) = \mathbf{R}^{\alpha}(t) = \begin{pmatrix} \cos\Omega t & s\ \sin\Omega t & 0 \\ -s\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{N}^{\alpha}(t) = \int_0^t\ dt'\ \mathbf{R}^{\alpha}(t) = \Omega^{-1} \begin{pmatrix} \sin\Omega t & s\ (1-\cos\Omega t) & 0 \\ s\ (\cos\Omega t - 1) & \sin\Omega t & 0 \\ 0 & 0 & \Omega t \end{pmatrix}$$

$$\Omega = \Omega_\alpha = \frac{|e_\alpha|B}{m_\alpha c}, \qquad s = s_\alpha = \frac{e_\alpha}{|e_\alpha|} = \pm 1$$

In the free motion limit: $\Omega \to 0$, $\quad \mathbf{N} \to\ t\,\mathbf{I}$, $\quad \mathbf{N}' \to\ \mathbf{I}$.

- Harmonic oscillator (1d):

...

- Free motion:

$$\{x_i(t), v_i(t)\} = \{x_i + v_i t, v_i\} \qquad i = 1, 2, 3$$

- General solution for $\lambda = 0$ (working hypothesis):

$$\begin{aligned} \mathbf{v}^{(0)}(t) &= \mathbf{v} + \frac{1}{m}\int_0^t dt' \; \mathbf{F_0}(t') = \mathbf{M}'(t)\,\mathbf{x} + \mathbf{N}'(t)\,\mathbf{v} \\ \mathbf{x}^{(0)}(t) &= \mathbf{x} + \int_0^t dt' \; \mathbf{v}(t') = \mathbf{M}(t)\,\mathbf{x} + \mathbf{N}(t)\,\mathbf{v} \end{aligned} \tag{10}$$

4 Statistical description

Liouville Equation:

$$\frac{\partial \rho}{\partial t} = L\,\rho \;= (L_R + L_\sigma + \lambda\, L_{int})\,\rho$$

The operators are defined as:

$$L_R = \sum_{n=1}^{N} L_n^{(0)} + \sum_{j<n}\sum_{n=1}^{N} L_{jn}\,, \qquad\qquad L_{int} = \sum_{n=1}^{N} L_{\sigma n} \tag{11}$$

- $L_j^{(0)}$: 1 particle Liouville operator *in the presence of the field*:

$$L_j^{(0)} = -\mathbf{v}_j \frac{\partial}{\partial \mathbf{x}_j} - \frac{1}{m_j}\mathbf{F}_j^{(0)} \frac{\partial}{\partial \mathbf{v}_j} \tag{12}$$

$(j = 1, 2, ..., N \; and \; \sigma)$,

- L_{ij}: *mutual interaction* term:

$$L_{ij} = \frac{\partial V(|\mathbf{x_i} - \mathbf{x_j}|)}{\partial \mathbf{x_i}} \left(\frac{1}{m_i}\frac{\partial}{\partial \mathbf{v_i}} - \frac{1}{m_j}\frac{\partial}{\partial \mathbf{v_j}}\right) \tag{13}$$

- The reservoir is in thermal equilibrium, i.e. $\partial_t\,\phi_R \;= L_R\,\phi_R \;= 0 \quad (\phi_R \;= \phi_{Maxwell})$.

4.1 Reduction of the Liouville Eq. – Perturbation theory – BBGKY hierarchy

1. We define *p*-particle *reduced distribution functions* (rpdf) f_p $(p = 1, 2, ..., N)$, e.g.

$$f_1 = \int d\mathit{\Gamma}_R \, \rho(\mathit{\Gamma}) \,, \qquad f_2 = \int d\mathit{\Gamma}^{c_{1,\sigma}} \, \rho(\mathit{\Gamma}) \,, \qquad ... \tag{14}$$

$(\Gamma^{c_{1,\sigma}} = \Gamma - \{\Gamma_1 \cup \Gamma_\sigma\}$, i.e. $\Gamma^{c_\sigma} = \Gamma - \Gamma_\sigma = \Gamma_R$, and so forth) ;

