Nonlinear Modulated Envelope Electromagnetic Excitations in Multi-Component Plasmas
Focus: Oblique Electromagnetic Wavepackets in Magnetized e-p-i Plasmas or Doped Pair Plasmas [*, †]

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Abstract

An exact theory for the nonlinear amplitude modulation of electromagnetic (EM) waves propagating in multi-component plasmas is presented. Focus is made on the nonlinear propagation of EM waves in a three-component plasma, consisting of two pair-ion populations and a third massive ion (assumed immobile). This model referes to electron-positron-ion (epi) plasmas, as well as pair-ion plasmas (p.p.) which are doped with a third massive charged species (e.g. dust defects).

Relying on a multi-fluid + Maxwell plasma description, and employing a multiple scales ("reductive pertubation") technique, the slow space and time evolution (modulation) of an EM wavepacket's envelope is shown to be governed by a set of coupled nonlinear Schrödinger-type equations (CNLSEs), governing the magnetic field (perturbation) components transverse to the ambient magnetic field. The conditions for modulation instability, and for the occurrence of localized EM modes (envelope solitons) are thus investigated.

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I. INTRODUCTION

Localized EM pulses, solitons. Space observations of electromagnetic (EM) plasma waves (by instruments on board satellite missions) provide abundant evidence for the existence of spatially localized propagating EM structures, e.g. in the Earth's magnetosphere [1–3]. These solitary EM waves are manifested as propagating localized excitations of the electric and magnetic fields, accompanied by co-propagating intrinsic plasma parameter (density, fluid velocity) perturbations, which vanish far from the center of the EM perturbation. Localized EM excitations typically take the form of a (one or more) localized hump(s) (a *pulse soliton*, or a train of such solitons). *Kink*-shaped transitions among two different potential regions (*double layers*, *DLs*, or kink solitons) have also been observed. Following a rather established theoretical paradigm, such structures are effectively modeled as quasi-solitons, i.e. localized entities (in fact, solutions of generic integrable partial derivative equations, *PDEs*) which owe their remarkable stability to a mutual balance among the wave dispersion and the intrinsic medium (plasma, here) nonlinearity [4]. In a plasma fluid-theoretical context, soliton excitations are typically modeled via a Sagdeev pseudo-potential (or a Bernoulli quasi-fluid) description and/or, in a long wavelength limit, by the Korteweg-de Vries (KdV) – or associated, e.g. modified KdV – equation paradigm(s) [5]. EM plasma soliton theories have successfully been tested – and confirmed – by laboratory plasma experiments.

Modulated wavepackets, envelope solitons. Apart from these widely studied localized EM forms, an apparently omnipresent type of localized excitation bears the form of a localized envelope pulse, which confines (*modulates*) a fast internal oscillatory structure (the EM carrier wave) and propagates at the EM wave group velocity. *Modulated wavepackets* of this form occur widely in realistic plasma situations, since nonlinearity generically results in wave amplitude variation in space and time. This *amplitude modulation* (AM), however weak, may potentially grow, as a result of random perturbations (noise), thus leading to strong energy localization and eventually to wave collapse into a "sea" of random excitations (*modulation instability*, MI). Alternatively, the perturbed system may evolve towards a stable final state in the form of a series of localized envelope pulses (*envelope solitons*), i.e. the envelope excitations we are interested in modeling here.

Self-modulation, theoretical framework. EM wave amplitude modulation may be due to

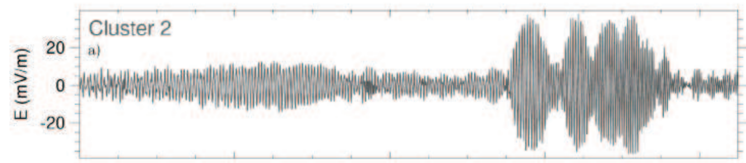

FIG. 1: Modulated structures, related to *'chorus'* (EM) emission in the magnetosphere (CLUSTER satellite data; reprinted from [1]).

