Nonlinear dust charge fluctuations in dusty (complex) plasmas: a Van der Pol-Mathieu model equation

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Abstract. The parametric excitation of dust acoustic oscillations due to dust grain charge fluctuations in a dusty plasma (DP) is investigated. A three-component model DP is considered, consisting of negative inertial dust grains (of constant size, mass and charge, for simplicity), in addition to a thermalized (Maxwellian) background of electrons and ions. By employing a fluid plasma description, and assuming a periodic fluctuation of the dust charge, a Van der Pol-Mathieu-type hybrid nonlinear oscillator model ordinary differential equation is obtained for the dust number density. An averaging technique provides the framework for an analysis of the dust density evolution in time, via an analytical reduction to an autonomous set of equations for a slowly varying pair of perturbation amplitudes. A phase space portrait is determined analytically and numerically, in terms of the DP frequency $\omega_{d,p}$ and the parametric excitation (charge fluctuation) frequency ω_{df}.

Keywords: Dust-acoustic plasma waves, Van der Pol equation, Mathieu equation, phase space dynamics, chaos.
PACS: 52.30.-q, 82.40.bj, 05.45.-a

Dusty (or complex) plasmas (DP) are characterized by the presence of massive mesoscopic (micron-sized, typically) particulates ("dust grains"), which is known to modify plasma properties substantially. The electric charge q_d which resides on dust grains is acquired dynamically (when dust grains are immersed in a gaseous plasma) and may fluctuate in time via a variety of charging processes, as a result of plasma particle flow onto their surface [1]. New modes and associated instabilities occur in DP, some of which are absent in ordinary e-i plasmas. A recent study was devoted to a dust charging instability modelled via the chaotic behavior of charged dust in the plasma [2]. A Van der Pol-Mathieu equation was introduced therein in order to model the dynamical behavior of the dust grain charge. Generic nonlinear oscillator model differential equations of this kind can be studied by existing analytical methods [3, 4]. The Van der Pol equation has been studied by many researchers [5, 6]. Siewe [7] has investigated a system consisting of an extended Duffin-Van der Pol oscillator, in which resonance and off-resonance oscillations are analyzed using the multiple time scale method, while Maccari [8] introduced a new asymptotic perturbation method in search of an exact solution.

The present work aims at investigating the dynamical behavior of dust dynamics near parametric resonance. Relying on a dust fluid model, a Van der Pol - Mathieu nonlinear equation is shown to govern the dust dynamics. The equation is analyzed in the vicinity of resonance, and a phase portrait is drawn, both analytically and numerically.

A VAN DER POL-MATHIEU (VDPM) EQUATION FOR DUST FLUCTUATIONS

We consider an unmagnetized collisionless dusty plasma, consisting of electrons (mass m_e, charge $-e$), ions (mass m_i, charge $+Z_i e$) and dust grains. The dust grain mass m_d is assumed to be constant, for simplicity, whilst the dust grain charge is a time-dependent variable $q_d(t) = -Z_d(t)e$.

The cold inertial dust fluid density n_d and velocity v_d are governed by the density evolution equation

$$\frac{\partial n_d}{\partial t} + n_{0d}\frac{\partial u_d}{\partial z} = \alpha n_d - \frac{1}{3}\beta n_d^3\,, \tag{1}$$

where t and z are independent time and (one-dimensional) space variables. The coefficients α and β entering the source term in the right-hand side correspond to a rate of charged dust grain production (by electron absorption) and loss (due to three body recombination, viz. $X^+ + e^- + Z \to X^* + Z$), respectively. The momentum equation reads

$$\frac{\partial u_d}{\partial t} = -\frac{q_d}{m_d}\frac{\partial \phi}{\partial z}\,. \tag{2}$$

The electric potential ϕ is determined by Poisson's equation

$$\frac{\partial^2 \phi}{\partial z^2} = -4\pi e(Z_i n_i - n_e - Z_d n_d)\,, \tag{3}$$

where n_i and n_e denote the ion and electron number density, respectively. The right-hand side of Eq. (3) cancels at equilibrium, thanks to the charge neutrality condition $Z_i n_{i0} - n_{e0} - Z_d n_{d0} = 0$, where n_{s0}, for $s = i, e, d$, denotes the ion, electron and dust particle number density at equilibrium, respectively. We shall assume a harmonic potential variation in space, characterized by a wave length $\lambda \equiv 2\pi/k$ (and a wave number k), i.e. $\phi(z,t) = \hat{\phi}(t)\exp(ikz)$. Being much lighter than dust particles, both electrons and ions are assumed to be in local thermodynamic equilibrium, so their number densities, n_e and n_i, obey a Boltzmann distribution, viz. $n_e = n_{e0}\exp\big(e\phi/k_B T_e\big)$ and $n_i = n_{i0}\exp\big(-Z_i e\phi/k_B T_i\big)$.