2. BBGKY hierarchy of equations for the rpdfs: integrating Eq. (1), we obtain a system of N coupled equations for f_p ;

3. We express the BBGKY hierarchy equations as a power series in λ;

4. Assuming that $\lambda \ll 1$, we keep only the lowest-order terms, up to λ^2, of the BBGKY hierarchy (truncation), and

5. we combine the first two members of the hierarchy, now decoupled from the rest, into a *closed* equation in terms of the rpdf $f = f_1$.

$$\left(\frac{\partial}{\partial t} - L_\sigma^{(0)}\right) f^\alpha = \lambda^2 \sum_{\alpha'} \int d^3\mathbf{x_1} \int d^3\mathbf{v_1}\, L_I\, g_{\alpha\alpha'} + \mathcal{O}(\lambda^3)$$

$$\left(\frac{\partial}{\partial t} - L_\sigma^{(0)} - L_1^{(0)}\right) g_{\alpha\alpha'} = \lambda\, L_I\, \phi_{eq}^{\alpha'}\, f^\alpha + \mathcal{O}(\lambda^2) \qquad (15)$$

- $g = f_2^{\alpha\alpha'} - \phi^{\alpha'} f^\alpha$: *correlation function.*

4.2 Collision term - Master Equation

$$\left(\frac{\partial}{\partial t} - L_\sigma^{(0)}\right) f^\alpha = \mathcal{K} = \sum_{\alpha'} n_{\alpha'} \int_0^t d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, L_I\, e^{L_0\,\tau}\, L_I\, \phi_{eq}^{\alpha'}(\mathbf{v_1})\, f^\alpha(\mathbf{x}, \mathbf{v}; t - \tau) \qquad (16)$$

Note the influence of:

(a) the interaction potential $V(r)$, through the operator $L_I = L_{1\sigma}$,

(b) the external (e.g. magnetic) field, via the operator $e^{L_0\,\tau} = e^{L_\sigma^{(0)}\,\tau}\, e^{L_1^{(0)}\,\tau}$,

(c) the previous *time history* of the function f (!): memory effect (*non-Markovian* Eq.).

5 A "pseudo-Markovian" approximation

— "*Markovianization*" hypothesis: $f(t-\tau) \approx e^{-L_0 \tau} f(t)$ & *asymptotic* limit: $t \to \infty$

$$\mathcal{K} \approx \sum_{\alpha'} n_{\alpha'} \int_0^{t\to\infty} d\tau \int d\mathbf{x_1} \int d\mathbf{v_1} \, L_I \, e^{L_0 \tau} \, L_I \, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \, f^{\alpha}(t) \equiv \Theta\{f\} \tag{17}$$

— Result: the PDE

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\Big[\mathbf{A}(\mathbf{v})\frac{\partial}{\partial \mathbf{v}} + \mathbf{G}(\mathbf{v})\frac{\partial}{\partial \mathbf{x}} + \frac{m}{m_1}\mathbf{a}(\mathbf{v})\Big] f \tag{18}$$

— for a spatially *homogeneous* plasma, i.e. $f = f(\mathbf{v}; t)$, one obtains:

$$\frac{\partial f}{\partial t} + m^{-1}\mathbf{F}^{(0)}\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i}(\mathcal{F}_i^{(V)} \, f) + \frac{\partial^2}{\partial v_i \partial v_j}(D_{ij} \, f) \tag{19}$$

i.e. a FOKKER-PLANCK (F.P.)-type *diffusion equation*;

— *drift* term:

$$\mathcal{F}_i^{(V)} = \Big(1 + \frac{m}{m_1}\Big)\frac{\partial D_{ij}}{\partial v_j} \tag{20}$$

= *Dynamical friction !*

In 1d, hence $D, \mathcal{F} \in \Re$ (in the absence of a field)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}(\mathcal{F}\ f) + \frac{\partial^2}{\partial v^2}(D\ f)\,.$$

5.1 Intermezzo: The Fokker-Planck eq. in the modelling of Brown motion

Basic form of the Fokker-Planck eq. (FPE) in 1d (in the absence of a force field)

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}(\eta\, v\ f) + D\frac{\partial^2 f}{\partial v^2}\ f \qquad (21)$$

i.e.

$$D = \eta\,\frac{k_B\,T}{m} = \text{const.} \qquad\qquad \mathcal{F} = -\eta\, v$$

($\eta = const. \in \Re$).