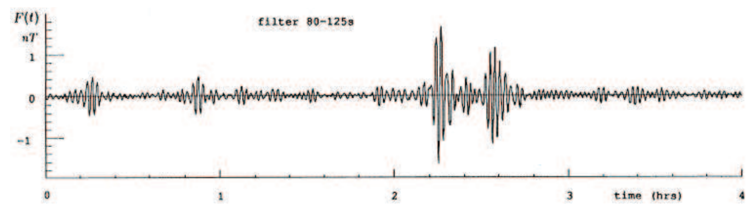

FIG. 2: Localized envelope structures in the magnetosphere (reprinted from [2]).

a variety of reasons. It often is the result of EM wave *ponderomotive coupling* to slow electrostatic (ES) plasma perturbations [6–9]. The plasma response then takes the form of a localized density variation, which accompanies the localized E/M field excitation; such bi-soliton forms are clearly seen in satellite recordings. However, AM may be simply due to nonlinear self-interaction of the carrier wave; this auto- (self-) modulation of EM modes generally leads to secondary *harmonic generation* in the Fourier spectrum and eventually to energy localization the MI mechanism described above. Nonlinear wave modulation thus leads to the formation of localized excitations which are entirely different (in structure and physics) from the constant profile pulses described above. These are typically modeled as envelope soliton solutions of nonlinear Schrödinger (NLS) type, or associated (differential NLS, DNLS) equations, which may be derived by appropriate perturbation schemes. Implementing the so-called reductive perturbation (multiple scales expansion) technique [10], a generic framework for modulated ES structures in a single-fluid plasma picture was elaborated in [11], and was recently extended to a number of multi-fluid problems [12]. A similar framework has been employed in the description of EM waves [13, 14], although no systematic general multi-species theory has so far been presented, to out knowledge.

Electron-positron (e-p) plasmas, pair-ion plasmas (p.p.): prerequisites. Magnetized *electron-positron (e-p) plasmas* exist in pulsar magnetospheres [15–19], in bipolar outflows (jets) in active galactic nuclei [20], at the center of our own galaxy [21], in the early universe

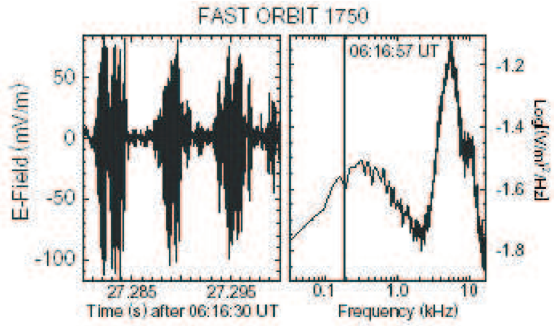

FIG. 3: Electrostatic noise wave forms, related to modulated electron-acoustic waves (FAST satellite data; figure reprinted from [3]). The co-existence of a high (carrier) and a low (modulated envelope) frequencies is clearly reflected in the Fourier spectrum, in the right.

[22], and in inertial confinement fusion scheme using ultraintense lasers [23]. Non-relativistic pair plasmas have also been created in experiments [24] for understanding the dynamics of pairs. Recently, Helander and Ward [25] has discussed the possibility of pair production in large tokamaks due to collisions between multi-MeV runaway electrons and thermal particles. Remarkably, *pair-ion plasmas* (p.p.), i.e. plasmas composed of (two populations of) fully ionized particles with same mass and absolute charges of opposite charge sign, have recently been created in laboratory [26] by creating a large ensemble of fullerene ions (C_{60}^+ and C_{60}^-, in equal numbers), thus allowing for a study of p.p. properties without having to bother for mutual annihilation (recombination), which is responsible for e-p plasma short lifetimes.

