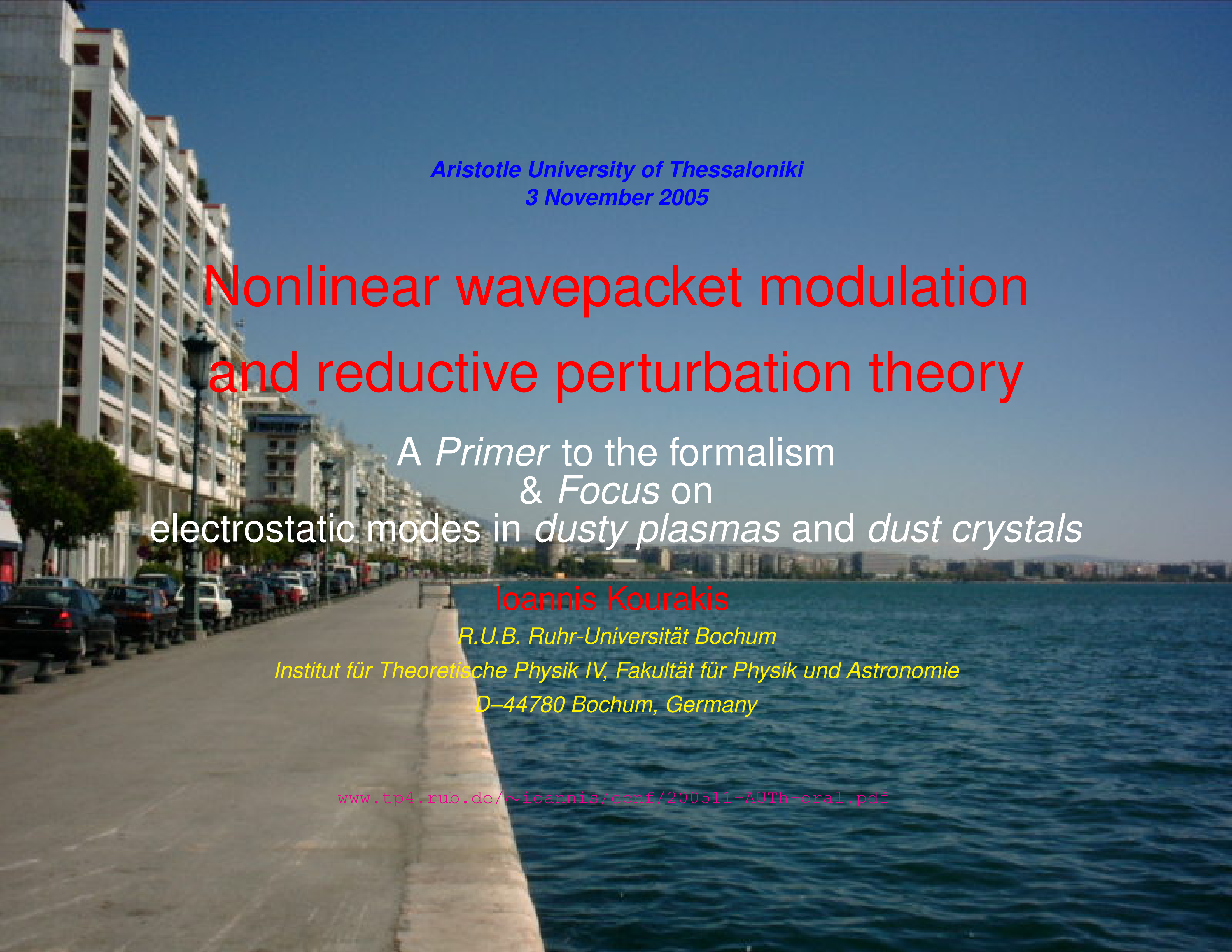

Aristotle University of Thessaloniki
3 November 2005

Nonlinear wavepacket modulation and reductive perturbation theory

A *Primer* to the formalism & *Focus* on electrostatic modes in *dusty plasmas* and *dust crystals*

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www.tp4.rub.de/~ioannis/conf/200511-AUTh-oral.pdf

Outline

- Introduction
  – *Amplitude Modulation*: a rapid overview of notions and ideas;
  – Relevance with space and laboratory observations;
  – *Intermezzo*: *Dusty Plasmas* (DP) (or *Complex Plasmas*) & dust crystals.

- Case study 1 (Part A): Fluid model for ES waves in *weakly-coupled* DP.
  – A pedagogical paradigm: Ion–acoustic plasma waves;
  – Other examples: EAWs, DAWs, ...

- The reductive perturbation *(multiple scales)* formalism.

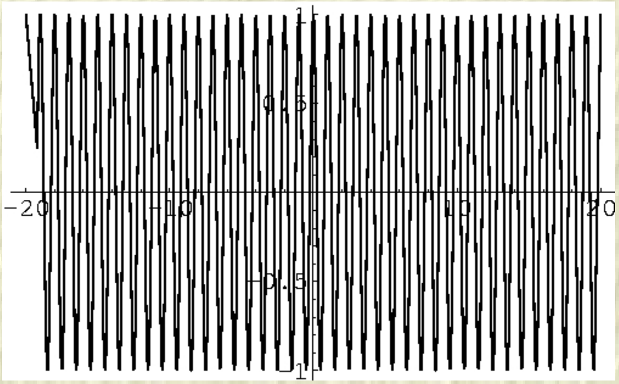
- *Modulational Instability (MI) & envelope excitations.*

- Case study 2 (Part B): Solitary excitations in dust (Debye) lattices.

- Conclusions.

1. Intro. The mechanism of *wave amplitude modulation*

The *amplitude* of a harmonic wave may vary in space and time:



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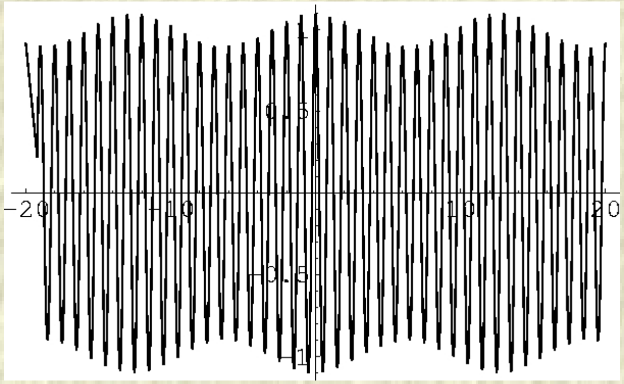

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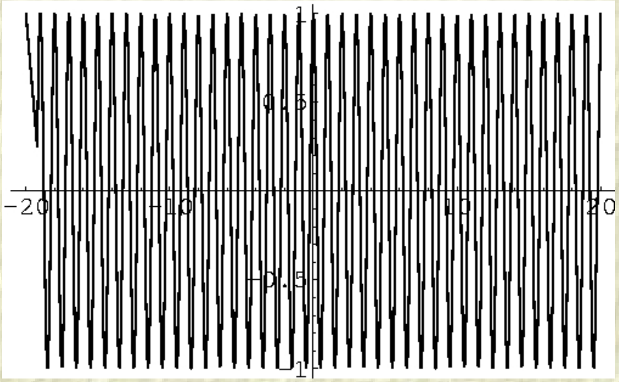

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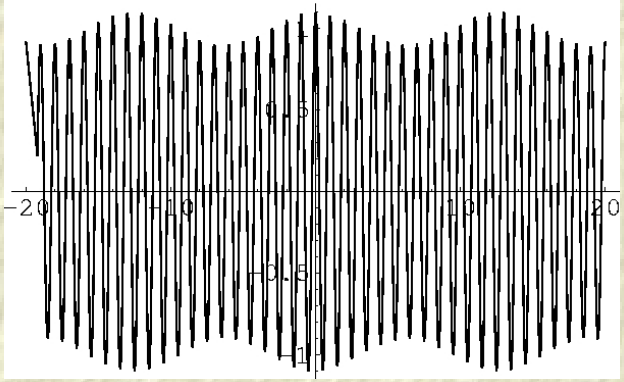
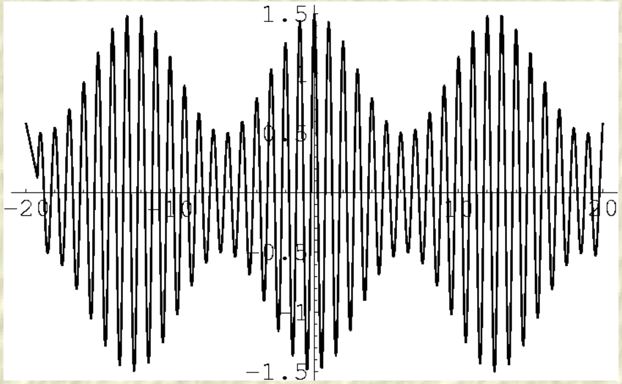


This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* (modulational instability) or ...



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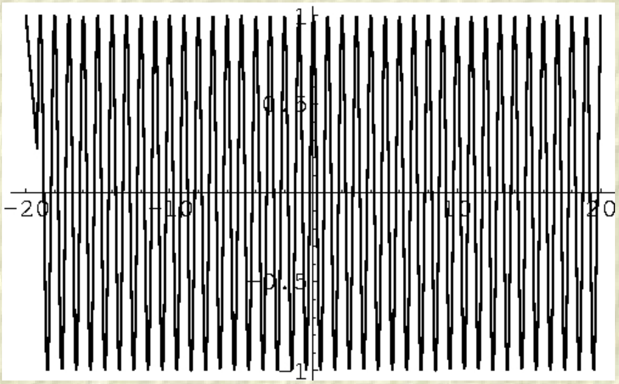

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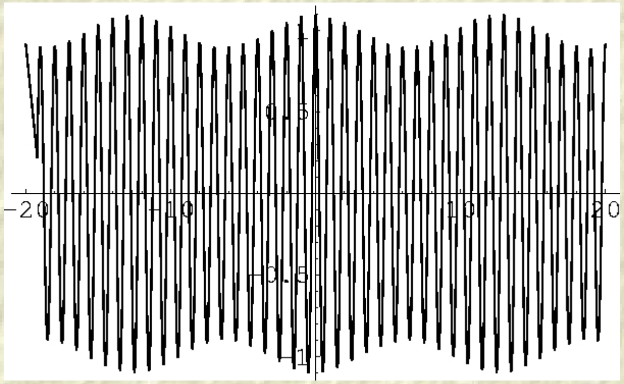
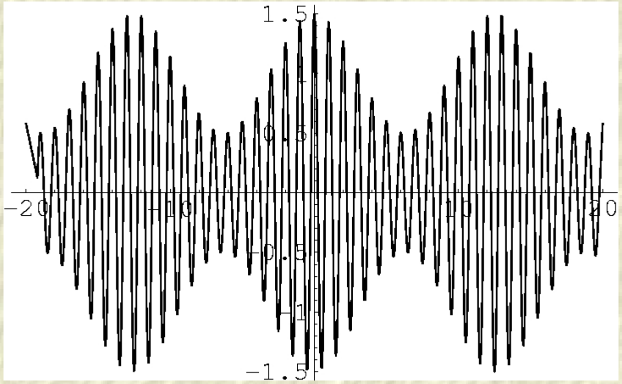


This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* (modulational instability) or to the formation of *envelope solitons*:



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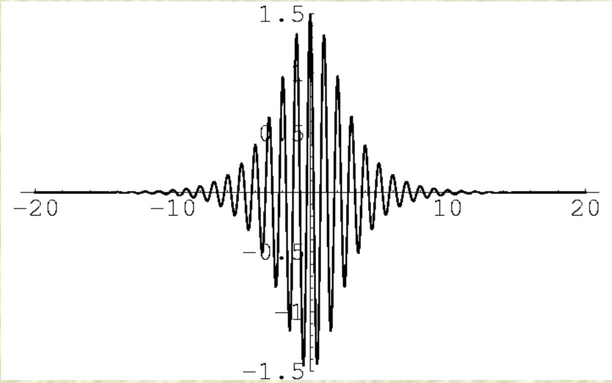


Modulated structures occur widely in Nature, e.g. in oceans (freak waves, or rogue waves) ...

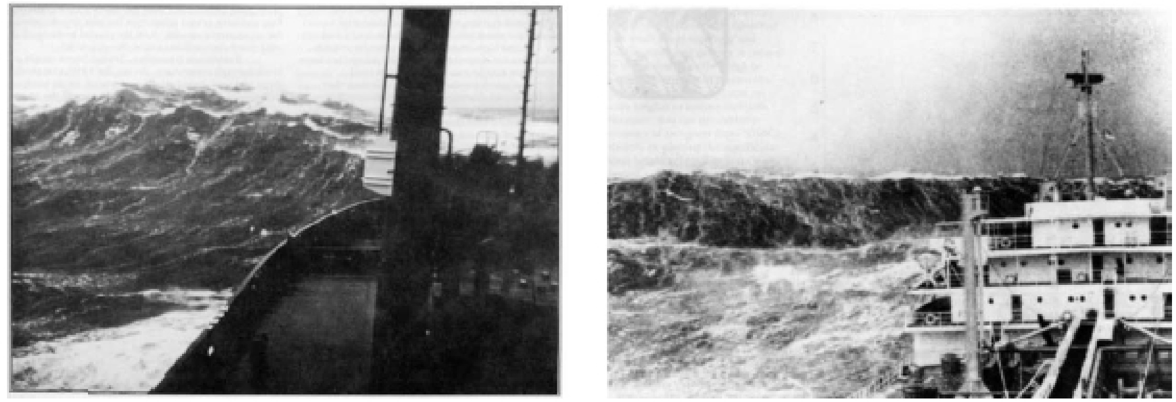


Fig. 2. Various photos of rogue waves.

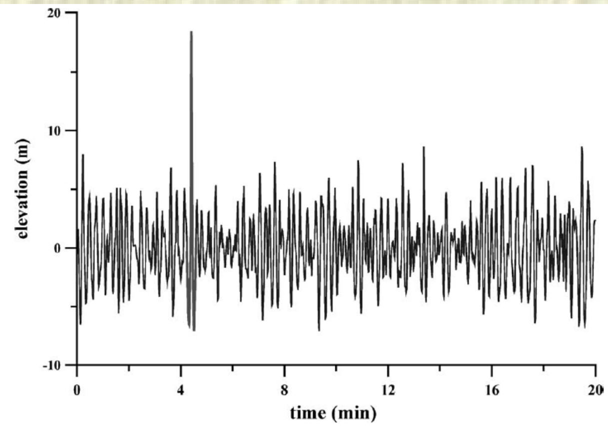


Fig. 3. Time record of the "New Year wave" in the North Sea.

(from: [Kharif & Pelinovsky, Eur. Journal of Mechanics B/Fluids **22**, 603 (2003)])

... during *freak wave* reconstitution in water basins, ...

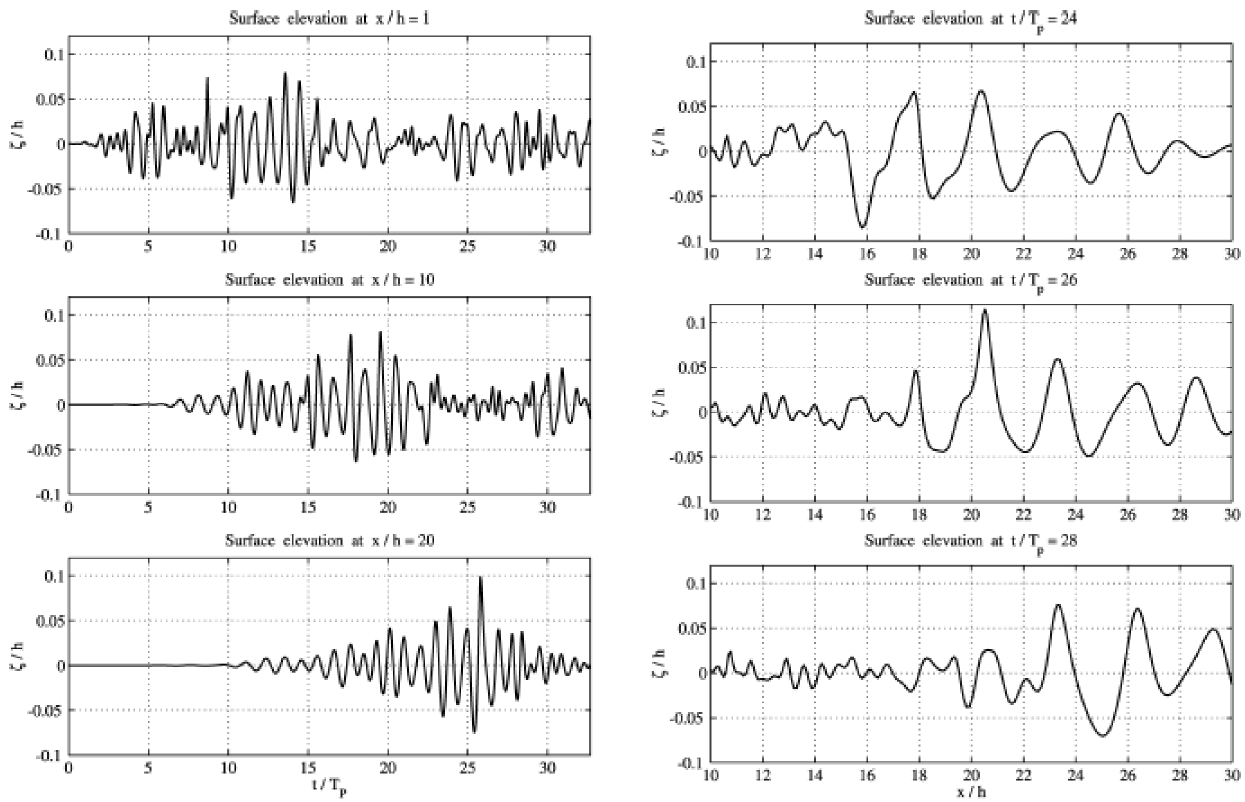
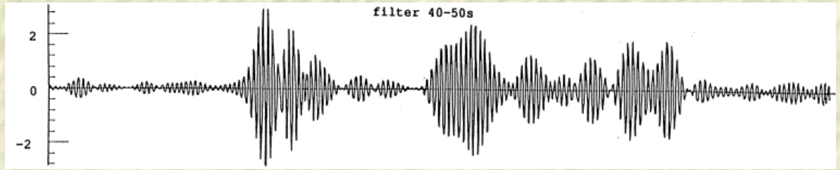
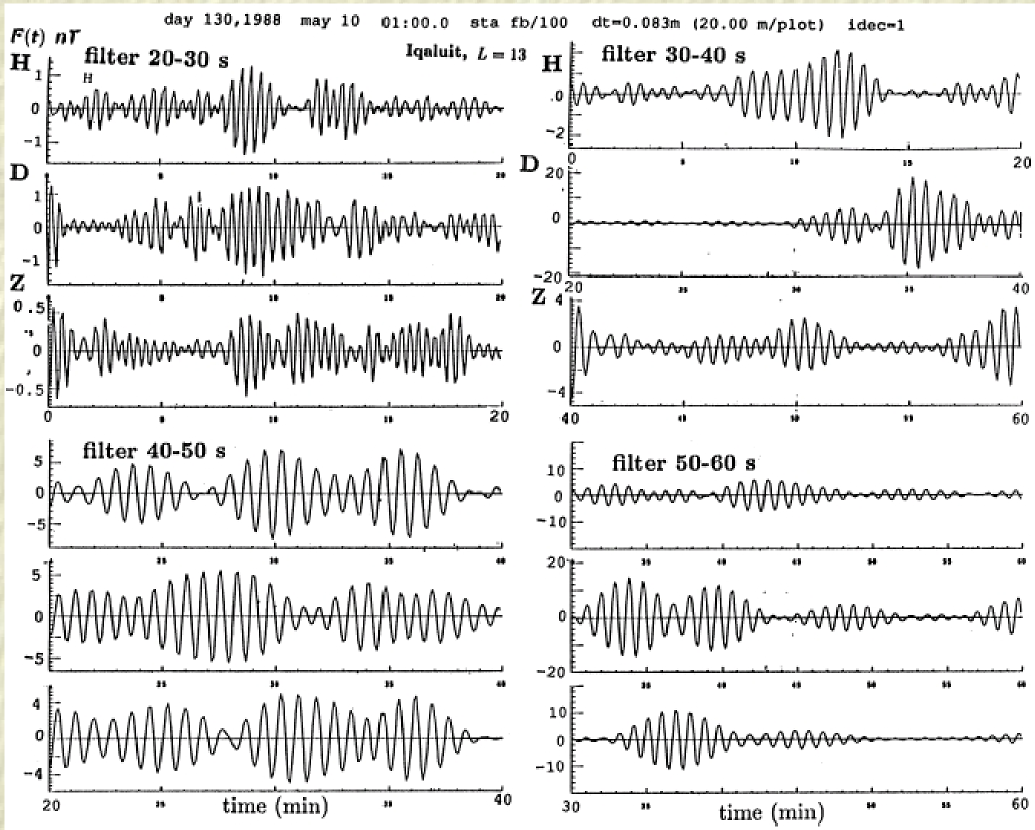