- Cf. A. Einstein/P. Langevin (Brown motion), Kramers (phase space dynamics), S. Chandrasekhar in Astronomy, etc.

5.2 Fokker-Planck Eq. in 6d phase space $\Gamma = \{\mathbf{x}, \mathbf{v}\}$

$$\frac{\partial f}{\partial t} \; + \; \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} \; + \; \frac{1}{m}\mathbf{F}_{\mathbf{ext}}\frac{\partial f}{\partial \mathbf{v}} \; = \; -\frac{\partial}{\partial q_i}(\mathcal{F}_i \; f) + \frac{\partial^2}{\partial q_i \partial q_j}(D_{ij} \; f) \equiv \Theta\{f\}$$

- 6×6 *diffusion matrix* $\mathbf{D}$, $6d$ friction vector:

$$\mathbf{D}^{\Theta}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{0} & \frac{1}{2}\mathbf{G}^T \\ \frac{1}{2}\mathbf{G} & \mathbf{A} \end{pmatrix} \qquad\qquad \vec{\mathcal{F}} = (\mathbf{0}, \vec{\mathcal{F}}^{(V)})^T$$

5.3 Mathematical properties of the kinetic evolution operator - the positivity issue

The d.f. f should remain, at all times (under the action of a kinetic evolution operator) (a) *real* ($f \in \Re$), (b) *normalized* ($\int f = 1$), and (c) *non negative* ($f \geq 0$) (def. *semi-group*); also: (d) (H-theorem) Monotonous convergence towards equilibrium.

$\longrightarrow$ Condition: Diffusion matrix *positive definite*: this criterion is *not* satisfied here!

6 An alternative approach: the Φ operator

– The quantum kinetic theory of open systems can “lend” us the operator:

$$\mathcal{A}_{t'} \cdot \, = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \, U^{(0)}(-t') \, \cdot \, U^{(0)}(t') \tag{22}$$

[E.B. Davies, *One-Parameter Semigroups* (1980); Davies (1974); Tzanakis (1988)]

– a Markovian operator: loss of the memory (non-locality) effect.

– (It has been proven that) the action of the Φ operator preserves the positivity of f!

7 Construction of the Φ operator for magnetized plasma

If $f = f(\mathbf{v}; t)$ (*homogeneous* plasma):

$$\frac{\partial f}{\partial t} + \frac{e}{mc} (\mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}} = \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\Big)\Big[D_\perp(\mathbf{v}) f\Big] + \frac{\partial^2}{\partial v_z^2}[D_\parallel(\mathbf{v}) f]$$
$$- \frac{\partial}{\partial v_x}\Big[\mathcal{F}_x(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_y}\Big[\mathcal{F}_y(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_z}\Big[\mathcal{F}_z(\mathbf{v})\, f\Big]$$

7.1 General form of the Φ kinetic operator: non-uniform plasma

- If $f = f(\mathbf{x}, \mathbf{v}; t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\frac{\partial f}{\partial \mathbf{v}} = \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\Big)\Big[D_\perp(\mathbf{v})f\Big] + \frac{\partial^2}{\partial v_z^2}[D_\parallel(\mathbf{v})f]$$

$$+ 2\,s\,\Omega^{-1}\Big[\frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x}\Big]\Big[D_\perp(\mathbf{v})f\Big] \quad + \Omega^{-2}\,D_\perp^{(XX)}(\mathbf{v})\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)f$$

