Extended modulational instability criteria for coupled Nonlinear Schrödinger (CNLS) equations

Ioannis Kourakis and Padma Kant Shukla

Institut für Theoretische Physik IV, Ruhr–Universität Bochum, D–44780 Bochum, Germany

1. Introduction

Amplitude modulation (AM) is a widely known nonlinear mechanism dominating wave propagation in dispersive media [1, 2]; it is related to mechanisms such as *modulational instability (MI)*, *harmonic generation* and *energy localization*, possibly leading to soliton formation. The study of AM generically relies on nonlinear Schrödinger (NLS) type equations [3]; a set of coupled NLS (CNLS) equations naturally occurs when interacting modulated waves are considered. CNLS equations are encountered in physical contexts as diverse as electromagnetic wave propagation in nonlinear media [4] optical fibers [5, 6], plasma waves [7], transmission lines [8], and left-handed (negative refraction index) metamaterials (LHM) [9, 10]. A similar paradigm (Gross-Pitaevskii equations) is employed in the mean-field statistical mechanical description of boson gases, to model the dynamics of *Bose-Einstein condensates* (formed at ultra low temperatures) [11, 12, 13].

Here, we shall investigate the (conditions for the) occurrence of MI in a pair of CNLS equations, from first principles, attempting to ***keep generality to a maximum*** (i.e. making no a priori simplifying hypothesis, e.g. on the magnitude and/or the sign of various parameters involved). A set of stability criteria are derived, to be tailor fit to any particular problem of coupled wave propagation. Details omitted here can be found in Ref. [14].

The formalism presented here applies to nonlinear optics [15], coupled BECs [13], left-handed media (LHM) [10], in H-bonded materials [16] and in various other contexts, where coupled wave propagation is of relevance.

2. The model: an asymmetric pair of CNLS Equations

Let us consider two coupled waves propagating in a dispersive and nonlinear medium. The wave functions ($j = 1, 2$) are $\psi_j \exp i(\mathbf{k_j r} - \omega_j t)$ + c.c. (complex conjugate), where the carrier wave number $\mathbf{k_j}$ and frequency ω_j of each wave are related by a dispersion relation function $\omega_j = \omega_j(\mathbf{k_j})$. The nonlinearity of the medium is manifested via a slow modulation of the wave amplitudes, in time and space, say along the x–axis. The amplitude evolution is described by the coupled NLS equations (CNLSE)

$$i\left(\frac{\partial\psi_1}{\partial t} + v_{g,1}\frac{\partial\psi_1}{\partial x}\right) + P_1\frac{\partial^2\psi_1}{\partial x^2} + Q_{11}\,|\psi_1|^2\psi_1 + Q_{12}\,|\psi_2|^2\psi_1 = 0\,,$$
$$i\left(\frac{\partial\psi_2}{\partial t} + v_{g,2}\frac{\partial\psi_2}{\partial x}\right) + P_2\frac{\partial^2\psi_2}{\partial x^2} + Q_{22}\,|\psi_2|^2\psi_2 + Q_{21}\,|\psi_1|^2\psi_2 = 0\,. \quad (1)$$

The *group velocity* $v_{g,j}$ and the *group-velocity-dispersion (GVD)* term P_j corresponding to the j–th wave is related to the dispersion curve via $v_{g,j} = \omega'$and $P_j = \omega''/2$ (in 2- or 3-D, the prime denotes differentiation *in the direction of modulation*, viz. $v_{g,j} = \partial\omega_j/\partial k_x$ and $P_j = \partial^2\omega_j/2\partial k_x^2$). The ***nonlinearity*** and ***coupling terms***, Q_{jj} and $Q_{jj'}$, express the effects of carrier self-modulation and interaction among amplitudes, respectively. No hypothesis holds, ***a priori***, on the sign and/or the magnitude of these coefficients, although specific simplifying assumptions may be relevant in certain problems. If(f) $v_{g,1} = v_{g,2}$, the corresponding terms are eliminated via a Galilean transformation. The combined assumption $P_1 = P_2$, $Q_{11} = Q_{22}$ and $Q_{12} = Q_{21}$ often holds [?, 15]. The case $P_1 = P_2$, $Q_{11} = Q_{21}$ and $Q_{12} = Q_{22}$ has also appeared, for EM waves in left-handed media (LHM) [9, 10].