Fig. 16. Evolution of rogue wave sequence—registrations at $x = 5$ m, 50 m and 100 m (left) as well as wave profiles at $t = 75$ s, 81 s and 87 s (right hand side) (water depth $h = 5$ m, $T_p = 3.13$ s).

(from: [Klauss, Applied Ocean Research **24**, 147 (2002)])

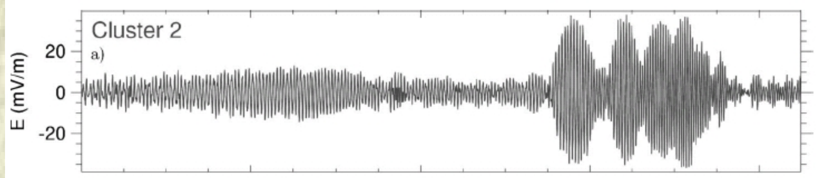
..., in EM field measurements in the magnetosphere, ...





(from: [Ya. Alpert, Phys. Reports **339**, 323 (2001)])

..., in satellite (e.g. CLUSTER, FAST, ...) observations:



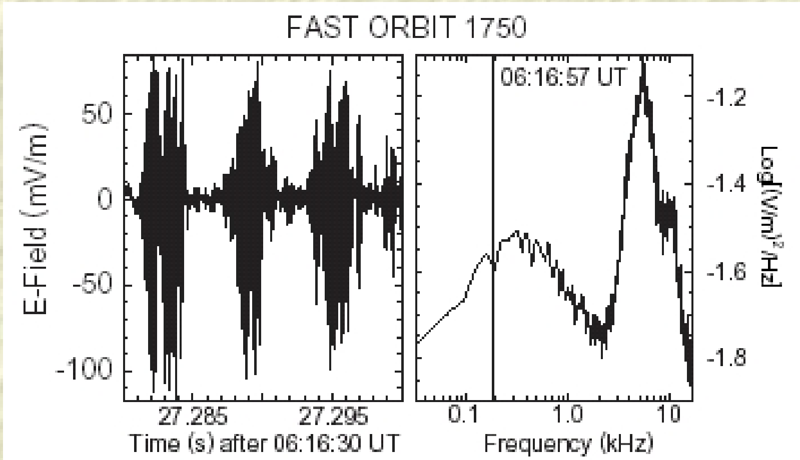


Figure 2. *Left*: Wave form of broadband noise at base of AKR source. The signal consists of highly coherent (nearly monochromatic frequency of trapped wave) wave packets. *Right*: Frequency spectrum of broadband noise showing the electron acoustic wave (at $\sim$ 5 kHz) and total plasma frequency (at $\sim$ 12 kHz) peaks. The broad LF maximum near 300 Hz belongs to the ion acoustic wave spectrum participating in the 3 ms modulation of the electron acoustic waves.

(*) From: O. Santolik *et al.*, *JGR* **108**, 1278 (2003); R. Pottelette *et al.*, *GRL* **26** 2629 (1999).

Modulational instability (MI) was observed in simulations,

e.g. early (1972) numerical experiments of EM cyclotron waves:

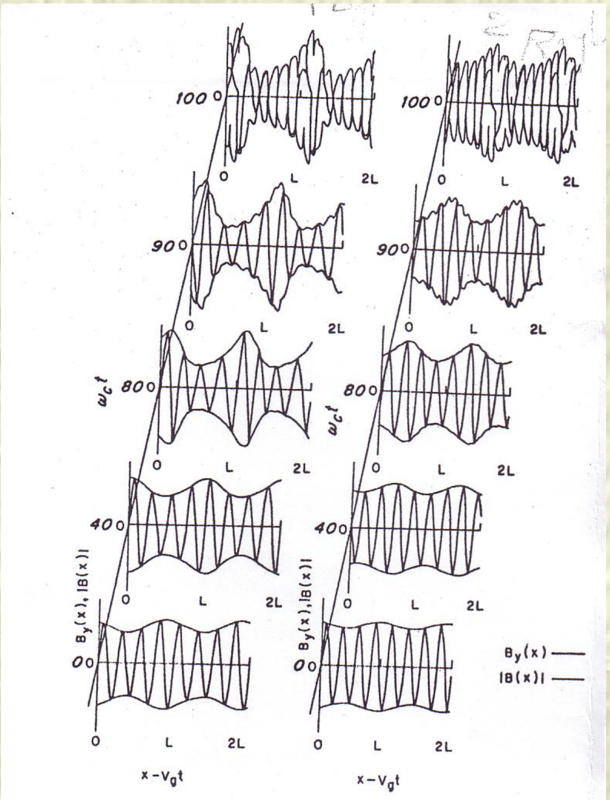


FIG. 2. Spatial variation of the y component of the wave magnetic field $B_y(x)$ (solid curve) and the magnitude of the total field $|B(x)|$ (thin curve) at different time of evolution for cases 1 (left) and 2 (right). Note the harmonic generation in $|B(x)|$ and the sudden collapse of the wave.

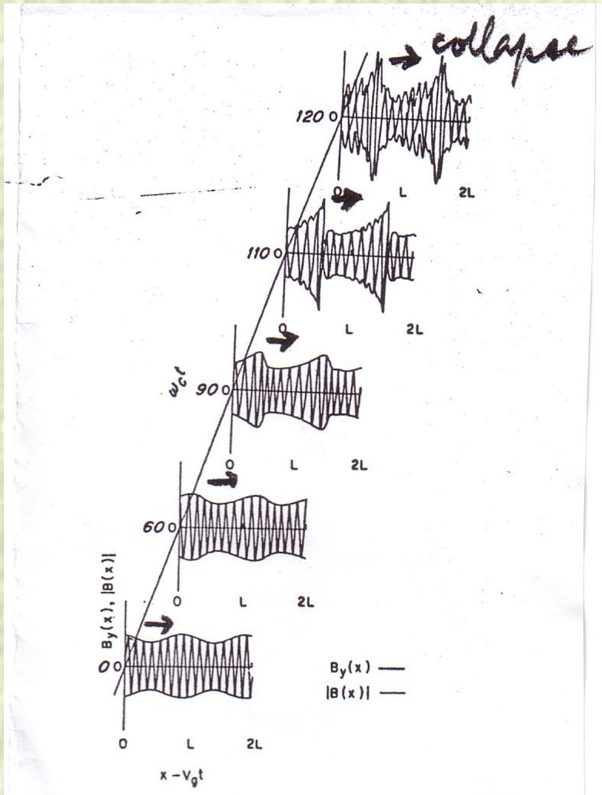


FIG. 3. $B_y(x)$ and $|B(x)|$ for case 4. Note the steepening of the wavefront which leads to the collapse of the wave.

[from: A. Hasegawa, *PRA* **1**, 1746 (1970); *Phys. Fluids* **15**, 870 (1972)].

Spontaneous MI has been observed in experiments,:

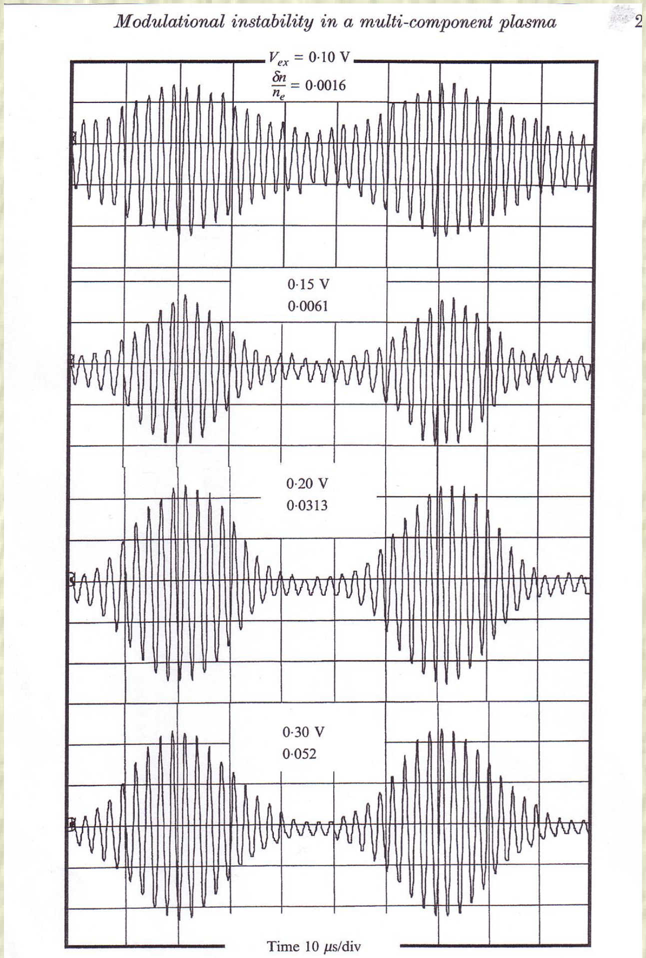


FIGURE 5. Oscilloscope traces of the detected signal for different excitation voltages. The probe was fixed at 14 cm from the grid. f_c = 400 kHz and f_m = 50 kHz.

e.g. on *ion acoustic waves*

[from: Bailung and Nakamura, *J. Plasma Phys.* **50** (2), 231 (1993)].

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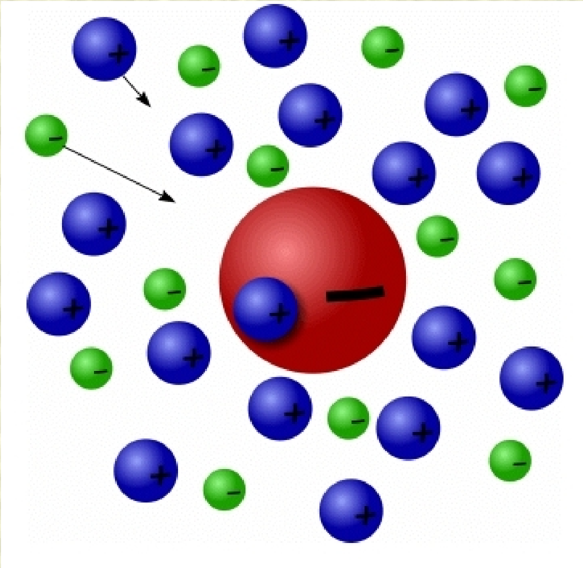
- *Focus*: *electrostatic* waves; e.g. *dust-ion acoustic* waves (DIAW); *electron acoustic* (EA), *dust acoustic* (DA) waves, ... also (*part B*): dust-lattice waves in Debye crystals.

Intermezzo: DP – Dusty Plasmas (or *Complex Plasmas*): definition and characteristics of a focus issue



❏ Ingredients:

– *electrons* e^- (charge $-e$, mass m_e),
– *ions* i^+ (charge $+Z_i e$, mass m_i), *and*
– charged micro-particles $\equiv$ *dust grains* d (most often d^-):

charge $Q = \pm Z_d e \sim \pm(10^3 - 10^4)\, e$,
mass $M \sim 10^9\, m_p \sim 10^{13}\, m_e$,
radius $r\ \sim\ 10^{-2}\, \mu m$ up to $10^2\, \mu$m.

Origin: Where does the dust come from?

- *Space:* cosmic debris (silicates, graphite, amorphous carbon), comet dust, man-made pollution (Shuttle exhaust, satellite remnants), ...

- *Atmosphere:* extraterrestrial dust (meteorites): $\geq 2 \cdot 10^4$ tons a year (!)(*), atmospheric pollution, chemical aerosols, ...

- *Fusion reactors:* plasma-surface interaction, carbonaceous particulates resulting from wall erosion-created debris (graphite, CFCs: Carbon Fiber Composites, ...)

- *Laboratory:* (man-injected) melamine–formaldehyde particulates (**) injected in *rf* or *dc* discharges; 3d ($=$ multiple 2d layers) or 1d (by appropriate experimental setting) crystals.

Sources: [P. K. Shukla & A. Mamun 2002], (*) [DeAngelis 1992], (**) [G. E. Morfill *et al.* 1998]

Some unique features of *the Physics of* Dusty Plasmas:

- Complex plasmas are *overall charge neutral*; most (sometimes *all*!) of the negative charge resides on the microparticles;

- The microparticles can be *dynamically dominant*: mass density $\approx 10^2$ times higher than the neutral gas density and $\approx 10^6$ times higher than the ion density !

- Studies in *slow motion* are possible due to high M i.e. *low Q/M ratio* (e.g. *dust plasma frequency*: $\omega_{p,d} \approx 10 - 100\,\mathrm{Hz}$);

- The (large) microparticles can be *visualised* individually and studied at the kinetic level (with a digital camera!) $\rightarrow$ video;

- Dust charge ($Q \neq \mathrm{const.}$) is now a dynamical variable, associated to a new *collisionless damping* mechanism;

(...continued) More “heretical” features are:

- Important *gravitational* (compared to the *electrostatic*) interaction effects; gravito-plasma physics; gravito-electrodynamics; Jeans-type (gravitational) plasma instabilities etc. [Verheest PPCF 41 A445, 1999]

- Complex plasmas can be *strongly coupled* and exist in *“liquid”* $(1 < \Gamma < 170)$ and *“crystalline”* $(\Gamma > 170$ [IKEZI 1986]) states, depending on the value of the *effective coupling (plasma) parameter* Γ;

$$\Gamma_{eff} = \frac{< E_{potential} >}{< E_{kinetic} >} \sim \frac{Q^2}{r\,T}\, e^{-r/\lambda_D}$$

(r: inter-particle distance, T: temperature, λ_D: Debye length).

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– DAW: dust grains ($\alpha = d$) against *thermalized* electrons and ions ($\alpha' = e, i$): $n_e = n_{e,0}\, e^{e\Phi/K_B T_e}$, $n_i = n_{i,0}\, e^{-Z_i e\Phi/K_B T_i}$.

Fluid moment equations:

Density n_α (*continuity*) equation:

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[(*) *Cold* fluid model]

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Pressure p_α equation: [(*) *Cold* vs. *Warm* fluid model]

$$\frac{\partial p_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla p_\alpha = -\gamma\, p_\alpha \nabla \cdot \mathbf{u}_\alpha$$

[$\gamma = (f+2)/f = c_P/c_V$: ratio of specific heats e.g. $\gamma = 3$ for 1d, $\gamma = 2$ for 2d, etc.].

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The potential Φ obeys *Poisson's* eq.:

$$\nabla^2 \Phi \; = \; -4\pi \sum_{\alpha''=\alpha, \{\alpha'\}} q_{\alpha''} \, n_{\alpha''} \; = \; 4\pi \, e \, (n_e - Z_i \, n_i + ...)$$

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Hypothesis: Overall charge *neutrality* at equilibrium:

$$q_\alpha \, n_{\alpha,0} = - \sum_{\{\alpha'\}} q_{\alpha'} \, n_{\alpha',0} \, ,$$

e.g. for DIAW: $n_{e,0} - Z_i \, n_{i,0} - s \, Z_d \, n_{d,0} = 0$ $\quad (s = q_d/|q_d| = \pm 1)$.