$$-\frac{\partial}{\partial v_x}\Big[\mathcal{F}_x(\mathbf{v})\,f\Big] - \frac{\partial}{\partial v_y}\Big[\mathcal{F}_y(\mathbf{v})\,f\Big] - \frac{\partial}{\partial v_z}\Big[\mathcal{F}_z(\mathbf{v})\,f\Big]$$

$$+ s\,\Omega^{-1}\,\mathcal{F}_y(\mathbf{v})\,\frac{\partial}{\partial x}f - s\,\Omega^{-1}\,\mathcal{F}_x(\mathbf{v})\,\frac{\partial}{\partial y}f$$

- Terms in $\partial^2/\partial z \partial v_z$, $\partial^2/\partial z^2$ have been omitted.

- *New diffusion term* $\perp$ **B**, *new* diffusion $X - V$- term ($\sim \partial^2 f/\partial v_i \partial v_j$).

8 Coefficients in the FPE - relation to microscopic dynamics

8.1 General form

$$\left\{ \begin{array}{c} \mathbf{A}(\mathbf{x},\mathbf{v}) \\ \mathbf{G}(\mathbf{x},\mathbf{v}) \end{array} \right\} = \frac{n}{m^2} \int_0^{t\to\infty} d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1})$$

$$\mathbf{F_{int}}(|\mathbf{x}-\mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau)-\mathbf{x_1}(-\tau)|) \left\{ \begin{array}{c} \mathbf{N'}^T(\tau) \\ \mathbf{N}^T(\tau) \end{array} \right\}$$

$$= \frac{n}{m^2} \int_0^{\infty} d\tau\, \mathbf{C}(\mathbf{x},\mathbf{v};t,t-\tau) \left\{ \begin{array}{c} \mathbf{N'}^T(\tau) \\ \mathbf{N^T}(\tau) \end{array} \right\} \tag{23}$$

– Friction vector:

$$F_i = (1 + \frac{m}{m_1}) \frac{\partial D_{ij}}{\partial v_j} \tag{24}$$

– Time correlation functions $C_{ij}(\tau)$ for the interaction forces (Kubo coefficients).

8.2 Exact form of the diffusion coefficients for magnetized plasma

Working hypotheses: (i) R in a Maxwellian state; (ii) Debye type interaction potential $V(r)$:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = D_0 \, \Lambda \int_0^t d\tau' \int_1^{x_{max}} dx \; e^{\Lambda^2 \, (1-x^2) \, \sin^2 \frac{\tau'}{2}} \Big(1 - \frac{1}{x^2}\Big)^{\{1,0\}} \, e^{-\tilde{v}_\parallel^2}$$

$$J_O(2\Lambda \sqrt{x^2 - 1} \, \tilde{v}_\perp \, \sin \frac{\tau'}{2}) \; \tilde{F}_{\{\perp, \parallel\}} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \, \cos \tau' \\ (-s^\alpha) \, \frac{1}{2} \, \sin \tau' \\ (1 + \frac{1}{2} \cos \tau') \end{array} \right\} \\ 1 \end{array} \right\}$$

where

$$x \equiv \frac{\tilde{k}_\perp}{k_D} = \Big(1 + \frac{k_\perp^2}{k_D^2}\Big)^{1/2}, \qquad \tau' = \Omega\tau, \qquad D_0 \equiv \frac{2\sqrt{2}\, n\, e^4}{m^2\, \sqrt{k_B T}}$$

(Spitzer plasma collision frequency).