3. Modulational (in)stability of single waves

A single modulated wave ψ, obeying a (single) NLS Eq., is *modulationally stable* (unstable) if $PQ < 0$ ($PQ > 0$) [1, 2, 3]. Indeed, a (linear) stability analysis around the plane wave (Stokes') solution $\psi(x,t) = \psi_0\, e^{iQ|\psi_0|^2 t}$ shows that a linear modulation, viz. $\psi_0 \to \psi_0 + \epsilon \exp i(Kx - \Omega t)$, obeys the dispersion relation

$$(\Omega - v_g K)^2 = P\,K^2\left(P\,K^2 - 2Q\,|\psi_0|^2\right), \quad (2)$$

which exhibits a purely growing unstable mode if $K \le K_{cr,0} = (2Q/P)^{1/2}\,|\psi_0|$ (only if $PQ > 0$). The growth rate $\sigma = \mathrm{Im}\Omega$ attains a maximum value $\sigma_{max} = |Q|\,|\psi_0|^2$ at $K_{cr,0}/\sqrt{2}$. For $PQ < 0$, the wave is *stable* to external perturbations.

4. Coupled waves: harmonic dispersion relation

Seeking an equilibrium state in the form $\psi_j = \psi_{j0}\exp[i\varphi_j(t)]$, one finds the monochromatic solution $\varphi_j(t) = \Omega_{j0}t$, where $\Omega_{j0} = Q_{jj}\psi_{j0}^2 + Q_{jl}\psi_{l0}^2$ (for $j \ne l = 1, 2$) Considering a small perturbation, we take $\psi_j = (\psi_{j0} + \epsilon\psi_{j1})\exp[i\varphi_j(t)]$ (where $\epsilon \ll 1$) into Eqs. (1), where $\psi_{j1} = \psi_{j1,0}\exp[i(Kx - \Omega t)]$ (K and Ω are the wave vector and the frequency of the perturbation).

One is thus led to a *generalized dispersion relation* in the form:

$$\left[(\Omega - v_{g,1}K)^2 - \Omega_1^2\right]\left[(\Omega - v_{g,2}K)^2 - \Omega_2^2\right] = \Omega_c^4 \quad (3)$$

where $\Omega_c^4 = M_{12}M_{21}$. Here, $M_{jj} = P_jK^2\left(P_jK^2 - 2Q_{jj}\psi_{j0}^2\right) \equiv \Omega_j^2$ and $M_{jl} = -2P_jQ_{jl}\psi_{j0}\psi_{l0}K^2$ (for $l \ne j = 1$ or 2).

5. Wave envelopes at equal group velocities: $v_{g,1} = v_{g,2}$

For $v_{g,1} = v_{g,2}$, one obtains the (reduced) dispersion relation

$$\Omega^4 - T\Omega^2 + D = 0\,, \quad (4)$$

where $T = \mathrm{Tr}\mathbf{M} \equiv \Omega_{11}^2 + \Omega_{22}^2$ and $D = \mathrm{Det}\mathbf{M} \equiv \Omega_{11}^2\Omega_{22}^2 - \Omega_c^4$ are the *trace* and the *determinant*, respectively, of the matrix $\mathbf{M}$. Equation (4) admits the solution

$$\Omega^2 = \frac{1}{2}\left[+T \pm (T^2 - 4D)^{1/2}\right], \quad (5)$$

or

$$\Omega_\pm^2 = \frac{1}{2}(\Omega_1^2 + \Omega_2^2) \pm \frac{1}{2}\left[(\Omega_1^2 - \Omega_2^2)^2 + 4\Omega_c^4\right]^{1/2}. \quad (6)$$

Stability is ensured (*for all* wavenumbers K) if (and only if) *both* of the (two) solutions of (4), say $\Omega_\pm^2$, are *positive* (real) numbers. This is tantamount to the requirements (simultaneously satisfied):

$$T > 0, \qquad D > 0 \qquad and \qquad \Delta = T^2 - 4D > 0\,.$$

(i) 1st road to MI: The positivity of the trace T:

$$T = K^2[K^2\sum_j P_j^2 - 2\sum_j P_jQ_{jj}\psi_{j0}^2] > 0$$

is ensured only if $q_1 \equiv \sum_j P_jQ_{jj}|\psi_{j0}|^2 < 0$. Thus, *absolute stability* ($\forall\,\psi_{j0}\,and\,K$) requires

$$P_1Q_{11} < 0 \quad and \quad P_2Q_{22} < 0 \qquad (Stability\,Criterion\,1,\,SC1)\,.$$