Reduced moment evolution equations:

Defining appropriate scales *(see next slide)* one obtains:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{u}) = 0\,,$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -s\,\nabla\phi - \frac{\sigma}{n}\nabla p\,,$$
$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p \;=\; -\gamma\, p\, \nabla \cdot \mathbf{u}\,;$$

also,

$$\nabla^2\phi \;=\; \phi - \alpha\,\phi^2 + \alpha'\,\phi^3 - s\,\beta\,(n-1)\,; \qquad (1)$$

i.e. *Poisson's Eq. close to equilibrium*: $\phi \ll 1$; $s = \mathrm{sgn} q_\alpha = \pm 1$.

- The dimensionless parameters α, α' and β must be determined exactly for any specific problem. They incorporate all the essential dependence on the plasma parameters.

We have defined the reduced (dimensionless) quantities:

- *particle density*: $n = n_\alpha / n_{\alpha,0}$;
- *mean (fluid) velocity*: $\mathbf{u} = [m_\alpha/(k_B T_*)]^{1/2} \mathbf{u}_\alpha \equiv \mathbf{u}_\alpha / c_*$;

where $c_* = (k_B T_*/m_\alpha)^{1/2}$ is a characteristic *"sound" velocity*$^{(*)}$;

- *dust pressure*: $p = p_\alpha/p_0 = p_\alpha/(n_{\alpha,0} k_B T_*)$;
- *electric potential*: $\phi = Z_\alpha e \Phi/(k_B T_*) = |q_\alpha| \Phi/(k_B T_*)$;

Also, *time* and *space* are scaled over:

- a characteristic time scale t_0, e.g. the inverse *DP plasma frequency*

$$\omega_{p,\alpha}^{-1} = (4\pi n_{\alpha,0} q_\alpha^2 / m_\alpha)^{-1/2}$$

- a characteristic length scale $r_0 = c_* t_0$, e.g. an *effective Debye length*

$$\lambda_{D,eff} = (k_B T_* / 4\pi n_{\alpha,0} q_\alpha^2)^{1/2} .$$

Finally, $\sigma = T_\alpha / T_*$ is the temperature (ratio)

$^{(*)}$ e.g. $T_* = T_e$ for DIAWs.

3. Reductive Perturbation Technique

– 1st step. Define *multiple scales* (*fast* and *slow*) i.e. (in 2d)

$$X_0 = x\,, \quad X_1 = \epsilon\, x\,, \quad X_2 = \epsilon^2\, x\,, \quad \ldots$$
$$Y_0 = y\,, \quad Y_1 = \epsilon\, y\,, \quad Y_2 = \epsilon^2\, y\,, \quad \ldots$$
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and modify operators appropriately:

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial X_0} + \epsilon \frac{\partial}{\partial X_1} + \epsilon^2 \frac{\partial}{\partial X_2} + \quad \ldots$$

$$\frac{\partial}{\partial y} \to \frac{\partial}{\partial Y_0} + \epsilon \frac{\partial}{\partial Y_1} + \epsilon^2 \frac{\partial}{\partial Y_2} + \quad \ldots$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \quad \ldots$$

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$$T_0 = t\,, \quad T_1 = \epsilon\, t\,, \quad T_2 = \epsilon^2\, t\,, \quad \ldots$$

– 2nd step. Expand near equilibrium:

$$n_\alpha \approx n_{\alpha,0} + \epsilon\, n_{\alpha,1} + \epsilon^2\, n_{\alpha,2} + \ldots$$

$$\mathbf{u}_\alpha \approx \mathbf{0} + \epsilon\, \mathbf{u}_{\alpha,\mathbf{1}} + \epsilon^2\, \mathbf{u}_{\alpha,\mathbf{2}} + \ldots$$

$$p_\alpha \approx p_{\alpha,0} + \epsilon\, p_{\alpha,1} + \epsilon^2\, p_{\alpha,2} + \ldots$$

$$\phi \approx 0 + \epsilon\, \phi_1 + \epsilon^2\, \phi_2 + \ldots$$

($p_{\alpha,0} = n_{\alpha,0} k_B T_\alpha$; $\epsilon \ll 1$ is a *smallness parameter*).

Reductive perturbation technique *(continued)*

– 3rd step. Project on Fourier space, i.e. consider $\forall m = 1, 2, ...$

$$S_m = \sum_{l=-m}^{m} \hat{S}_l^{(m)} \, e^{il(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \hat{S}_0^{(m)} + 2\sum_{l=1}^{m} \hat{S}_l^{(m)} \cos l(\mathbf{k}\cdot\mathbf{r}-\omega t)$$

for $S_m = (n_m, \{u_{x,m}, u_{y,m}\}, p_m, \phi_m)$, i.e. *essentially*:

$$n_1 = n_0^{(1)} + \tilde{n}_1^{(1)} \cos\theta \,, \qquad n_2 = n_0^{(2)} + \tilde{n}_1^{(2)} \cos\theta + \tilde{n}_2^{(2)} \cos 2\theta, \text{ etc.}$$

Reductive perturbation technique *(continued)*

– 3rd step. Project on Fourier space, i.e. consider $\forall m = 1, 2, ...$

$$S_m = \sum_{l=-m}^{m} \hat{S}_l^{(m)} e^{il(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \hat{S}_0^{(m)} + 2\sum_{l=1}^{m} \hat{S}_l^{(m)} \cos l(\mathbf{k}\cdot\mathbf{r}-\omega t)$$

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– 4rth step. (for multi-dimensional propagation) *Modulation obliqueness*: the slow amplitudes $\hat{\phi}_l^{(m)}$, etc. vary *only along* the x-axis:

$$\hat{S}_l^{(m)} = \hat{S}_l^{(m)}(X_j, T_j)\,, \qquad j = 1, 2, ...$$

while the fast carrier phase $\theta = \mathbf{k}\cdot\mathbf{r} - \omega t$ is now:

$$k_x\, x + k_y\, y - \omega t = k\, r \cos\alpha - \omega t \; .$$

First-order solution ($\sim \epsilon^1$)

Substituting in the model Eqs., and isolating terms in $m = 1$, we obtain:

❑ The *dispersion relation* $\omega = \omega(k)$:

$$\omega^2 = \frac{\beta k^2}{k^2 + 1} + \gamma\, \sigma\, k^2 \qquad (2)$$

e.g. for DIAWs

$$\omega \approx \left(\frac{n_{i,0}}{n_{e,0}}\right)^{1/2} \left(\frac{k_B T_e}{m_i}\right)^{1/2} k = \left(1 - s Z_d \frac{n_{d,0}}{n_{e,0}}\right)^{1/2} \left(\frac{k_B T_e}{m_i}\right)^{1/2} k$$

❑ The *solution(s)* for the 1st–harmonic amplitudes (e.g. $\propto \phi_1^{(1)}$):

$$n_1^{(1)} = s\,\frac{1 + k^2}{\beta} \phi_1^{(1)} = \frac{1}{\gamma}\, p_1^{(1)} = \frac{k}{\omega\, \cos\theta}\, u_{1,x}^{(1)} = \frac{k}{\omega\, \sin\theta}\, u_{1,y}^{(1)} \qquad (3)$$

Second-order solution ($\sim \epsilon^2$)

- From $m = 2, l = 1$, we obtain the relation:

$$\frac{\partial \psi}{\partial T_1} + v_g \frac{\partial \psi}{\partial X_1} = 0 \tag{4}$$

where
– $\psi = \phi_1^{(1)}$ is the potential correction ($\sim \epsilon^1$);

– $v_g = \frac{\partial \omega(k)}{\partial k_x}$ is the group velocity along $\hat{x}$;

– the wave's envelope satisfies: $\psi = \psi(\epsilon(x - v_g t)) \equiv \psi(\zeta)$.

- The solution, up to $\sim \epsilon^2$, is of the form:

$$\phi \approx \epsilon\, \psi \cos\theta + \epsilon^2 \left[\phi_0^{(2)} + \phi_1^{(2)} \cos\theta + \phi_2^{(2)} \cos 2\theta\right] + \mathcal{O}(\epsilon^3)\,,$$

etc. (+ similar expressions for n_d, u_x, u_y, p_d): $\rightarrow$ *Fourier harmonics!*.

Third-order solution ($\sim \epsilon^3$)

- Compatibility equation (from $m = 3, l = 1$), in the form of:

$$i\,\frac{\partial \psi}{\partial \tau} + P\,\frac{\partial^2 \psi}{\partial \zeta^2} + Q\,|\psi|^2\,\psi = 0\,. \tag{5}$$

i.e. a *Nonlinear Schrödinger–type Equation* (NLSE) .

- Variables: $\zeta = \epsilon(x - v_g t)$ and $\tau = \epsilon^2\,t$;

- *Dispersion coefficient* P:

$$P = \frac{1}{2}\frac{\partial^2 \omega}{\partial k_x^2} = \frac{1}{2}\left[\omega''(k)\cos^2\alpha + \omega'(k)\,\frac{\sin^2\alpha}{k}\right]; \tag{6}$$

- *Nonlinearity coefficient* Q: ...
  A *(lengthy!)* function of k, angle α and T_e, T_i, ... $\rightarrow$ *(omitted)*.

4a. Modulational (in)stability analysis

- The NLSE admits the *harmonic wave solution*:

$$\psi = \hat{\psi}\, e^{iQ|\hat{\psi}|^2\tau} + \text{c.c.}$$

- *Perturb* the amplitude by setting: $\hat{\psi} = \hat{\psi}_0 + \epsilon\, \hat{\psi}_{1,0} \cos(\tilde{k}\zeta - \tilde{\omega}\tau)$

- We obtain the *(perturbation)* dispersion relation:

$$\tilde{\omega}^2 = P^2\, \tilde{k}^2 \left(\tilde{k}^2 - 2\frac{Q}{P}|\hat{\psi}_{1,0}|^2\right). \qquad (7)$$

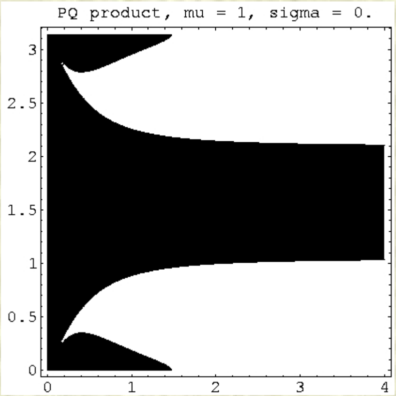
- If $PQ < 0$: the amplitude ψ is *stable* to external perturbations;

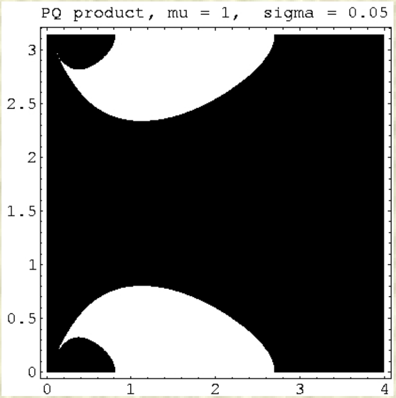
- If $PQ > 0$: the amplitude ψ is *unstable* for $\tilde{k} < \sqrt{2\frac{Q}{P}}|\psi_{1,0}|$.

Stability profile (IAW): Angle α versus wavenumber k

Typical values: $Z_i = +1$ (hydrogen plasma), $\gamma = 2$.

– *Ion-acoustic waves*; cold ($\sigma = 0$) vs. warm ($\sigma \neq 0$) fluid:





Stability profile (IAW): Angle α versus wavenumber k

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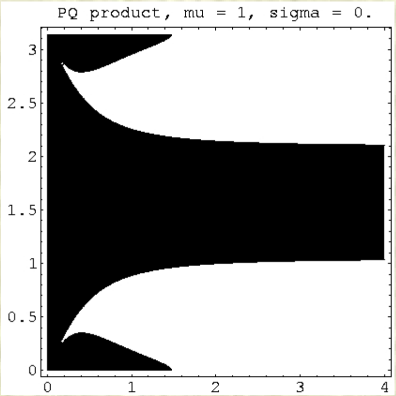
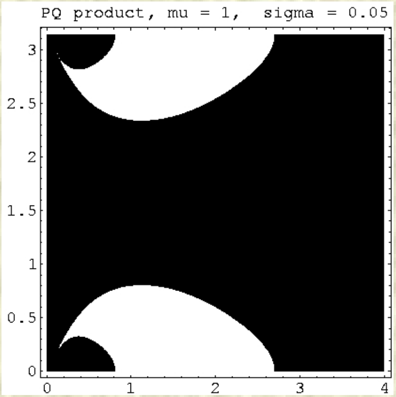
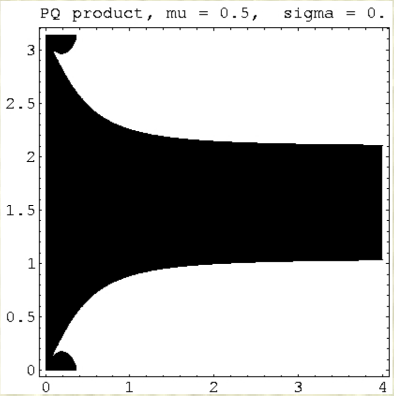
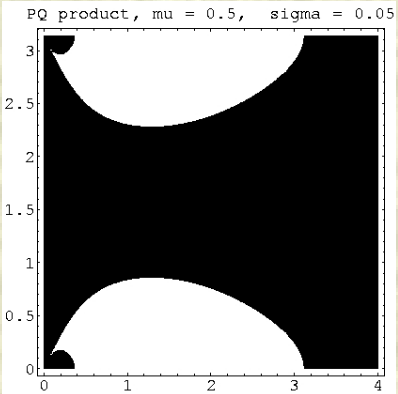




– *Dust-ion acoustic waves*, i.e. in the presence of *negative dust* ($n_{d,0}/n_{i,0} = 0.5$):





Stability profile (IAW): Angle α versus wavenumber k

Typical values: $Z_i = +1$ (hydrogen plasma), $\gamma = 2$.

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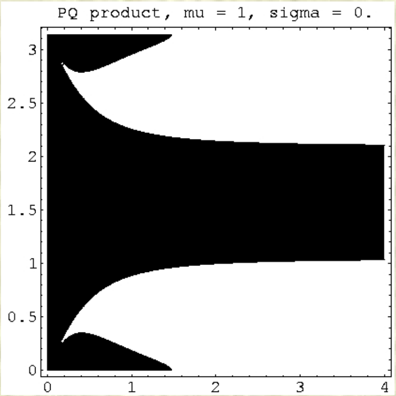
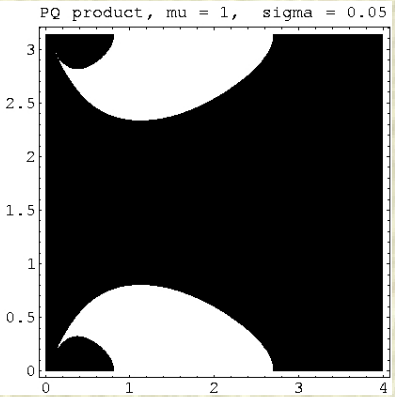
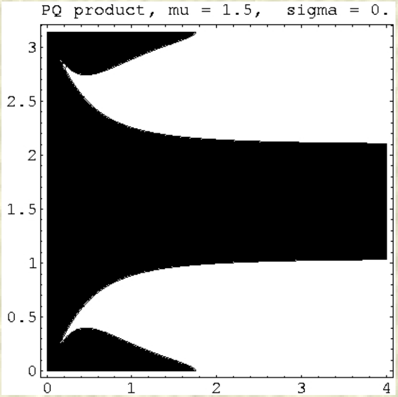
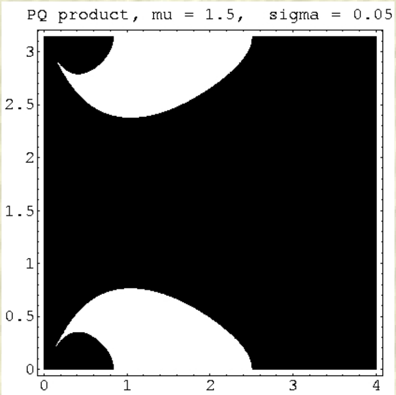




– *Dust-ion acoustic waves*, i.e. in the presence of *positive dust* ($n_{d,0}/n_{i,0} = 0.5$):

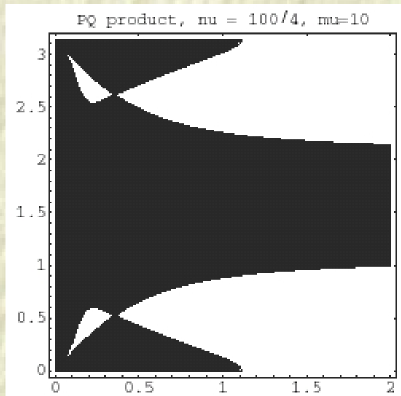


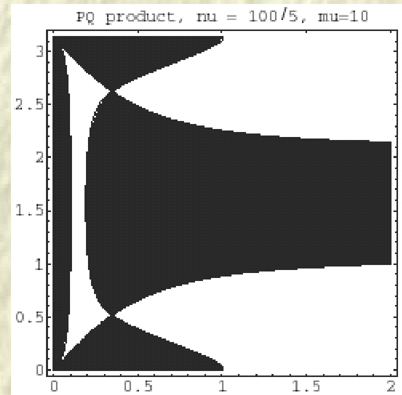


Stability profile (IAW/EAW): Angle α versus wavenumber k

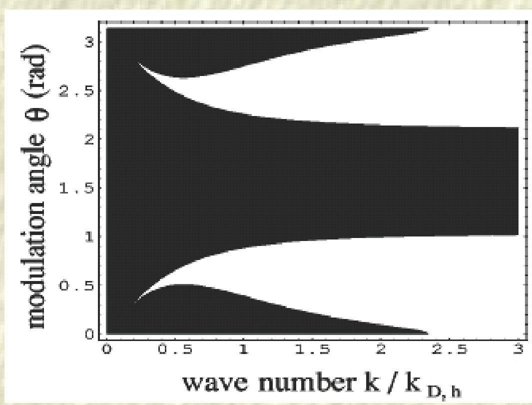
Typical values: $Z_i = +1$ (hydrogen plasma), $\gamma = 2$.