Assuming $\phi \ll \{k_B T_e/e, k_B T_i/Z_i e\}$, and considering a harmonic dust charge fluctuation, i.e. $q_d = q_{d0}(1 + h\cos\gamma t)^{1/2}$ (where the real parameter $h \ll 1$ and the frequency γ are real constants), Eqs. (1) to (3) are combined into a closed evolution equation for the dust density, in the form of a HYBRID MATHIEU-VAN DER POL EQUATION

$$\frac{d^2 x}{dt^2} - (\alpha - \beta x^2)\frac{dx}{dt} + \omega_0^2(1 + h\cos\gamma t)\,x = 0\,, \tag{4}$$

where we have defined the dimensionless parameter $x = n_d/n_{d0} - 1$, and the characteristic oscillation frequency $\omega_0 = \omega_{pd}\,k/(k^2 + k_D^2)^{1/2}$; the dust plasma frequency reads $\omega_{pd} = (4\pi n_{d0} q_{d0}^2/m_d)^{1/2}$. The Debye wave number k_D is defined via the effective Debye length $\lambda_{Deff} = (\lambda_{De}^{-2} + \lambda_{Di}^{-2})^{-1/2}$, where $\lambda_{De} = (k_B T_e/4\pi n_{e0} e^2)^{\frac{1}{2}}$ and $\lambda_{Di} = (k_B T_i/4\pi Z_i n_{i0} e^2)^{\frac{1}{2}}$ are the electron and ion Debye radii, respectively. The inertialess electrons and ions affect the dust acoustic oscillations via a dynamical charge balance. In the limit $k \gg k_D$, $\omega_0 \approx \omega_{pd}$. The dust density is assumed to be uniform in space.

The second term in the left-hand side of equation (4) is characteristic of the Van der Pol (VdP) nonlinear oscillator model equation; indeed, the VdP Equation, which generically describes a self-sustained nonlinear oscillation, is exactly recovered for $h = 0$. On the other hand, for $\alpha = \beta = 0$, one recovers a Mathieu-type equation, which describes a parametric type oscillation. The ordinary differential equation (ODE) (4) is a hybrid equation, combining the features of the Van der-Pol and the Mathieu equations.

REDUCTION TO A SET OF COUPLED AMPLITUDE ODE'S

The averaging method. In this section, we shall investigate the dynamical behavior of the nonlinear Van der Pol-Mathieu (VdPM) oscillator (4), under the effect of parametric resonance. The dynamical profile is determined by an interplay among the parameters α, β and $\omega^2(t) = \omega_0^2(1 + h\cos\gamma t)$ (a time dependent oscillation frequency). Since parametric resonance is stronger for a frequency $\omega(t)$ nearly twice the eigenfrequency ω_0 (see e.g. in Ref. [3], §27), we shall consider the parametric excitation frequency to be $\gamma = 2\omega_0 + \varepsilon$, where $\varepsilon \ll 1$ is a (small) real parameter.

We assume a solution given by the ansatz

$$x = a(t)\,\cos\left(\omega_0 + \frac{\varepsilon}{2}\right)t + b(t)\,\sin\left(\omega_0 + \frac{\varepsilon}{2}\right)t\,, \tag{5}$$

where the (real) coefficients a and b vary *slowly* with time.

Substituting Eq. (5) into (4) and keeping only first order terms ε and h, we obtain the system of equations

$$\frac{da}{dt} = \frac{\alpha}{2}a - \frac{b}{2}\left(\varepsilon + \frac{h\omega_0}{2}\right) - \frac{\beta}{8}(a^3 + ab^2) \equiv f(a,b)\,, \tag{6}$$

$$\frac{db}{dt} = \frac{\alpha}{2}b + \frac{a}{2}\left(\varepsilon - \frac{h\omega_0}{2}\right) - \frac{\beta}{8}(b^3 + a^2 b) \equiv g(a,b)\,. \tag{7}$$

These equations represent a system of first order, autonomous, ordinary differential equations, governing the amplitudes of the approximate solution expressed in (5). Note that Eqs. (6) and (7) are invariant under $(a,b) \to (-a,-b)$.

Stability analysis. Assuming that $a \sim exp(st)$ and $b \sim exp(st)$ are small harmonic perturbations, one obtains

$$\left(s - \frac{\alpha}{2}\right)^2 = \frac{1}{4}\left[\left(\frac{h\omega_0}{2}\right)^2 - \varepsilon^2\right], \tag{8}$$

which leads to the reality condition: $-\frac{h\omega_0}{2} < \varepsilon < \frac{h\omega_0}{2}$.