Otherwise, either $\Omega_-^2 < 0 < \Omega_+^2$ or $\Omega_-^2 < \Omega_+^2 < 0$, hence MI for

$$K < K_{cr,1} = \left(2\sum_j P_jQ_{jj}\psi_{j0}^2/\sum_j P_j^2\right)^{1/2};$$

this is always possible for a sufficiently large perturbation amplitude $|\psi_{20}|$ if, say, $P_2Q_{22} > 0$ (even if $P_1Q_{11} < 0$). Therefore, *only a pair of two individually stable waves can be stable; the sole presence of a single unstable wave may de-stabilize its counterpart (even if the latter would be individually stable).*

(ii) 2nd road to MI: The positivity of the determinant D depends on the quantities $q_2 \equiv \sum K_{D,j}^2 = 2P_1P_2(P_2Q_{11}\psi_{0,1}^2 + P_1Q_{22}\psi_{0,2}^2)$ and $q_3 \equiv \prod K_{D,j}^2 = 4P_1P_2(Q_{11}Q_{22} - Q_{12}Q_{21})\psi_{0,1}^2\psi_{0,2}^2$. The condition $D > 0$ $(\forall K, \psi_{j0})$ requires that $q_2 < 0$ *and* $q_3 > 0$. If the former inequality alone is not satisfied (i.e. $q_3 > 0$ and $q_2 > 0$), then the two roots of D, $K_{D,1/2}^2$, will be positive ($0 < K_{D,1}^2 < K_{D,2}^2$) and the wave pair will be unstable to a perturbation with intermediate K, i.e. $K_{D,1}^2 < K^2 < K_{D,2}^2$ (this defines a finite, i.e. non-zero, unstable K "window"). If $q_3 < 0$ (regardless of q_2), then the two roots $K_{D,1/2}^2$ will be of opposite sign, with $K_{D,1}^2 < 0 < K_{D,2}^2$, and the wave pair will be unstable if $K^2 < K_{D,2}^2$.

Absolute stability requires, in addition to (SC1) above (hence $q_2 < 0\ \forall\,\psi_{j0}\,and\,K$), that $q_3 > 0$, i.e.

$$P_1P_2(Q_{11}Q_{22} - Q_{12}Q_{21}) > 0 \qquad (Stability\,Criterion\,2,\,SC2)\,.$$

In the case where $P_1 = P_2$ (e.g. in symmetric BEC pairs [13], where $P_j = \hbar^2/2m$), this criterion reduces to:

$$Q_{11}Q_{22} - Q_{12}Q_{21} > 0 \qquad (Stability\,Criterion\,2',\,SC2')\,.$$

Different combinations (in terms of the amplitude ratio $|\psi_{20}|/|\psi_{10}|$) prescribing (in)stability exist, yet are omitted here for brevity [13].

(iii) 3rd road to MI: The last stability condition requires the positivity of the discriminant quantity $\Delta = T^2 - 4D$ (automatically ensured if $D < 0$). Let us here assume that $D > 0$, i.e. that criterion (C2) *is* satisfied. We consider the inequality:

$$\Delta = K^4\left(d_4K^4 - d_2K^2 + d_0\right) > 0$$

where $d_4 = (P_1^2 - P_2^2)^2$, $d_2 = 4(P_1^2 - P_2^2)(P_1Q_{11}\psi_{10}^2 - P_2Q_{22}\psi_{20}^2)$ and $d_0 = 4[(P_1Q_{11}\psi_{10}^2 - P_2Q_{22}\psi_{20}^2)^2 + 4P_1P_2Q_{12}Q_{21}\psi_{10}^2\psi_{20}^2]$.

Two cases may be distinguished here.

If $P_1 = P_2 = P$, this condition reduces to $d_0 > 0$. Absolute stability is thus only ensured if

$$Q_{12}Q_{21} > 0 \qquad (Stability\,Criterion\,3',\,SC3')\,.$$

See that (C3) is always fulfilled for a symmetric wave pair. If (C3) is violated, the wave pair will be unstable in a range of parameter values, to be determined by solving $d_0' < 0$.

Let us assume that $P_1 > P_2$ (no loss of generality implied). Since $\Delta'' = d_2^2 - 4d_4d_0 = -64P_1P_2Q_{12}Q_{21}(P_1^2 - P_2^2)^2\psi_{10}^2\psi_{20}^2$, the stability condition $\Delta > 0$ is satisfied *for all* K, ψ_{j0} if(f) $\Delta'' < 0$, i.e.

$$P_1P_2Q_{12}Q_{21} > 0 \qquad (Stability\,Criterion\,3,\,SC3)\,.$$

(this is always true for symmetric wave pairs).