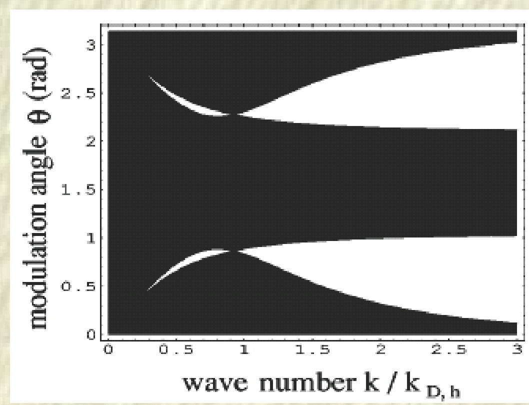
– *Ion acoustic waves*, in the presence of *2 electron populations*:





– *Electron acoustic waves (+ cold electrons)*:

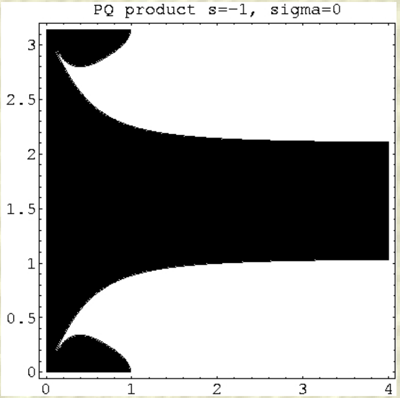


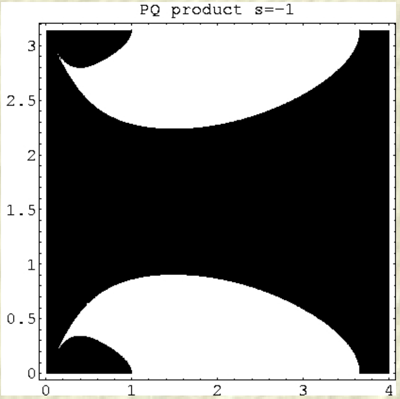


Stability profile (DAW): Angle α versus wavenumber k

Typical values: $Z_d/Z_i \approx 10^3$, $T_e/T_i \approx 10$, $n_{d,0}/n_{i,0} \approx 10^{-3}$, $\gamma = 2$.

– *Negative dust*: $s = -1$; cold $(\sigma = 0)$ vs. warm $(\sigma \neq 0)$ fluid:

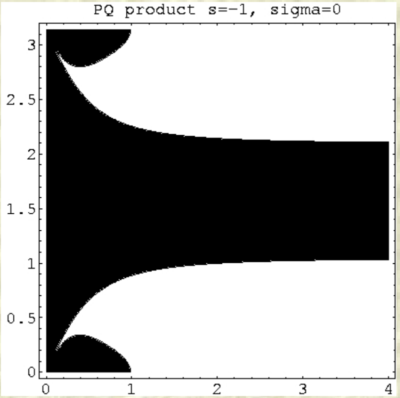


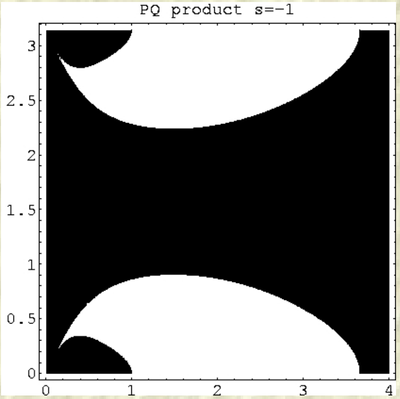


Stability profile (DAW): Angle α versus wavenumber k

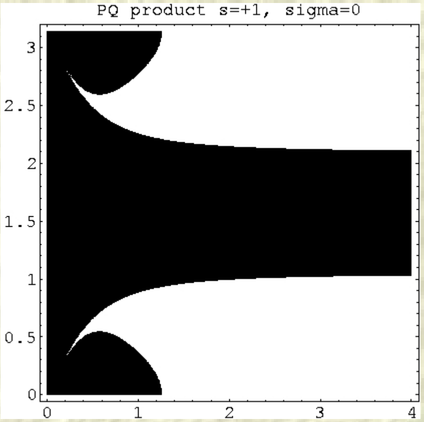
Typical values: $Z_d/Z_i \approx 10^3$, $T_e/T_i \approx 10$, $n_{d,0}/n_{i,0} \approx 10^{-3}$, $\gamma = 2$.

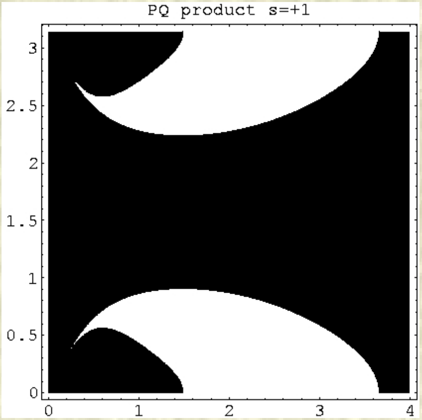
– *Negative dust*: $s = -1$; cold $(\sigma = 0)$ vs. warm $(\sigma \neq 0)$ fluid:





– The same plot for *positive dust* $(s = +1)$:





4b. Localized envelope excitations (solitons)

- The NLSE:

$$i\,\frac{\partial\psi}{\partial\tau} + P\,\frac{\partial^2\psi}{\partial\zeta^2} + Q\,|\psi|^2\,\psi = 0$$

accepts various solutions in the form: $\psi = \rho\, e^{i\Theta}$;
the *total* electric potential is then: $\phi \approx \epsilon\,\rho\,\cos(\mathbf{kr} - \omega t + \Theta)$ where the amplitude ρ and phase correction Θ depend on ζ, τ.

4b. Localized envelope excitations (solitons)

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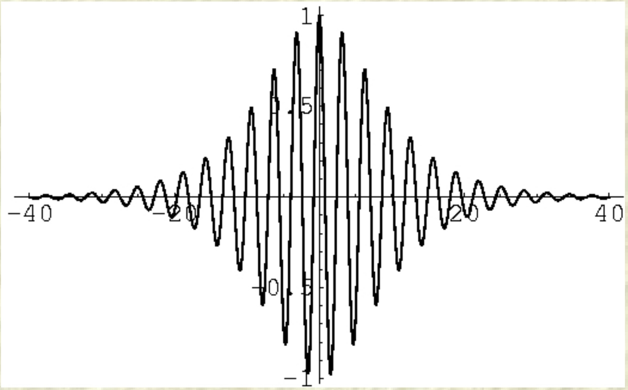
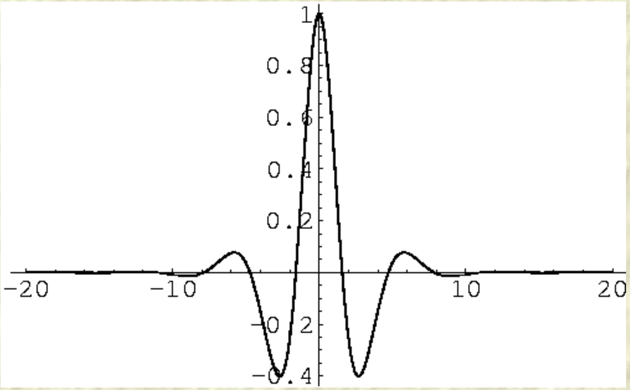
- Bright–type envelope soliton (pulse):

$$\rho = \rho_0\, \mathrm{sech}\left(\frac{\zeta - v\,\tau}{L}\right), \qquad \Theta = \frac{1}{2P}\left[v\,\zeta\, - (\Omega + \frac{1}{2}v^2)\tau\right]. \qquad (8)$$

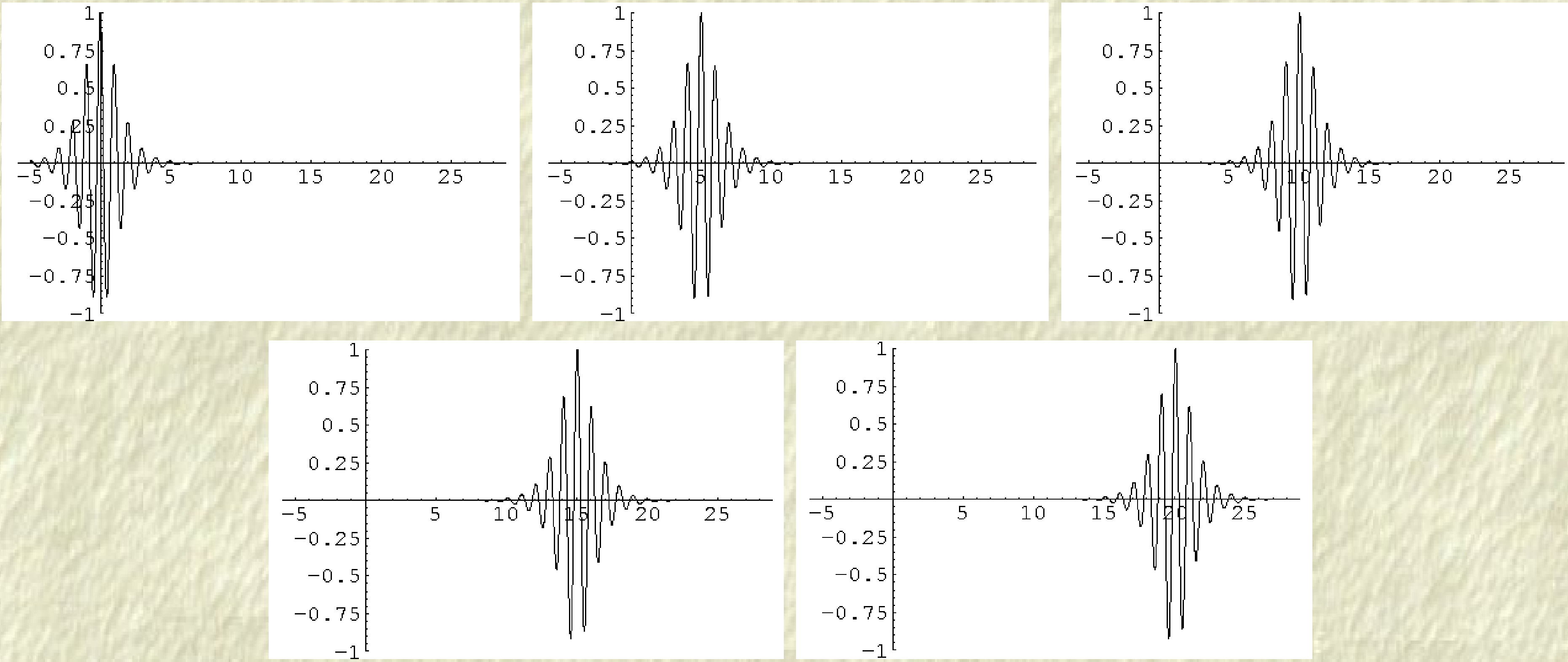
$L = \sqrt{\frac{2P}{Q}\frac{1}{\rho_0}}$

This is a propagating (and *oscillating*) localized pulse:

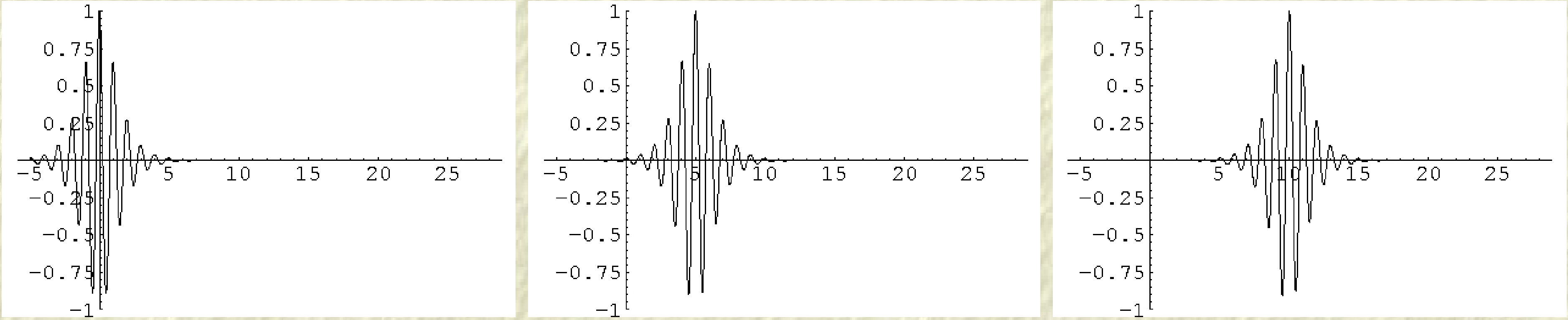




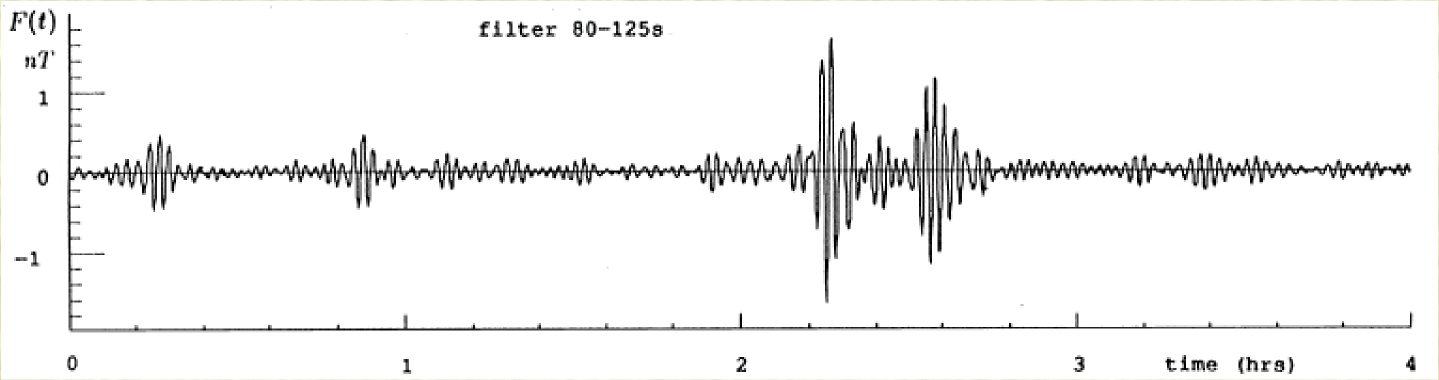
Propagation of a bright envelope soliton (pulse)



Propagation of a bright envelope soliton (pulse)

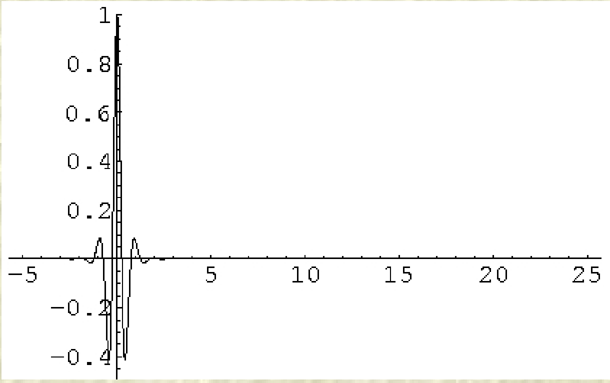


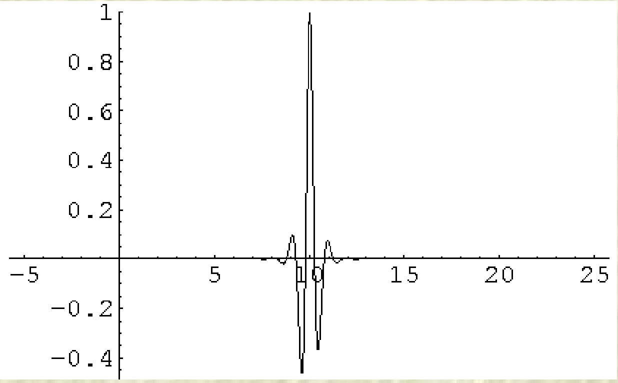
Cf. electrostatic plasma wave data from satellite observations:

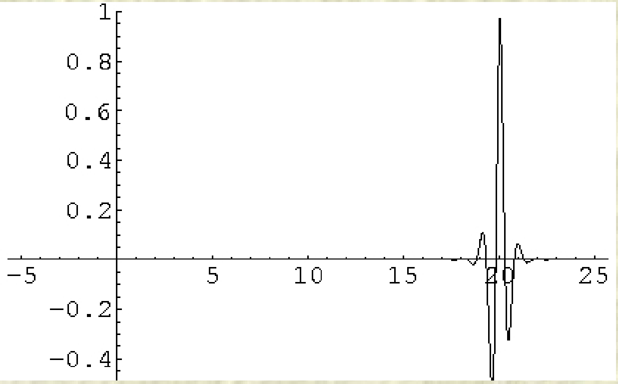


(from: [Ya. Alpert, Phys. Reports **339**, 323 (2001)])

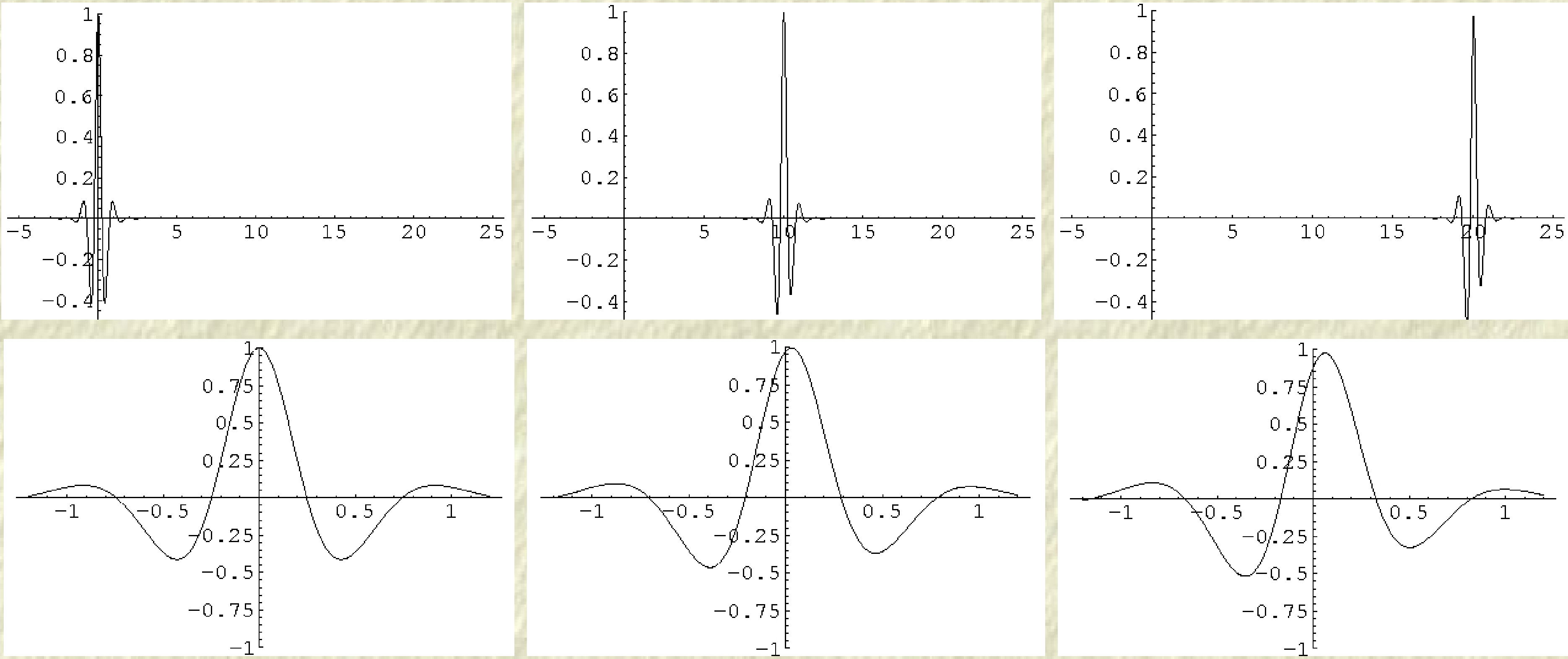
Propagation of a bright envelope soliton (continued...)