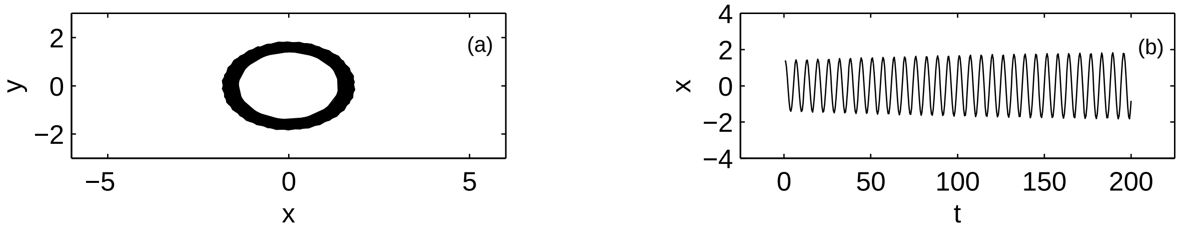

FIGURE 1. Phase diagram in (a) x-y and (b) t-x plane for $\alpha = \beta = 0.01$.

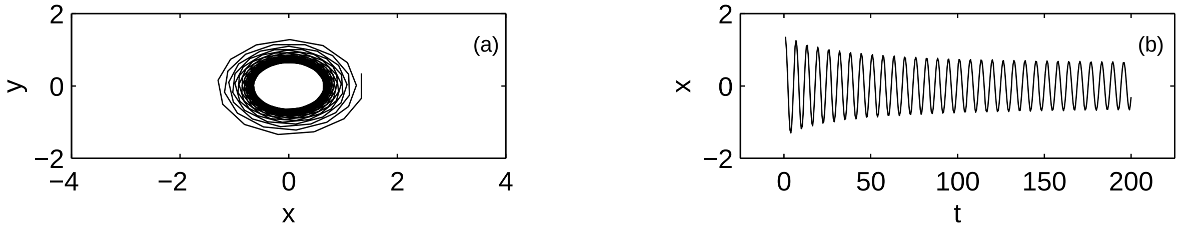

FIGURE 2. Phase diagram in (a) x-y and (b) t-x plane for $\alpha = 0.01$ and $\beta = 0.1$.

Equilibrium state. It can be deduced from Eqs. (6) and (7) that $a = b = 0$ (i.e. $x = 0$) determines an equilibrium state, i.e. a *fixed point* in phase space (a, b). The stability of the fixed point is determined by the eigenvalues of the Jacobian matrix of the vector fields in Eqs. (6) and (7), which yields the following characteristic polynomial

$$p(\lambda) = \lambda^2 - \alpha\lambda + \frac{\alpha^2}{4} + \frac{1}{4}\left(\varepsilon^2 - \frac{h^2\omega_0^2}{4}\right). \tag{9}$$

The Jacobian matrix of the vector field at $(0,0)$ therefore has two complex (conjugate) eigenvalues, λ_1 and λ_2, as $\lambda_{1,2} = A \pm iB$, where $A = \alpha/2$ and $B = (h^2\omega_0^2/4 - \varepsilon^2)^{1/2}/2$, provided that $|h\omega_0/2| > \varepsilon > 0$. One thus obtains $a, b \sim e^{At}e^{iBt}$, which is locally stable (unstable) for $\alpha < 0$ ($\alpha > 0$). For $\alpha = 0$ the eigenvalues are purely imaginary, so the equilibrium state is a center. A Hopf bifurcation point is thus determined, where $\alpha = 0$. Since we assume that $\alpha > 0$ here, the system never approaches the equilibrium state. Thus, even orbits initially chosen close to the equilibrium state will be repelled by the origin $(0,0)$, which therefore determines an unstable saddle point. We conclude that the dust production rate α plays a major role in the system dynamics.

Periodic solutions. The Jacobian matrix $\mathbf{J} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ derived from Eqs. (6) and (7) consists of

$$\begin{aligned} a_{11} &= \frac{\partial f}{\partial a} = \frac{\alpha}{2} - \frac{\beta}{8}(3a^2 + b^2), & a_{12} &= \frac{\partial f}{\partial b} = -\frac{1}{2}(\varepsilon + \frac{h\omega_0}{2}) + \frac{\beta}{4}ab, \\ a_{21} &= \frac{\partial g}{\partial a} = \frac{1}{2}(\varepsilon - \frac{h\omega_0}{2}) - \frac{\beta}{4}ab, & a_{22} &= \frac{\partial g}{\partial b} = \frac{\alpha}{2} - \frac{\beta}{8}(3b^2 + a^2). \end{aligned} \tag{10}$$

The eigenvalue problem $\mathrm{Det}(\mathbf{J} - \lambda\mathbf{I}) = 0$ leads to the characteristic polynomial $p(\lambda) = \lambda^2 - T\lambda + D$, where $T \equiv a_{11} + a_{22}$ and $D \equiv a_{11}a_{22} - a_{21}a_{12}$. Since a stable solution can only exist if $T = \lambda_1 + \lambda_2 < 0$ and $D = \lambda_1\lambda_2 > 0$, both eigenvalues $\lambda_{1,2}$ need to be negative for stability. One of the critical boundaries is determined by $T = 0$ and $D > 0$, i.e. by a pair of imaginary eigenvalues $\lambda_{1,2} = \pm i\sqrt{D}$. We see that the periodic solution may lose its stability via a Hopf Bifurcation at the critical boundary, i.e. where $T = 0$.