- The functions $\tilde{F} = \tilde{F}(\phi(x,\tau'), \tilde{v}_\parallel)$ are given by:

$$\tilde{F}^{\alpha'}_{\{\perp,\parallel\}} = \pm\sqrt{\pi}\,\phi + \frac{\pi}{4} \sum_{s=+1,-1} \Big[(1 \mp 2\,\phi^2 \mp s2\,\phi\,\tilde{v}_\parallel)\, e^{(\phi + s\,\tilde{v}_\parallel)^2}\, Erfc(\phi + s\,\tilde{v}_\parallel)\Big]\,,$$

where

$$\phi = \frac{1}{2}\,\Lambda\,\tau'\,x\,, \qquad \Lambda = \sqrt{2}\,\frac{\omega_p}{\Omega}\,, \qquad \tilde{v}_* = (\frac{m v_*^2}{2k_B T})^{1/2}\,, \qquad * \in \{\perp, \parallel\}$$

$$k_D = (\frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha})^{1/2} \qquad \omega_{p,\alpha} = (\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha})^{1/2}$$

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\,.$$

9 Diffusion coefficients: a parametric study

Considering an electron plasma characterized by

– a temperature $T = 10\,KeV$,

– density $n = 10^{14}\,cm^{-3} = 10^{20}\,m^{-3}$,

– plasma frequency $\omega_{p,e} = 5.64 \cdot 10^{11}\,s^{-1}$, and

– cyclotron frequency $\Omega_e = 1.76\,10^{11} \times B \quad s^{-1}$ (B in Tesla),

we have studied

the correlation function $C_\perp(\tau)$ vs. time τ, and

the coefficients $D_{\perp,\|}$ vs. the test-particle velocity.

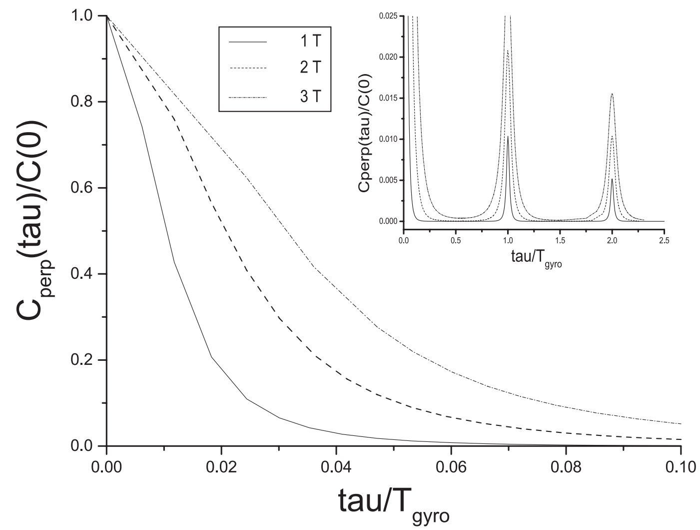

Figure 3: The transverse element of the interaction correlation function $C_{\perp}(\tau; v_{\perp}, v_{\parallel}, B)$ vs. time τ (in gyration periods $T_c = 2\pi/\Omega$), for different values of B ($\sim \Omega$). We have considered as typical values $v_{\perp} = v_{\parallel} = v_{th} = (T/m)^{1/2}$. Notice the peaks (attenuated) at every period.

– Field dependence.

– Space *confinement* due to the field - particles stick to their helicoidal trajectories.

– Velocity dependence.