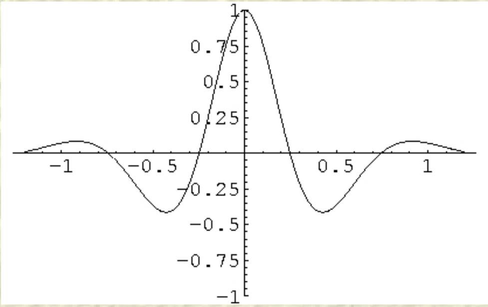


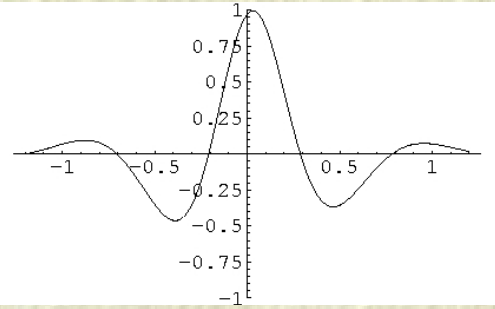


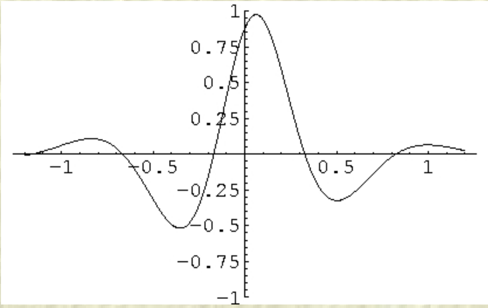
Propagation of a bright envelope soliton (continued...)



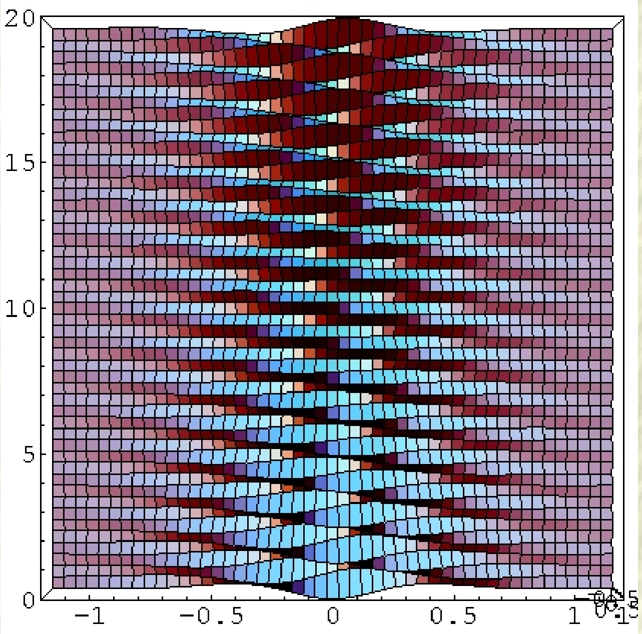
Propagation of a bright envelope soliton (continued...)







Rem.: *Time-dependent phase* → *breathing* effect (at rest frame):



Localized envelope excitations

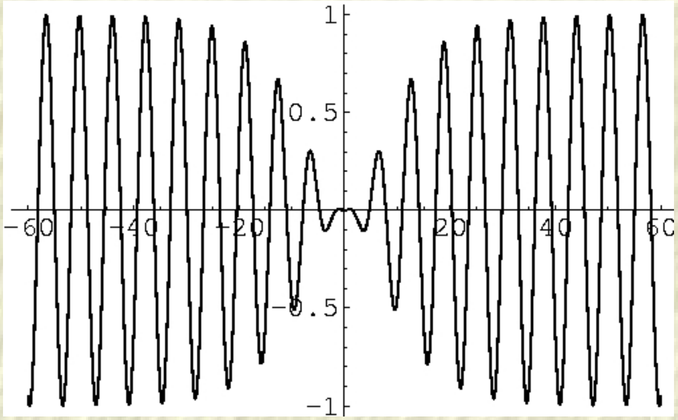
❑ Dark–type envelope solution (*hole soliton*):

$$\rho = \pm\rho_1 \left[1 - \operatorname{sech}^2\left(\frac{\zeta - v\tau}{L'}\right)\right]^{1/2} = \pm\rho_1 \tanh\left(\frac{\zeta - v\,\tau}{L'}\right),$$

$$\Theta = \frac{1}{2P}\left[v\,\zeta - \left(\frac{1}{2}v^2 - 2PQ\rho_1^2\right)\tau\right]$$

$$L' = \sqrt{2\left|\frac{P}{Q}\right|\frac{1}{\rho_1}} \qquad (9)$$

This is a *propagating localized* *hole* (zero density void):



Localized envelope excitations

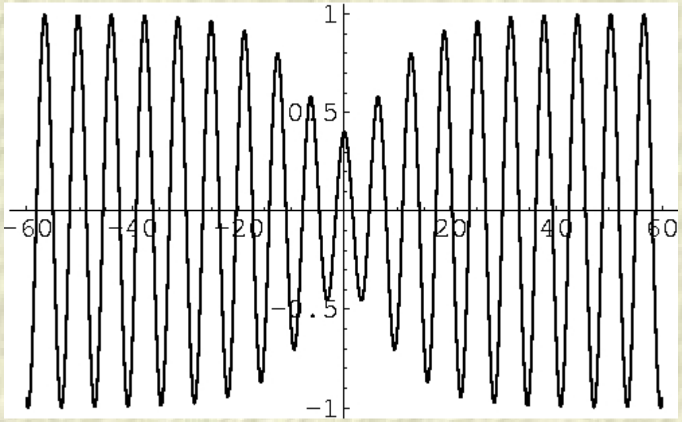
❏ Grey–type envelope solution (*void soliton*):

$$\rho = \pm\rho_2 \left[1 - a^2 \,\mathrm{sech}^2\left(\frac{\zeta - v\,\tau}{L''}\right)\right]^{1/2}$$

$$\Theta = \ldots$$

$$L'' = \sqrt{2\left|\frac{P}{Q}\right|}\frac{1}{a\rho_2} \qquad (10)$$

This is a propagating *(non zero-density)* void:



5. (Part B): Focusing on 1d DP crystals:

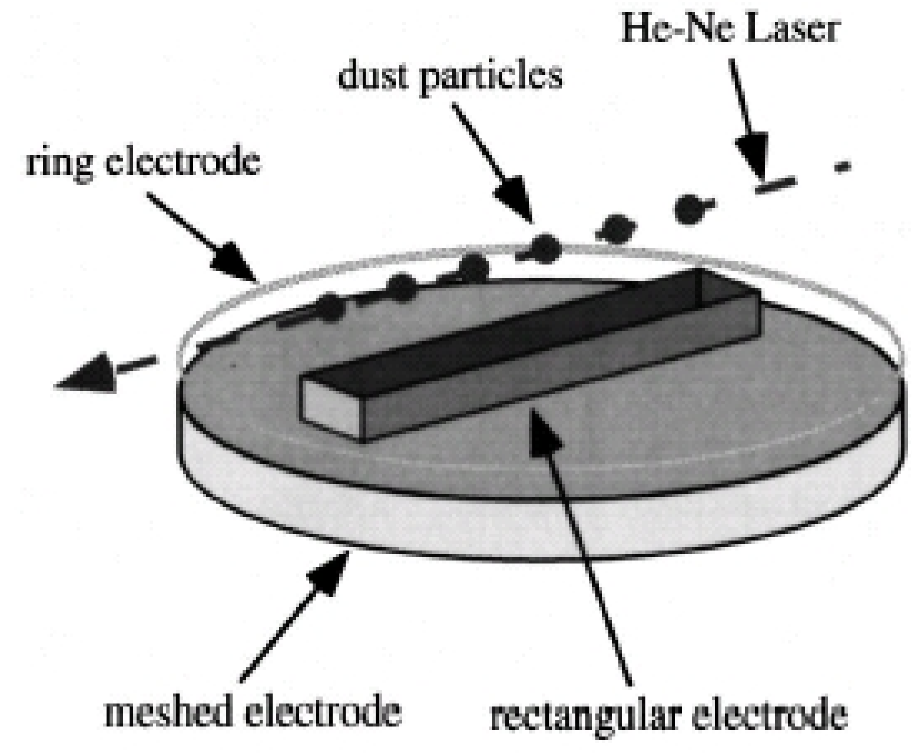


FIG. 10. Experimental configuration for forming a linear dust chain above a long rectangular box on a negatively biased mesh electrode.

[Figure from: S. Takamura *et al.*, Phys. Plasmas **8**, 1886 (2001).]

Focusing on 1d DP crystals: known linear modes.

- *Longitudinal Dust Lattice (LDL)* mode:
  - *Horizontal* oscillations ($\sim \hat{x}$): cf. *phonons* in atomic chains;
  - *Acoustic* mode: $\omega(k = 0) = 0$;
  - *Restoring force* provided by electrostatic interactions.

- *Transverse Dust Lattice (TDL)* mode:
  - *Vertical* oscillations ($\sim \hat{z}$);
  - *Optical* mode:
  $$\omega(k = 0) = \omega_g \neq 0$$
  (center of mass motion);
  - *Single* grain vibrations (propagating $\sim \hat{x}$ for $k \neq 0$): *Restoring force* provided by the *sheath electric potential* (*and* interactions).

- Transverse ($\sim \hat{y}$, in-plane, optical) d.o.f. *suppressed*.

Model *Hamiltonian*:

$$H = \sum_n \frac{1}{2} M \left(\frac{d\mathbf{r}_n}{dt} \right)^2 + \sum_{m \neq n} U_{int}(r_{nm}) + \Phi_{ext}(\mathbf{r}_n)$$

where:

– *Kinetic Energy* (1st term);

– $U_{int}(r_{nm})$ is the (binary) *interaction potential energy*;

– $\Phi_{ext}(\mathbf{r}_n)$ accounts for *'external' force fields*:
may account for *confinement potentials* and/or *sheath electric* forces, i.e.

$$F_{sheath}(z) = -\partial\Phi/\partial z \,.$$

Q.: Nonlinearity: *Origin: where from ?* *Consequence(s) ?*

Nonlinearity: Where does it come from?

- (i) *Interactions* between grains: Intrinsically *anharmonic*!
  - Electrostatic (e.g. Debye), long-range, screened ($r_0/\lambda_D \approx 1$); typically:

$$U_{Debye}(r) = \frac{q^2}{r} \exp\left(-r/\lambda_D\right).$$

  - Expanding $U_{pot}(r_{nm})$ near equilibrium:

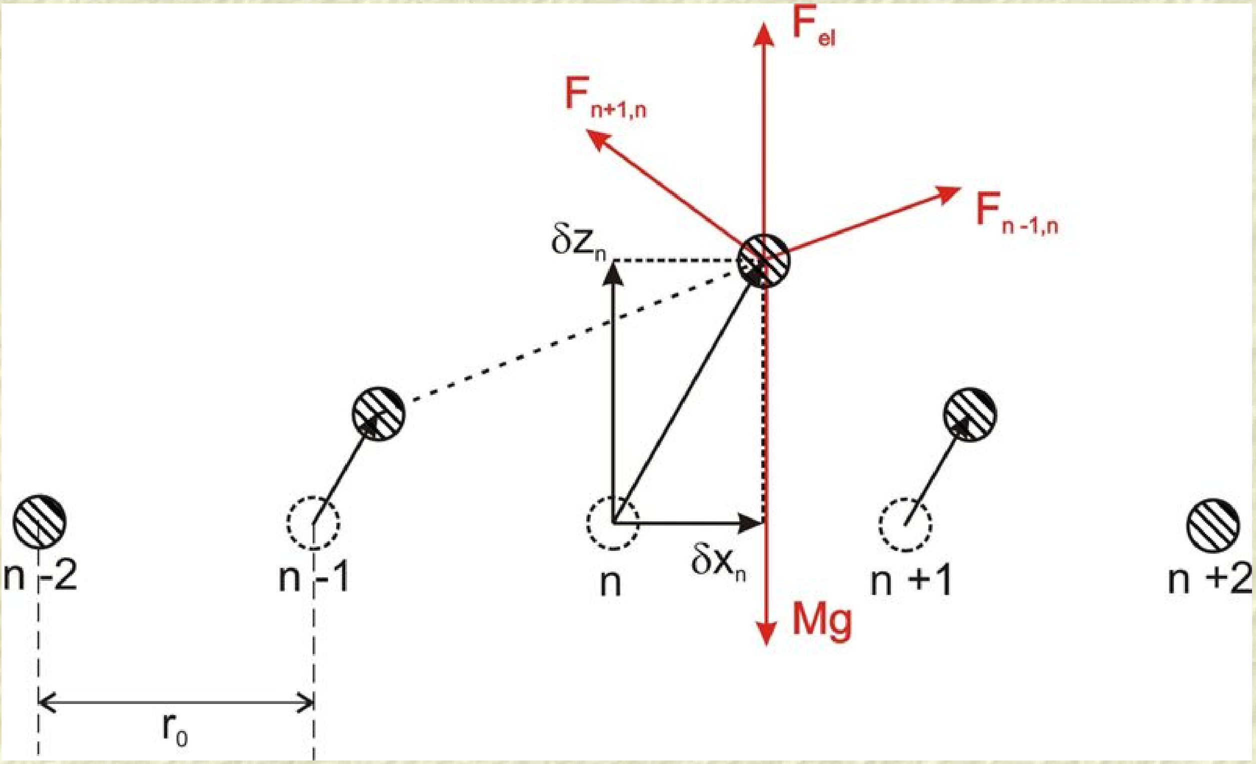
$$\Delta x_n = x_n - x_{n-m} = m r_0, \qquad \Delta z_n = z_n - z_{n-m} = 0$$

one obtains:

$$\begin{aligned} U_{nm}(r) \approx\ & \frac{1}{2} M \omega_{L,0}^2 (\Delta x_n)^2 + \frac{1}{2} M \omega_{T,0}^2 (\Delta z_n)^2 \\ & + \frac{1}{3} u_{30} (\Delta x_n)^3 + \frac{1}{4} u_{40} (\Delta x_n)^4 + \ldots + \frac{1}{4} u_{04} (\Delta z_n)^4 + \ldots \\ & + \frac{1}{2} u_{12} (\Delta x_n)(\Delta z_n)^2 + \frac{1}{4} u_{22} (\Delta x_n)^2 (\Delta z_n)^2 + \ldots \end{aligned}$$

Nonlinearity: Where from? *(continued ...)*

- (ii) Mode *coupling* also induces non linearity:
  anisotropic motion, *not* confined along one of the main axes ($\sim \hat{x}$, $\hat{z}$).