If, on the other hand, $q_5 \equiv P_1P_2Q_{12}Q_{21} < 0$, then one needs to investigate the signs of $d_2 = K_{\Delta,1}^2 + K_{\Delta,2}^2 \equiv q_6$ and $d_0 = K_{\Delta,1}^2K_{\Delta,2}^2 \equiv q_7$, in terms of the amplitudes ψ_{j0}. The only possibility for stability ($\forall K$) is provided by the combination $d_2 < 0$ and $d_0 > 0$ (hence $K_{\Delta,1}^2 < K_{\Delta,2}^2 < 0$). The possibility for *instability* arises either for $K_{\Delta,1}^2 < 0 < K^2 < K_{\Delta,2}^2$ (if $d_0 < 0$), or for $0 < K_{\Delta,1}^2 < K^2 < K_{\Delta,2}^2$ (if $d_0 > 0$ *and* $d_2 > 0$). As above, we may thus obtain an instability "window" far from $K = 0$.

The stability criteria (SC1-3) may be summarized as

$$P_1Q_{11} < 0 \qquad and \qquad 0 < P_1P_2Q_{12}Q_{21} < P_1P_2Q_{11}Q_{22}\,,$$

which is the *generalized stability criterion* to be retained.

The above (in)stability regions and wavenumber thresholds need to be determined in detail for any physical problem considered.

MI is manifested if either of the above conditions are violated.

If (SC1) or (SC2) is violated, then one (at least) of the solutions of (3) is negative [cf. (6)]; the maximum instability growth rate is then given by $\sigma \equiv \sqrt{-\Omega_-^2}$, and occurs in the ranges $[0, K_{cr,1}]$ - cf. (SC1) - or, either $[0, K_{D,2}]$ or $[K_{D,1}, K_{D,2}]$ - cf. (SC2).

If (SC3) is violated, all solutions of (3) are complex, viz. $\mathrm{Im}(\Omega_\pm^2) = \pm\sqrt{|\Delta|}/2$, so $\sigma = \mathrm{Max}\{\mathrm{Im}(\Omega_\pm)\} = \mathrm{Max}\{\mathrm{Im}(\Omega_\pm^2)^{1/2}\}$. This is possible either for $[0, K_{\Delta,2}]$ or $[K_{\Delta,1}, K_{\Delta,2}]$.

Finally, *up to three unstable wavenumber "windows" may exist*, either partially superposed, or distinct from each other.

6. Different group velocities: $v_{g,1} \ne v_{g,2}$

The role of the group velocity difference may be investigated qualitatively, by casting Eq. (3) in the form: $f_1(x) = f_2(x)$, where

$$f_1(x) = (x - x_1)^2 + A\,, \qquad f_2(x) = \frac{C}{(x - x_2)^2 + B} \quad (7)$$

and $x = \Omega$, $x_j = Kv_{g,j}$, $A = -\Omega_1^2 = -M_{11}$, $B = -\Omega_2^2 = -M_{22}$, and $C = \Omega_c^4 = M_{12}M_{21}$. The stability profile is determined by the number of *real* solutions of this equation: for stability, we need 4 real solutions; otherwise, MI occurs and the (imaginary parts of the) remaining (complex) solutions determine its growth rate.

See that $A < 0$ ($B < 0$) implies that wave 1 (2) *alone* is stable.

Various parameter combinations exist; see in [14]b for details.

Among other results, one finds that:

(i) a group velocity misfit *destabilizes* a *stable* pair of *stable* waves:

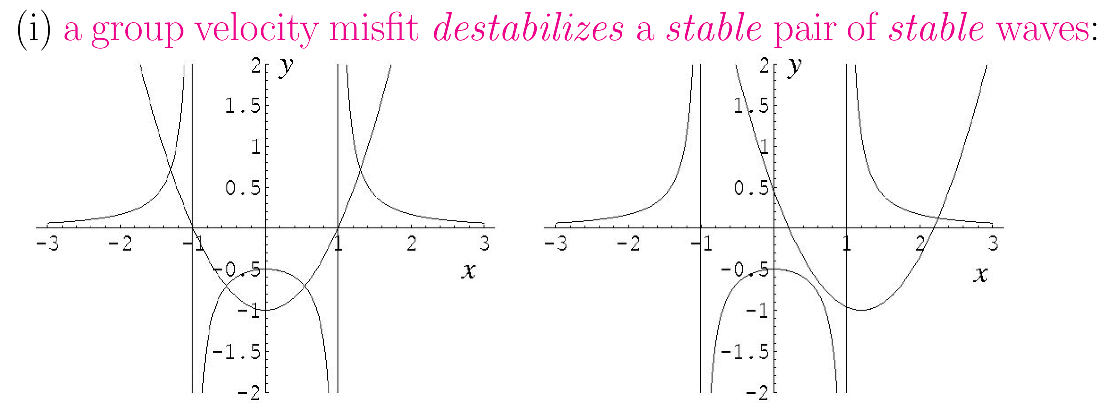

Figure 1. The functions $f_1(x)$ (parabola) and $f_2(x)$ (rational function, two vertical asymptotes) are depicted, vs. x, for $A = B = -1$, $C = 0.5$ (so that $D = AB - C = +0.5 > 0$), $x_1 = x_2 = 0$ (equal group velocities).