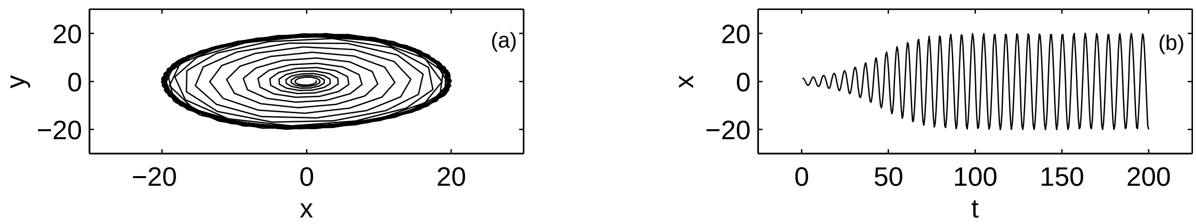

FIGURE 3. Phase diagram in (a) x-y and (b) t-x plane for $\alpha = 0.1$ and $\beta = 0.001$.

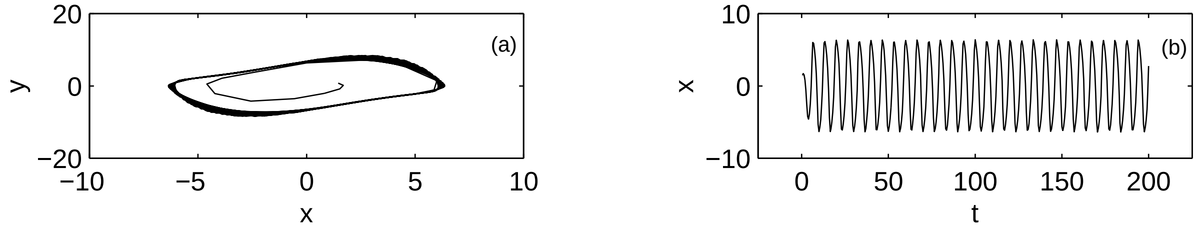

FIGURE 4. Phase diagram in (a) x-y and (b) t-x plane for $\alpha = 1$ and $\beta = 0.01$.

NUMERICAL RESULTS AND DISCUSSION

The VdPM-type nonlinear ODE (4) possesses an oscillatory (periodic) solution, which is a periodic attractor. Every nontrivial solution tend to this periodic solution. The periodic solution may be sought by varying α and β parameters.

In order to investigate the dynamical profile of Eq. (4) numerically, we have chosen a set of fixed values: $\omega = 1.0$, $\omega_0 = 1.0$ and $h = 0.01$, in addition to the initial conditions $\{x_0, y_0; t_0\} = \{1.0, 1.0; 0.0\}$. Employing a fourth-order Runge-Kutta method, we have solved Eq. (4). The system was found to possess various stable and unstable limit cycles. The phase diagram in (x, y) and (t, x) planes, for different values of α and β, is depicted in the Figures. Periodic states occur when we choose $a_{11} + a_{22} = 0$, i.e. $\alpha = \beta$; see in Fig. 1. For $\alpha < \beta$ the system exhibits a stable limit cycle: large amplitude initial states are attracted to the limit cycle; cf. Fig. 2. In Fig. 3 (for $\alpha = 100\beta = 0.1$, the system's behavior is initially unstable, and a typical chaotic limit cycle picture is obtained; the solution later tends to a limit cycle from inside. As α increases, the system attains a stable state in a deformed limit cycle; cf. Fig. 4.

The Van der-Pol oscillator (with no external force) is known to converge to a limit cycle. Here, we see that lower values of α lead to a limit cycle similar to that of the (stable) Mathieu equation, while for higher values of α, a profile similar to the limit cycle of the Van der-Pol equation is recovered. We conclude that this system features a balance among an instability region, where it behaves according to the Mathieu equation, and a stability region, where it follows the Van der-Pol equation profile (recall that the VdP equation always possesses a periodic solution).

These results complement previous findings on dust charging instabilities occurring in dusty plasmas.

Acknowledgements M.M. would like to thank Prof. J. Mahmodi for helpful discussions. His research was partially supported by the University of Tabriz (Iran). I.K. acknowledges partial support by the Deutsche Forschungsgemeinschaft (Bonn, Germany) through the Sonderforschungsbereich (SFB) 591 Programme.

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