[cf. A. Ivlev *et al.*, PRE **68**, 066402 (2003); I. Kourakis & P. K. Shukla, Phys. Scr. (2004)]

Nonlinearity: Where from? *(continued ...)*

- (iii) *Sheath* environment: *anharmonic* vertical potential:

$$\Phi(z) \approx \Phi(z_0) + \frac{1}{2} M \omega_g^2 (\delta z_n)^2 + \frac{1}{3} M \alpha \, (\delta z_n)^3 + \frac{1}{4} M \beta \, (\delta z_n)^4 + \ldots$$

cf. experiments [Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)];

$\delta z_n = z_n - z_{(0)}$; α, β, ω_g are defined via $E(z)$, $[B(z)]^\dagger$ and $Q(z)$; (in fact, functions of n and P) [† V. Yaroshenko *et al.*, NJP 2003; PRE 2004]

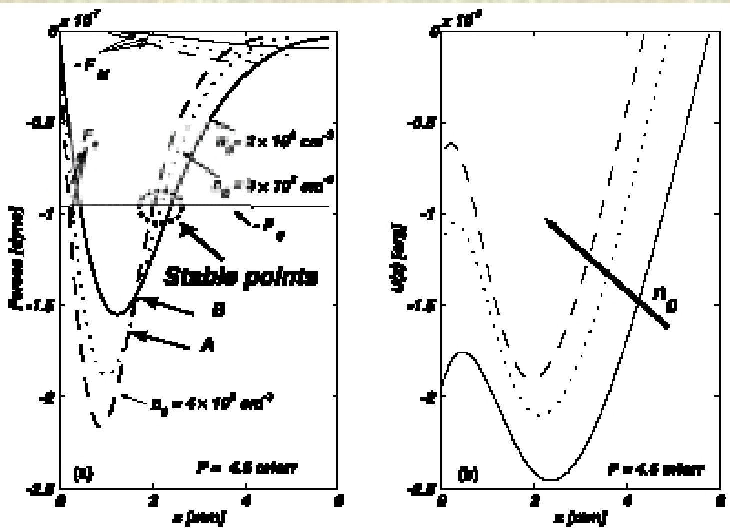


Figure 3: (a) Forces and (b) trapping potential profiles $U(z)$ as function of distance from the electrode for: $n_0 = 2\times 10^8 cm^{-3}$ (solid line), $n_0 = 3\times 10^8 cm^{-3}$ (dashed line), $n_0 = 4 \times 10^8 cm^{-3}$ (dotted line). The parameters are: P = 4.6 mtorr, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$.

Source: Sorasio *et al.* (2002).

Overview: Localized excitations in 1d dust (Debye) crystals

- B1. *Transverse* degree of freedom ($\sim \hat{z}$): *envelope solitons, breathers* (continuum theory)

- B2. *Longitudinal* d.o.f. ($\sim \hat{x}$): *asymmetric* envelope solitons, ...

- B3. *Longitudinal* solitons: *Korteweg-deVries* (KdV) vs. *Boussinesq* theories, ...

  $\rightarrow$ *Appendix*

- B4. *Discrete Breathers* (*Intrinsic Localized Modes*).

  $\rightarrow$ *Appendix*

B1. *Transverse* oscillations

The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys (*):

$$\frac{d^2(\delta z_n)}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 (\delta z_{n+1} + \delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n = 0 \qquad (11)$$

- TDL eigenfrequency:

$$\omega_{T,0} = \left[-qU'(r_0)/(Mr_0)\right]^{1/2} = \omega_{DL}^2 \exp(-\kappa)\,(1+\kappa)/\kappa^3$$

(*for Debye interactions*); $\kappa = r_0/\lambda_D$ is the *lattice parameter*;

- $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$ is the characteristic DL frequency scale;
- λ_D is the *Debye length*.

(*) [Vladimirov, Shevchenko and Cramer, PRE 1997]

Transverse **oscillations** ***(linear)***

The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys

$$\frac{d^2(\delta z_n)}{dt^2} + \nu \frac{d(\delta z_n)}{dt} + \omega_{T,0}^2 \,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n = 0 \qquad (12)$$

- Neglect dissipation, i.e. set $\nu = 0$ in the following;
- *Continuum* analogue: $\delta z_n(t) \to u(x,t)$, where

$$\frac{\partial^2 u}{\partial t^2} + c_T^2 \frac{\partial^2 u}{\partial x^2} + \omega_g^2\,u = 0$$

where $c_T = \omega_{T,0}\,r_0$ is the *transverse "sound" velocity*.

Transverse **oscillations *(linear, "undamped")***

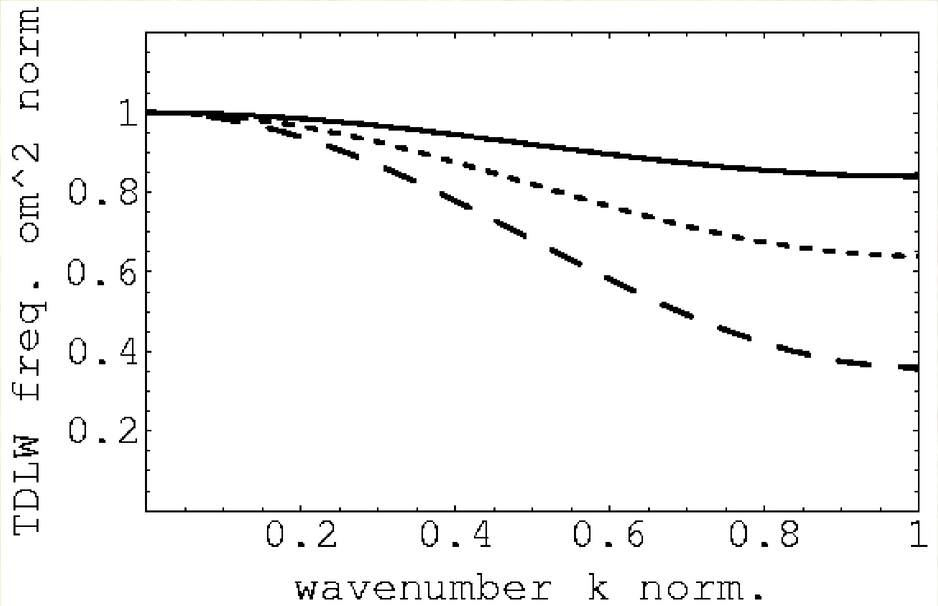
The *(linear)* vertical $n-$th grain displacement $\delta z_n = z_n - z_{(0)}$ obeys

$$\frac{d^2(\delta z_n)}{dt^2} + \omega_{T,0}^2 \left(\delta z_{n+1} + \delta z_{n-1} - 2\,\delta z_n \right) + \omega_g^2 \,\delta z_n = 0 \qquad (13)$$

Optical dispersion relation

(*backward* wave, $v_g < 0$) †:

$$\omega^2 = \omega_g^2 - 4\omega_{T,0}^2 \sin^2\left(k r_0/2\right)$$



† Cf. experiments: T. Misawa *et al.*, *PRL* **86**, 1219 (2001); B. Liu *et al.*, *PRL* **91**, 255003 (2003).

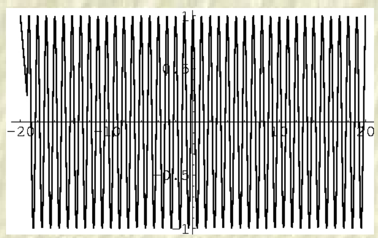
What if *nonlinearity* is taken into account?

$$\frac{d^2\delta z_n}{dt^2}+\nu\,\frac{d(\delta z_n)}{dt}+\omega_{T,0}^2\,(\,\delta z_{n+1}+\,\delta z_{n-1}-2\,\delta z_n)+\omega_g^2\,\delta z_n$$
$$+\alpha\,(\delta z_n)^2+\beta\,(\delta z_n)^3=0\,. \qquad (14)$$

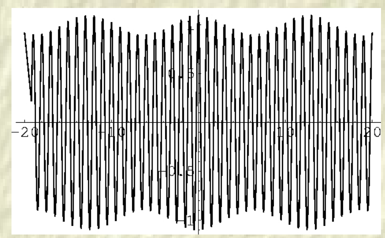
What if *nonlinearity* is taken into account?

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n$$
$$+\alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \qquad (15)$$

* *Intermezzo:* The mechanism of *wave amplitude modulation*:
The *amplitude* of a harmonic wave may vary in space and time:



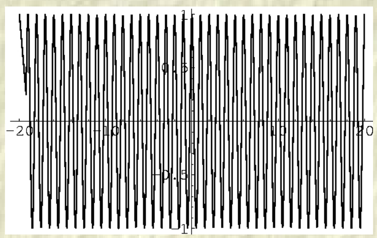
$\longrightarrow$



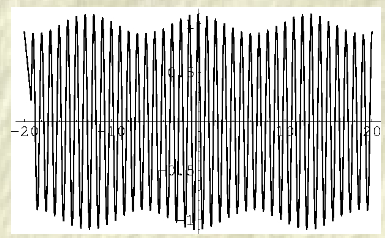
What if *nonlinearity* is taken into account?

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\,(\,\delta z_{n+1} + \,\delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n$$
$$+\alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \qquad (16)$$

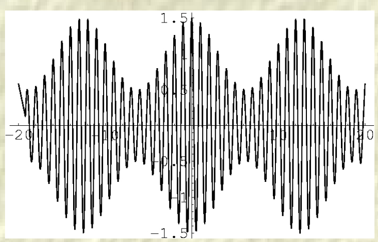
* *Intermezzo:* The mechanism of *wave amplitude modulation*:
The *amplitude* of a harmonic wave may vary in space and time:



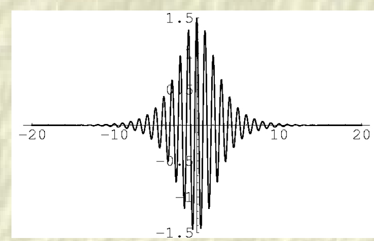
$\longrightarrow$



This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* or formation of *envelope solitons*:



$\longrightarrow$?



Large amplitude oscillations - envelope structures

A reductive perturbation (multiple scale) technique, viz.

$$t \to \{t_0, t_1 = \epsilon t, t_2 = \epsilon^2 t, ...\},\; x \to \{x_0, x_1 = \epsilon x, x_2 = \epsilon^2 x, ...\}$$

yields ($\epsilon \ll 1$; damping omitted):

$$\delta z_n \approx \epsilon\,(A\,e^{i\phi_n} + \text{c.c.}) \,+\, \epsilon^2\,\alpha\left[-\frac{2|A|^2}{\omega_g^2} + \left(\frac{A^2}{3\omega_g^2}\,e^{2i\phi_n} + \text{c.c.}\right)\right] + ...$$

Here,

- $\phi_n = nkr_0 - \omega t$ is the (fast) TDLW carrier phase;
- the amplitude $A(X, T)$ depends on the (*slow*) variables

$$\{X,\, T\} = \{\epsilon(x - v_g t),\, \epsilon^2 t\}\,.$$

Transverse oscillations - the envelope evolution equation

The amplitude $A(X, T)$ obeys the *nonlinear Schrödinger equation* (NLSE):

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \qquad (17)$$

where

- The *dispersion coefficient* ($\rightarrow$ see dispersion relation)

$$P = \frac{1}{2}\,\frac{d^2\omega_T(k)}{dk^2} = ...$$

  is negative/positive for low/high values of k.

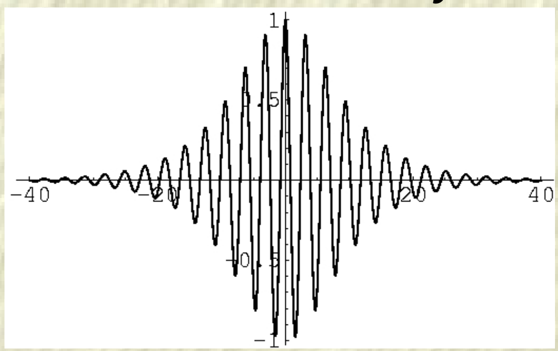
- The *nonlinearity coefficient* is $Q = \left[10\alpha^2/(3\omega_g^2) - 3\,\beta\right]/2\omega$.

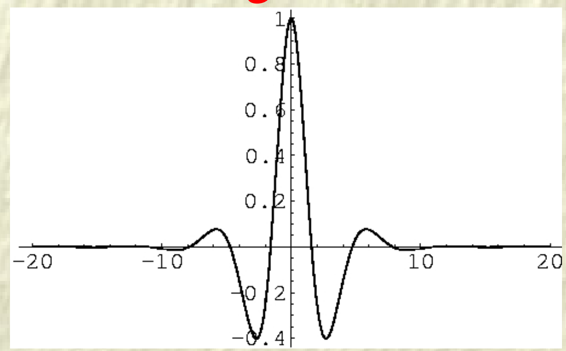
- Cf.: known properties of the NLS Eq. (cf. previous part).

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 2322 (2004); also *PoP*, **11**, 3665 (2004).]

Modulational stability analysis & envelope structures

❑ $PQ > 0$: *Modulational instability* of the carrier, *bright solitons*:



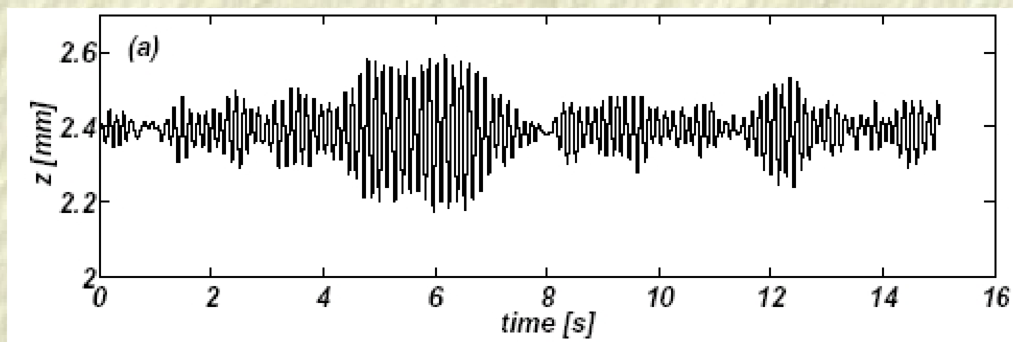


→ *TDLW*s: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

Rem.: $Q > 0$ for *all* known experimental values of α, β.

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)]

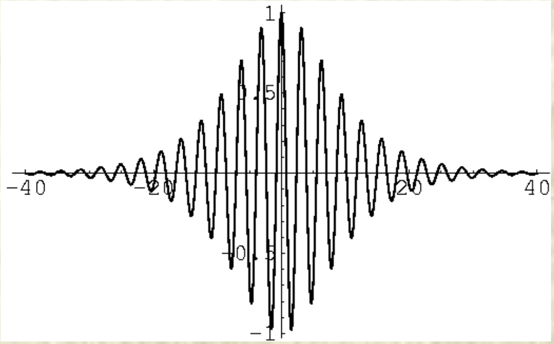
Figure 9: Dust grain oscillations induced by a 1% fluctuation in plasma density. The simulation parameters are: $P = 0.9\ mtorr$, $n_0 = 0.8 \times 10^8\ cm^{-3}$, $T_e = 1\ eV$, $T_i = T_n = 0.05\ eV$, $R = 2.5\ \mu m$, $\rho_d = 1.5\ g\,cm^{-3}$, $\phi_w = 6\ V$, $\varsigma_t = 0.06$, $\varsigma_p = 1\% n_0$

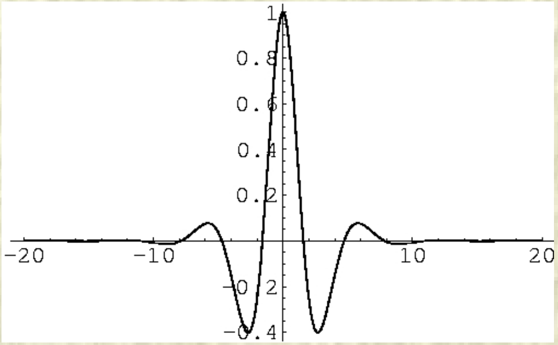


Source: G. Sorasio *et al.* (2002).

Modulational stability analysis & envelope structures

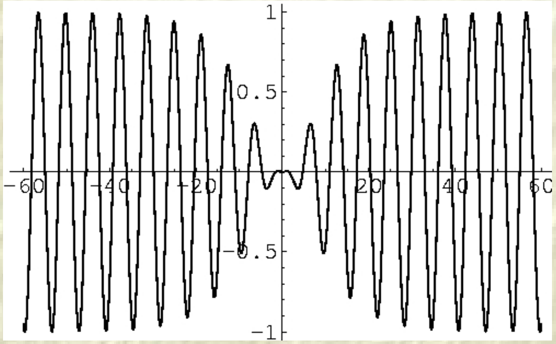
- $PQ > 0$: *Modulational instability* of the carrier, *bright solitons*:

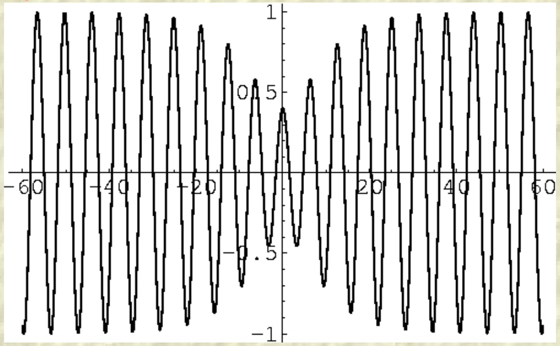




$\rightarrow$ *TDLW*s: possible for *short* wavelengths i.e. $k_{cr} < k < \pi/r_0$.

- $PQ < 0$: Carrier wave is *stable*, *dark/grey solitons*:





$\rightarrow$ *TDLW*s: possible for *long* wavelengths i.e. $k < k_{cr}$.

Rem.: $Q > 0$ for *all* known experimental values of α, β

[Ivlev *et al.*, PRL **85**, 4060 (2000); Zafiu *et al.*, PRE **63** 066403 (2001)] *(end of TDL)*.

B2. *Longitudinal* excitations

The (*linearized*) equation of *longitudinal* ($\sim \hat{x}$) motion reads (*):

$$\frac{d^2(\delta x_n)}{dt^2} + \nu \frac{d(\delta x_n)}{dt} = \omega_{0,L}^2 \, (\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n)$$

– $\delta x_n = x_n - n r_0$: longitudinal dust grain displacements

– *Acoustic* dispersion relation:

$$\omega^2 = 4\omega_{L,0}^2 \sin^2\bigl(k r_0/2\bigr) \equiv \omega_L^2(k)$$

– LDL eigenfrequency: $\omega_{0,L}^2 = U''(r_0)/M = 2\,\omega_{DL}^2 \exp(-\kappa)\,(1+\kappa+\kappa^2/2)/\kappa^3$ $(*)$

$^{(*)}$ *for Debye interactions*; Rem.: $\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$.