(ii) A pair of *unstable* waves is always unstable:

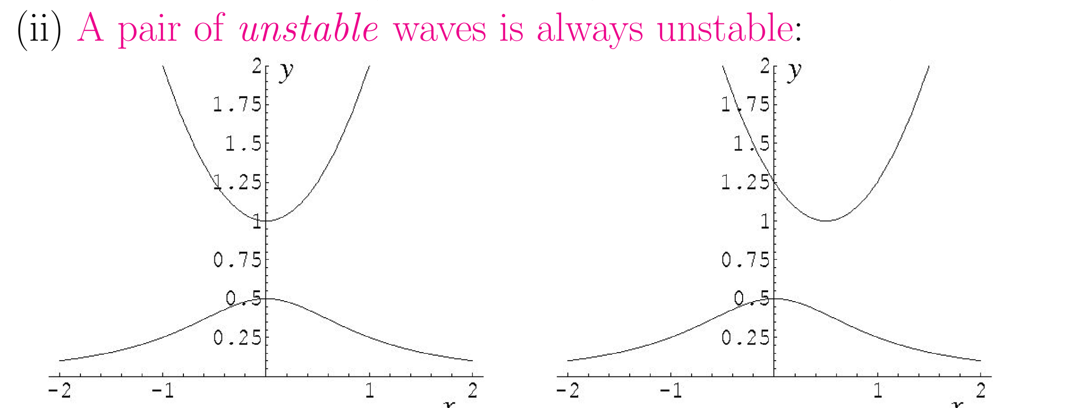

Figure 2. The functions $f_1(x)$ and $f_2(x)$ are depicted, for $A = B = +1$, $C = 0.5$ (so that $D = AB - C > 0$), and $x_1 = x_2 = 0$. At most 2 intersection points may occur by translation, either vertically (A, B) or horizontally $(v_{g1/2})$.

(iii) A *stable* pair of *stable-unstable* waves may always be destabilized by a group velocity mismatch, while an *unstable* pair of (stable-stable) waves remains unstable:

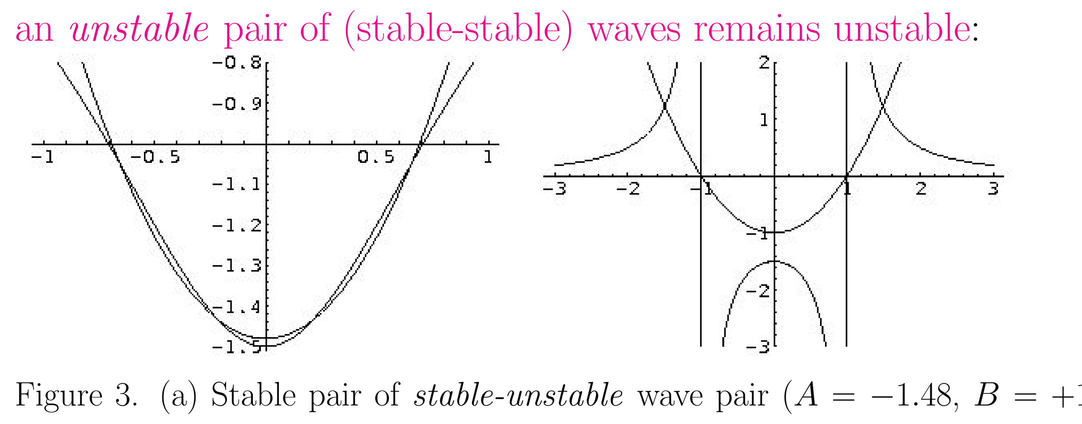

Figure 3. (a) Stable pair of ***stable-unstable*** wave pair ($A = -1.48$, $B = +1$, $C = -1.5$, and $x_1 = x_2 = 0$); (b) Unstable pair of ***Stable-stable*** waves ($A = B = -1$, $C = 1.5$ (so that $D = AB - C = -0.5 < 0$), $x_1 = x_2 = 0$).

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