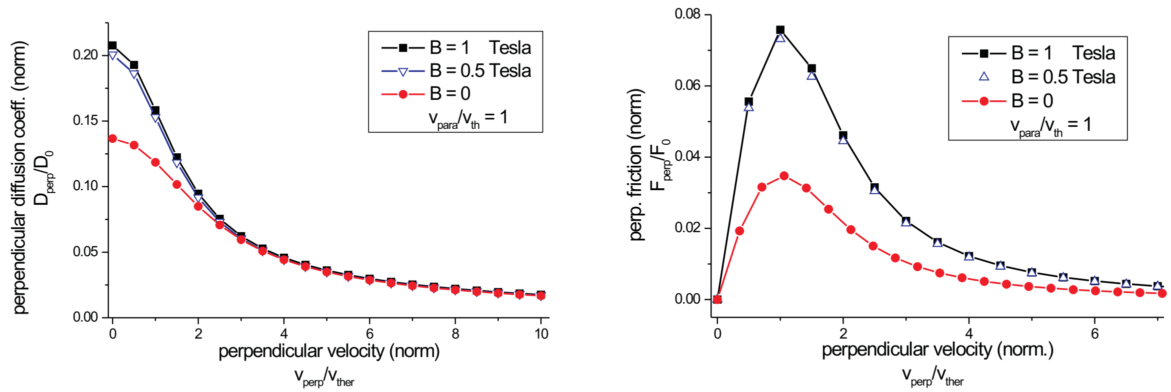

Figure 4: (a) The transverse diffusion coefficient $D_\perp$ and (b) the friction vector (norm) $\mathcal{F}_\perp$ (normalized) vs. the test-particle velocity $v_\perp$ ($\perp$ field) (scaled by the sound velocity) for a *magnetized electrostatic plasma.* All coefficients increase with the field.

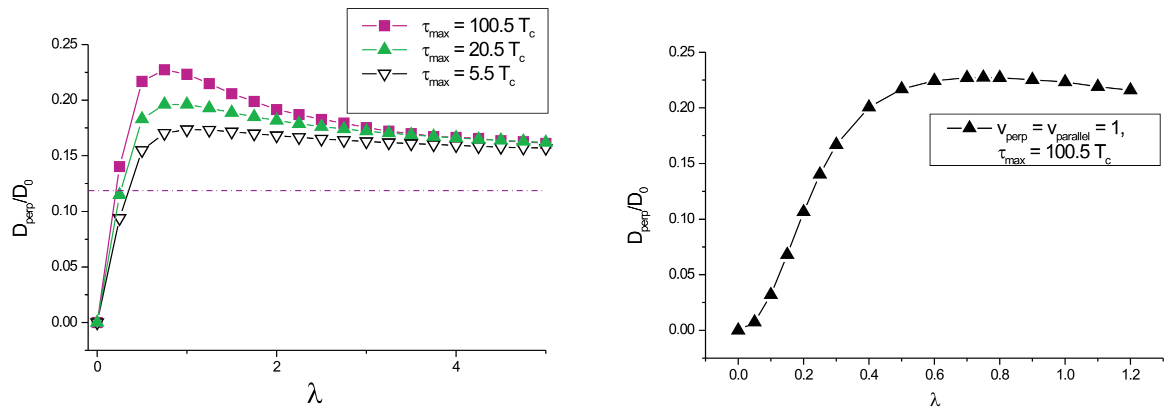

Figure 5: The transverse diffusion coefficient $D_\perp$ vs. the dimensionless parameter Λ ($\sim 1/\Omega$ - see def. above). The asymptotic value (dashed line) corresponds to the limit $\Lambda \to \infty$, i.e un-magnetized plasma ($\Lambda \to \infty$ implies $\Omega \to 0$). In (c), we have focused in the region near $\Lambda \approx 1$. We have taken $v_\perp = v_\parallel = v_{th} = (T/m)^{1/2}$. The different curves in (a) correspond to different values of the upper time integration limit t - cf. p. 23 above.

– for $\Lambda \gg 1$ (weak field): the *Landau* description is sufficient (for $\Omega \longrightarrow 0$).

– near $\Lambda \approx 1$ (important field, $\Omega \approx \omega_p$): strong field dependence of the collision term!

10 Conclusions

- Relying on first principles of Non-Equilibrium Statistical Mechanics, we have presented a method for the description of the (macroscopic) behavior of large (N particle) systems, as results from the microscopic laws of motion.

- We have focused on:

  1. the space dependence of the d.f, and
  2. the dependence of the collision term on the external field.

- We have shown that a widely adopted "Markovianization" hypothesis (the Θ operator) leads to erroneous (physically unacceptable) results.

- By adopting an alternative markovianization approach (the Φ operator), we have succeeded in deriving a correct FP-type kinetic equation for a t.p. in magnetized plasma.

- A numerical investigation has shown a strong dependence of $\mathcal{K}$ on the magnetic field.

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