– Neglect damping in the following, viz. $\nu \to 0$.

(*) [Melandsø PoP 1996, Farokhi *et al*, PLA 1999]

Longitudinal **excitations (linear, “undamped”)**

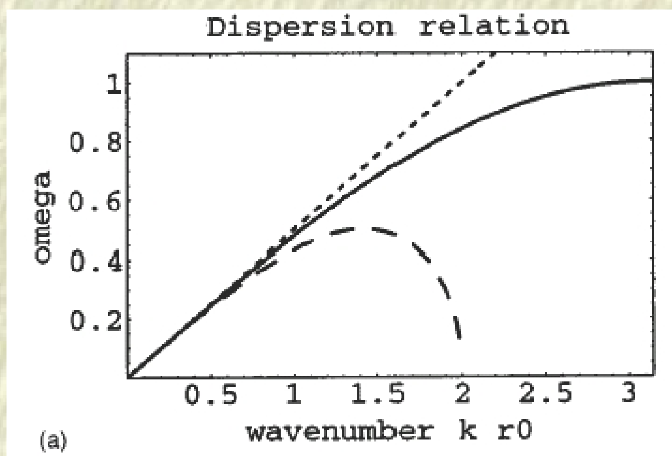
The (*linearized*) equation of longitudinal motion reads:

$$\frac{d^2(\delta x_n)}{dt^2} = \omega_{0,L}^2 \left(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n\right)$$

or, in the continuum approximation:

$$\frac{\partial^2(\delta x_n)}{\partial t^2} - c_L^2 \frac{\partial^2(\delta x_n)}{\partial t^2} = 0$$

$(c_L = \omega_{0,L} r_0)$



Longitudinal **excitations (*nonlinear*).**

The *nonlinear* equation of longitudinal motion reads:

$$\frac{d^2(\delta x_n)}{dt^2} = \omega_{0,L}^2 \left(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n\right)$$
$$- a_{20} \left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2\right]$$
$$+ a_{30} \left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3\right] \quad (18)$$

– $\delta x_n = x_n - nr_0$: longitudinal dust grain displacements

– Cf. *Fermi-Pasta-Ulam (FPU) problem*: anharmonic spring chain model:

$$U_{int}(r) \approx \frac{1}{2} M \omega_{0,L}^2 r^2 - \frac{1}{3} M a_{20} r^3 + \frac{1}{4} M a_{30} r^4 \,.$$

Longitudinal **envelope structures.**

The reductive perturbation technique (cf. above) now yields:

$$\delta x_n \approx \epsilon \left[u_0^{(1)} + \left(u_1^{(1)} e^{i\phi_n} + \text{c.c.}\right)\right] + \epsilon^2 \left(u_2^{(2)} e^{2i\phi_n} + \text{c.c.}\right) + \ldots ,$$

[Harmonic generation; Cf. experiments: K. Avinash PoP 2004].

Longitudinal **envelope structures.**

The reductive perturbation technique (cf. above) now yields:

$$\delta x_n \approx \epsilon \left[u_0^{(1)} + (u_1^{(1)}\, e^{i\phi_n} + \text{c.c.})\right] + \epsilon^2\, (u_2^{(2)}\, e^{2i\phi_n} + \text{c.c.}) + \dots ,$$

where the amplitudes obey the coupled equations:

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L\, \frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\, |u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\, u_1^{(1)} \frac{\partial u_0^{(1)}}{\partial X} = 0\,,$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - c_L^2}\, \frac{\partial}{\partial X}\, |u_1^{(1)}|^2 \equiv R(k) \frac{\partial}{\partial X}\, |u_1^{(1)}|^2$$

– $Q_0 = -\frac{k^2}{2\omega} \left(q_0\, k^2 + \frac{2p_0^2}{c_L^2 r_0^2} \right)$; $v_{g,L} = {\omega_L}'(k)$; $\{X, T\}$: *slow* variables;

– $p_0 = -U'''(r_0) r_0^3/M \equiv 2a_{20} r_0^3$, $q_0 = U''''(r_0) r_0^4/(2M) \equiv 3a_{30} r_0^4$.

– $R(k) > 0$, since $\forall\, k$ $v_{g,L} < \omega_{L,0}\, r_0 \equiv c_L$ *(subsonic LDLW envelopes)*.

Asymmetric *longitudinal* envelope structures.

– The system of Eqs. for $u_1^{(1)}$, $u_0^{(1)}$ may be combined into a closed (*NLSE*) equation (for $A = u_1^{(1)}$, here);

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0$$

– $P = P_L = \omega_L''(k)/2 < 0$;

– $Q > 0$ (< 0) prescribes *stability* (instability) at *low* (high) k.

Asymmetric *longitudinal* envelope structures.

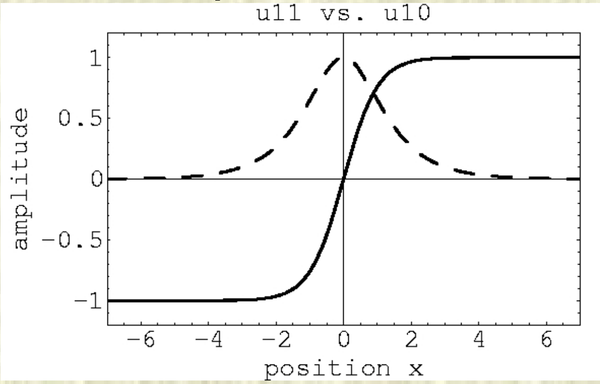
– The system of Eqs. for $u_1^{(1)}$, $u_0^{(1)}$ may be combined into a closed (*NLSE*) equation (for $A = u_1^{(1)}$, here);

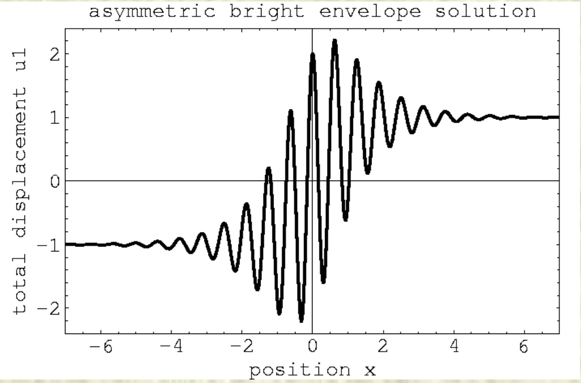
$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0$$

– $P = P_L = \omega_L''(k)/2 < 0$;

– $Q > 0$ (< 0) prescribes *stability* (instability) at *low* (high) k.

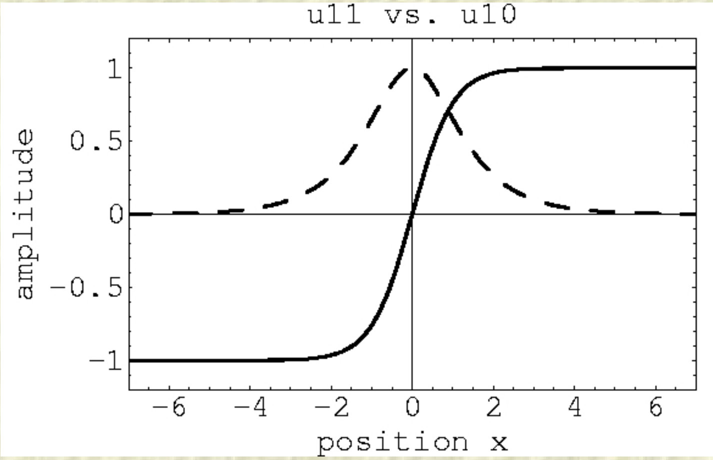
– Envelope excitations are now *asymmetric*:

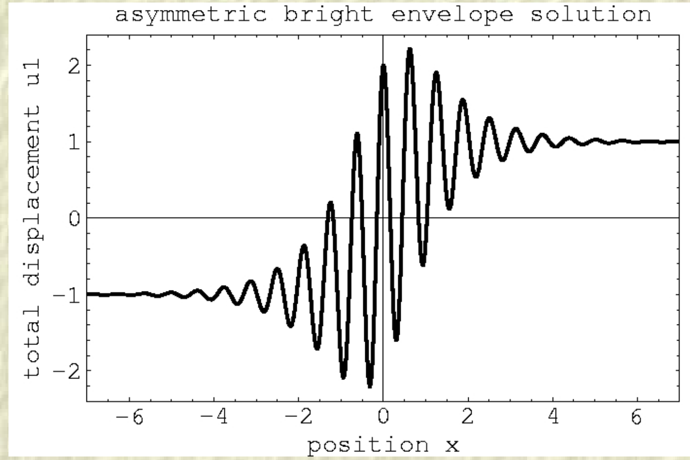




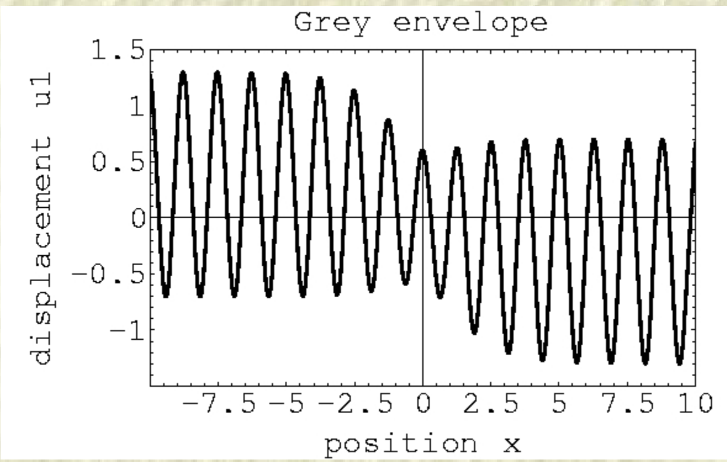
(at high k, i.e. $PQ > 0$).

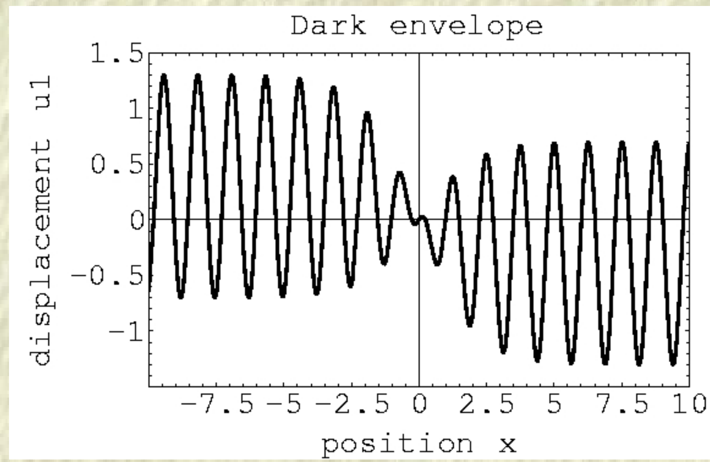
Asymmetric *longitudinal* envelope structures.





(at high k)





(at low k)

[I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **11**, 1384 (2004).] *(end of L-Part)*.

6. Conclusions

- *Amplitude Modulation* (due to carrier self-interaction) is an inherent feature of oscillatory mode dynamics in dynamical systems;
- modulated waves may undergo spontaneous *modulational instability*; this is an intrinsic feature of nonlinear dynamics, which ...
- ... may lead to the formation of *envelope localized structures* (envelope solitons), in account of *energy localization* phenomena observed in Nature;
- *Modulated electrostatic (ES) plasma wave packets* observed in Space and in the lab, in addition to *dust-lattice excitations*, may be modelled this way.
- The RP analytical framework permits modelling of these mechanisms in terms of intrinsic physical (plasma) parameters.

  $\rightarrow$ *an efficient model for a weakly nonlinear dynamical system in* $\{x, t\}$*.*

Thank You !

Ioannis Kourakis

Acknowledgments:

Padma Kant Shukla and co-workers at RUB (Germany)

Material from:

I. Kourakis & P. K. Shukla, *Phys. Plasmas*, **10** (9), 3459 (2003);
idem, *PRE*, **69** (3), 036411 (2003).
idem, *J. Phys. A* , **36** (47), 11901 (2003).
idem, *European Phys. J. D*, **28**, 109 (2004).

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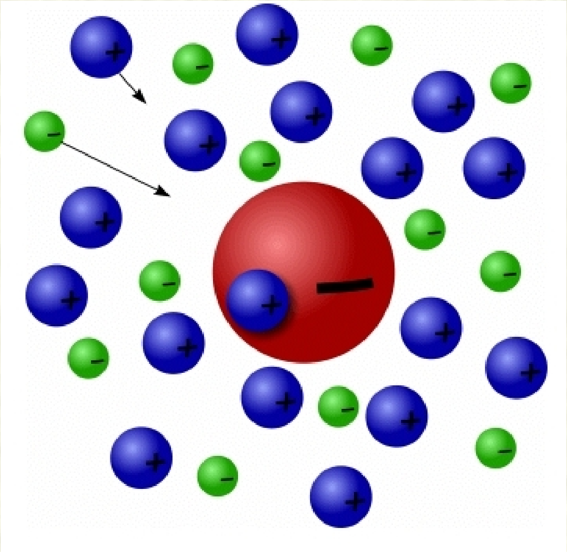
Appendix: DP – Dusty Plasmas (or *Complex Plasmas*): definition and characteristics

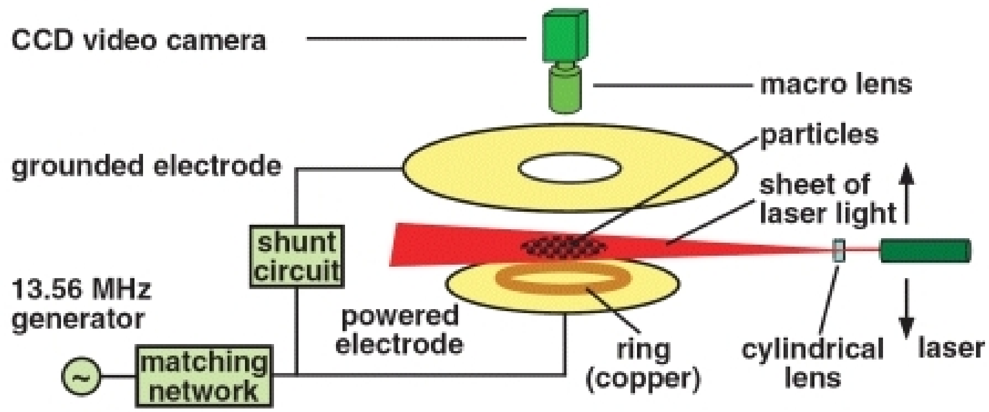


❑ Ingredients:

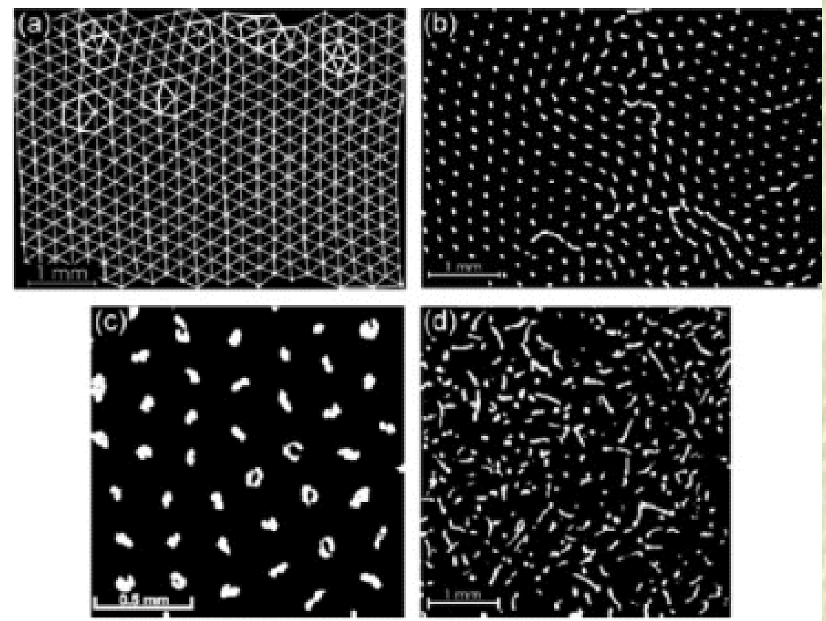
– *electrons* e^- (charge $-e$, mass m_e),
– *ions* i^+ (charge $+Z_i e$, mass m_i), *and*
– charged micro-particles $\equiv$ *dust grains* d (most often d^-):

charge $Q = \pm Z_d e \sim \pm(10^3 - 10^4)\, e$,
mass $M \sim 10^9\, m_p \sim 10^{13}\, m_e$,
radius $r \sim 10^{-2}\, \mu m$ up to $10^2\, \mu$m.

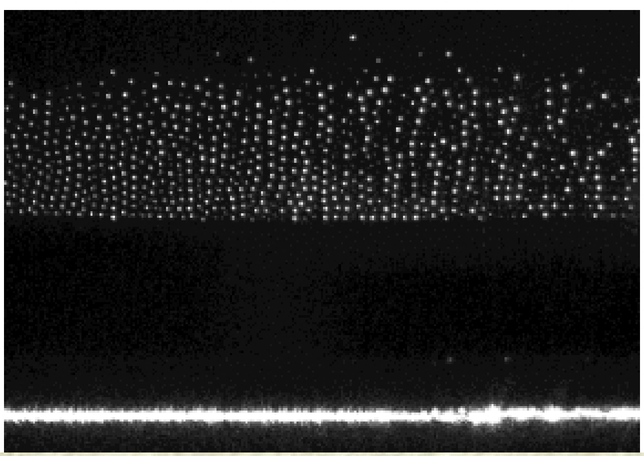
Dust laboratory experiments on Earth:



top view

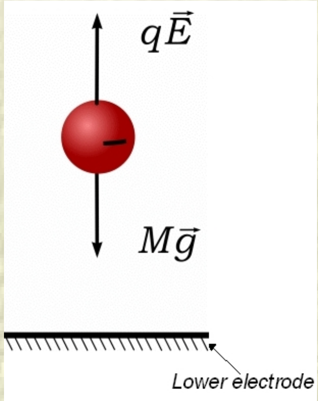


side view



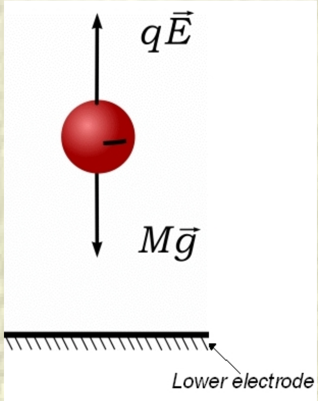
G E Morfill et al,
Plasma Phys. Control. Fusion **44** (2002) B263

Earth experiments are subject to gravity:



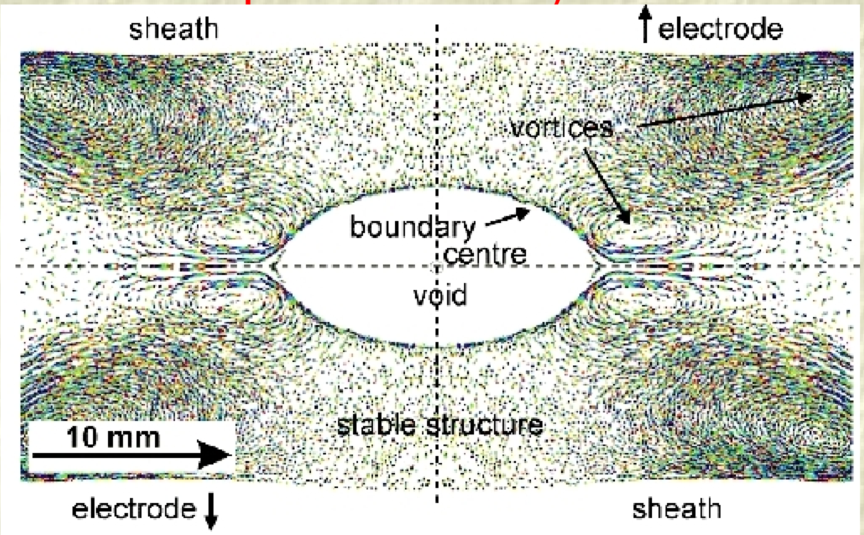
levitation in strong sheath electric field

Earth experiments are subject to gravity:



levitation in strong sheath electric field

thus ...: Dust experiments in ISS (International Space Station)



void formed in the middle of discharge under microgravity conditions

(Online data from: Max Planck Institüt - CIPS).

B3. *Longitudinal* soliton formalism.

Q.: *A link to soliton theories: the Korteweg-deVries Equation.*

– *Continuum approximation*, viz. $\delta x_n(t) \to u(x,t)$.

– *"Standard"* description: keeping *lowest order* nonlinearity,

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx}$$

$c_L = \omega_{L,0}\,r_0$; $\quad \omega_{L,0}$ and p_0 were defined above.

– Let us neglect damping ($\nu \to 0$), once more.

– For *near-sonic propagation* (i.e. $v \approx c_L$), slow profile evolution in time τ and defining the *relative displacement* $w = u_\zeta$, one obtains the KdV equation:

$$w_\tau - a\,w\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0$$

(for $\nu = 0$); $\zeta = x - vt$; $a = p_0/(2c_L) > 0$; $b = c_L r_0^2/24 > 0$.

– This KdV Equation yields soliton solutions, ... ($\to$ *next page*)

The KdV description

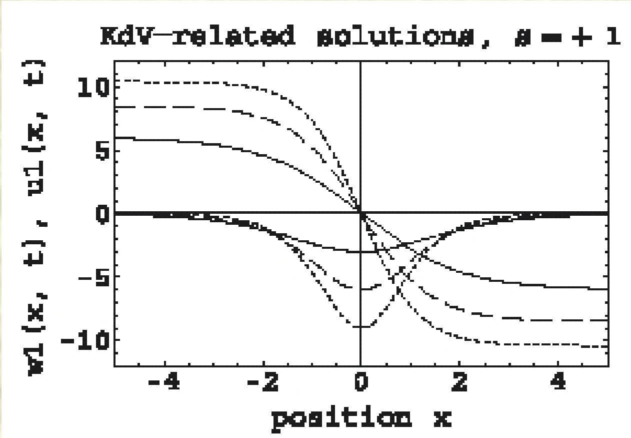
The Korteweg-deVries (KdV) Equation

$$w_\tau - a\,w\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0$$

yields *compressive* (*only*, here) solutions, in the form (here):

$$w_1(\zeta,\tau) = -w_{1,m} sech^2\Big[(\zeta - v\tau - \zeta_0)/L_0\Big]$$

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.



The KdV description

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$$w_\tau - a\,w\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0$$

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$$w_1(\zeta, \tau) = -w_{1,m} sech^2\Big[(\zeta - v\tau - \zeta_0)/L_0\Big]$$

– Pulse amplitude: $w_{1,m} = 3v/a = 6vv_0/|p_0|$;

– Pulse width: $L_0 = (4b/v)^{1/2} = [2v_1^2 r_0^2/(vv_0)]^{1/2}$;

– Note that: $w_{1,m}L_0^2 = \text{constant}$ (cf. experiments)[†].

– This solution is a negative pulse for $w = u_x$,
describing a *compressive* excitation for the *displacement* $\delta x = u$,
i.e. a localized increase of *density* $n \sim -u_x$.
– This is the standard treatment of dust-lattice solitons today ... [†]

[†] F. Melandsø 1996; S. Zhdanov *et al.* 2002; K. Avinash *et al.* 2003; V. Fortov *et al.* 2004.

Characteristics of the KdV theory

The *Korteweg - deVries* theory, as applied in DP crystals:

– provides a *correct qualitative description* of *compressive* excitations observed in experiments;

– benefits from the KdV *"artillery"* of analytical know-how obtained throughout the years: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

Characteristics of the KdV theory

The *Korteweg - deVries* theory, as applied in DP crystals:

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– benefits from the KdV *"artillery"* of analytical know-how obtained throughout the years: *integrability*, *multi-soliton* solutions, *conservation laws*, ... ;

but possesses a few drawbacks:

– *approximate* derivation: (i) propagation velocity v near (longitudinal) sound velocity c_L, (ii) time evolution terms omitted *'by hand'*, (iii) higher order nonlinear contributions omitted;

– *only* accounts for *compressive* solitary excitations (for Debye interactions); nevertheless, the existence of *rarefactive* dust lattice excitations is, *in principle*, *not excluded*.

Longitudinal **soliton formalism** ***(continued)***

Q.: *What if we also kept the next order in nonlinearity ?*

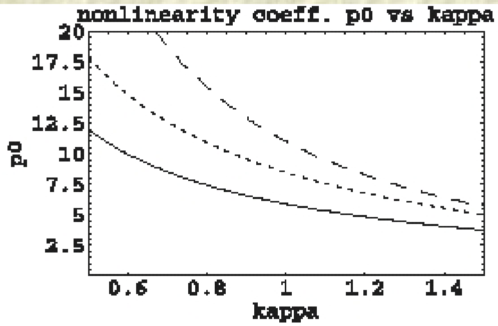
Longitudinal* soliton formalism *(continued)

Q.: *What if we also kept the next order in nonlinearity ?*

– *"Extended"* description: :

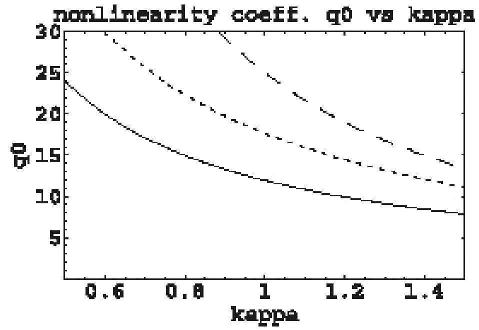
$$\ddot{u} - c_L^2\, u_{xx} - \frac{c_L^2}{12}\, r_0^2\, u_{xxxx} = -\, p_0\, u_x\, u_{xx} + q_0\, (u_x)^2\, u_{xx}$$

$c_L = \omega_{L,0}\, r_0; \quad \omega_{L,0},\ p_0 \sim -U'''(r)$ and $q_0 \sim U''''(r)$ (cf. above).



(b)

Fig. 4. (a) The nonlinearity coefficient p_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.



(b)

Fig. 5. (a) The nonlinearity coefficient q_0 (normalized over $Q^2/(M\lambda_D)$) is depicted against the lattice constant κ for $N = 1$ (first-neighbor interactions: —), $N = 2$ (second-neighbor interactions: - - -), $N = \infty$ (infinite-neighbors: – – –), from bottom to top. (b) Detail near $\kappa \approx 1$.

Rq.: q_0 is *not* negligible, compared to p_0! (instead, $q_0 \approx 2p_0$ practically, for $r_0 \approx \lambda_D$!)

Longitudinal soliton formalism *(continued)*

Q.: *What if we also kept the next order in nonlinearity ?*

– *"Extended"* description: :

$$\ddot{u} + \nu \, \dot{u} - c_L^2 \, u_{xx} - \frac{c_L^2}{12} r_0^2 \, u_{xxxx} = - p_0 \, u_x \, u_{xx} + q_0 \, (u_x)^2 \, u_{xx}$$

$c_L = \omega_{L,0} \, r_0$; $\omega_{L,0}$, p_0 and q_0 were defined above.

– For *near-sonic propagation* (i.e. $v \approx c_L$), and defining the *relative displacement* $w = u_\zeta$, one obtains the E-KdV equation:

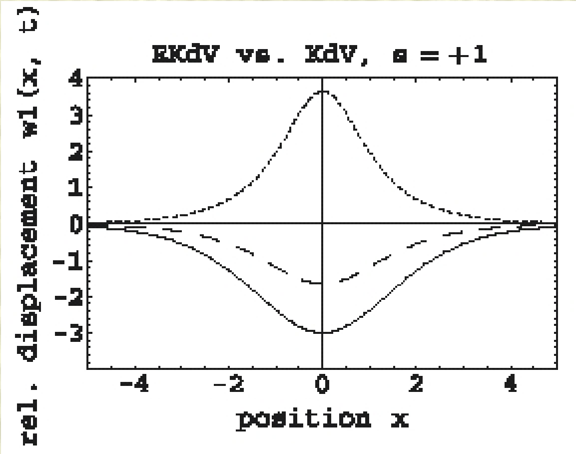
$$w_\tau - a \, w \, w_\zeta + \hat{a} \, w^2 \, w_\zeta + b \, w_{\zeta\zeta\zeta} = 0 \qquad (19)$$

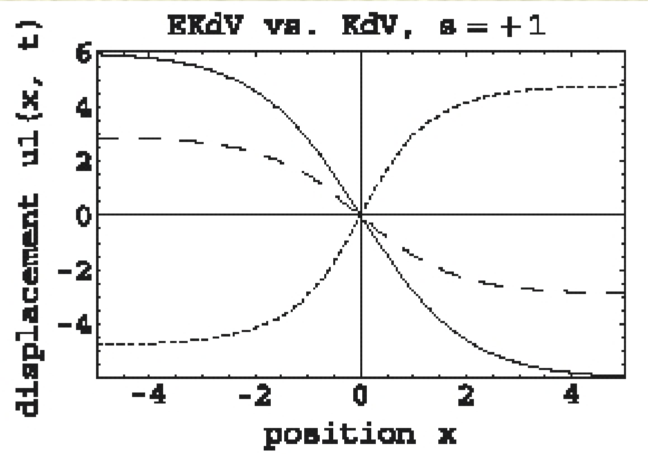
(for $\nu = 0$); $\zeta = x - vt$; $a = p_0/(2c_L) > 0$; $b = c_L r_0^2/24 > 0$;
$\hat{a} = q_0/(2c_L) > 0$.

Characteristics of the EKdV theory

The *extended Korteweg - deVries* Equation:

– accounts for *both compressive and rarefactive* excitations;





(*horizontal grain displacement* $u(x, t)$)

– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– is previously widely studied, in literature;

Still, ...

– It was derived under the *assumption*: $v \approx c_L$.

One more alternative: the Boussinesq theory

The *Generalized Boussinesq* (Bq) Equation (for $w = u_x$):

$$\ddot{w} - c_L^2 \, w_{xx} = \frac{c_L^2 r_0^2}{12} \, w_{xxxx} - \frac{p_0}{2} \, (w^2)_{xx} + \frac{q_0}{2} \, (w^3)_{xx}$$

– predicts *both compressive and rarefactive* excitations;

– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– has been widely studied in literature;

and, ...

One more alternative: the Boussinesq theory

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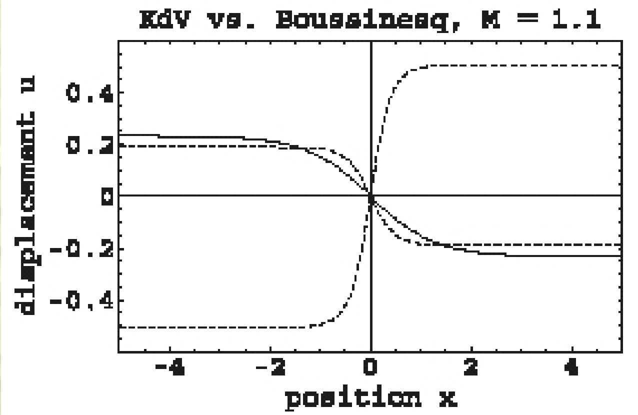
– predicts *both compressive and rarefactive* excitations;

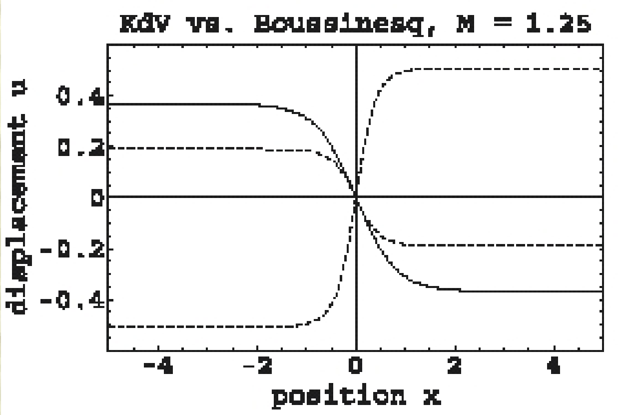
– reproduces the *correct qualitative character* of the KdV solutions (amplitude - velocity dependence, ...);

– has been widely studied in literature;

and, ...

– relaxes the velocity assumption, i.e. is valid $\forall \quad v > c_L$.





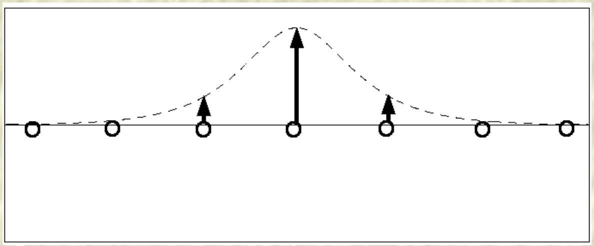
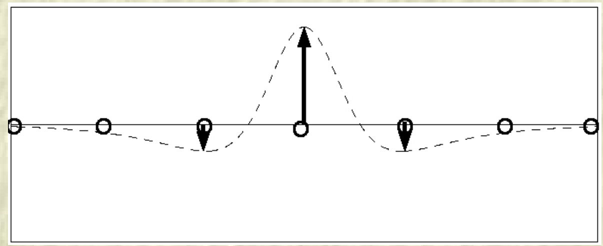
(end of L-Part 4.)

B4. Transverse Discrete Breathers (DB)

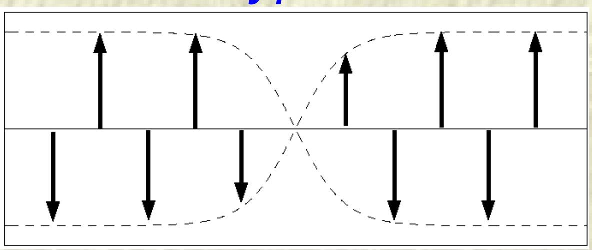
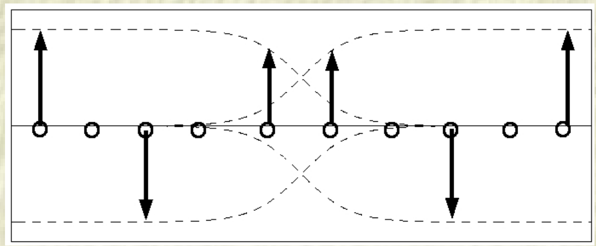
- DBs are *highly discrete* oscillations (*Intrinsic Localized Modes, ILMs*);
- Looking for DB solutions in the *transverse* direction, viz.

$$\frac{d^2 u_n}{dt^2} + \omega_{T,0}^2 \,(\, u_{n+1} + \, u_{n-1} - 2\, u_n) + \omega_g^2 \,\delta z_n + \alpha\, u^2 + \beta\, u^3 = 0$$

one obtains the *bright-type* DB solutions (localized pulses):

as well as the *dark-type* excitations (holes; *Kivshar dark* modes):

- Similar modes may be sought in the longitudinal direction.

Transverse Discrete Breathers (DB)

- Existence and stability criteria still need to be examined.

- It seems established that DBs exist if the *non-resonance criterion*:

$$n\,\omega_B \neq \omega_k \qquad\qquad \forall n \in \mathcal{N}$$

  is fulfilled, where:

  – ω_B is the *breather frequency*;

  – ω_k is the *linear ("phonon") frequency* (cf. dispersion relation).

- If ω_B (or its harmonics) enter(s) into resonance with the linear spectrum ω_k, discrete oscillations will decay into a "sea" of linear lattice waves.

- The DB existence condition is satisfied in *all* known lattice wave experiments.