Pair plasmas: physics, previous works. The physics of a pair plasma is markedly different from the electron-ion (e-i) plasma in that many of the time and space scales, which are present in an e-i plasma, are simply absent in a pair plasma due to equal masses of the pairs [27–29]. For example, in an unmagnetized pair plasma, the two distinct normal modes are the high-frequency electromagnetic and Langmuir waves, which interact neither linearly nor nonlinearly. In a magnetized pair plasma, besides the electrostatic upper-hybrid waves, we have the perpendicularly propagating ordinary and extraordinary modes as well as magnetic field-aligned circularly polarized EM waves. Iwamoto [30] has presented an elegant description of numerous linear collective modes in a non-relativistic pair magnetoplasma. Zank and Greaves [31] have discussed the linear properties of various electrostatic and electromagnetic modes in unmagnetized and magnetized pair plasmas, in addition to discussing the two-stream instability and non-envelope solitary wave solutions. Magnetic field-aligned nonlinear Alfvén waves in an ultra-relativistic pair plasma have also been investigated by Sakai and Kawata [32] and Verheest [33]. Zhao *et al.* [34] have performed three-dimensional electromagnetic particle simulations of nonlinear Alfvén waves in an electron-positron magnetoplasma.

Pair plasmas: nonlinear theories. Nonlinear excitations in p.p. have been studied quite extensively, yet mostly relying on the pseudo-potential (and associated) (see e.g. [35] and Refs. therein), or the KdV [36] approaches. However, apart from direct extrapolations from general multi-species approaches (cf. [14] and [37]), no systematic theory has so far been presented for modulated envelope wavepackets in pair-ion plasmas. We aim at filling this gap here.

Formulation of the problem. Our aim here is to model a three-component uniform magnetoplasma, consisting of a pair-species population, i.e. two particle species of equal mass and absolute change, yet of opposite charge sign, in addition to a massive third component, which may be considered immobile (frozen) at the frequency scale of interest. This description covers, e.g.

(i) e-p-i plasmas (1=positrons, 2=electrons, 3=ions),

(ii) doped pair ion plasmas, e.g. fullerene pair plasmas (1=C_{60}^+, 2=C_{60}^-) enriched with an extra massive charged component, i.e. defects or dust (3=X^+)

(iii) multi-species dusty plasmas with opposite polarity dust grains (1=d^+, 2=d^-, 3=d'^+, with $m_3 \gg m_2 = m_1$).

II. THE MODEL

To make the notation clear, we consider a plasma composed of three distinct particle species, namely:

- positive ions (mass $m_1 = m$, charge $q_1 = s_1 Z_1 e = +Ze$), here referred to as species 1,
- negative ions (mass $m_2 = m$, charge $q_2 = s_2 Z_2 e = -Ze$), here referred to as species 2, and
- (heavier) positive ions (mass $m_3 = m_i \gg m$, charge $q_3 = s_3 Z_3 e = +Z_i e$), alias species 3.

We have defined the charge state(s) Z_j ($j = 1, 2, 3$), the charge sign $s_j = q_j/|q_j| = \pm 1$ and the absolute electron charge e; we shall denote the respective equilibrium number densities by $n_{j,0}$. In specific, we aim at modeling e-p-i plasmas ($Z_1 = Z_2 = 1$, $s_1 = -s_2 = s_3 = +1$) or, alternatively, pair plasmas of the type, say, $X_A^{+Z} X_A'^{-Z} Y_{A'}^{\pm Z'}$, i.e. enriched with a third massive ion (or dust) component ($Z_1 = Z_2 = Z$, $s_1 = -s_2 = +1$ and $s_3 = \pm 1$ — either case may be possible). Pair fullerene-ion [viz. 1 (2) $= C_{60}^{+(-)}$] contaminated, say, by charged defects (e.g. 3 = dust) are thus covered, for $Z = 1$. The pure p.p. case is also implicitly covered, for $n_{3,0} = 0$. Although the parameter space (s_j, Z_j, m_j) is thus somewhat prescribed, given the physical system of interest, we may keep the general notation (i.e. indices $j = 1, 2, 3$ everywhere) wherever appropriate, for generality.

We shall consider the (two-) fluid plasma moment evolution (density and momentum) equations:

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \, \mathbf{u}_j) \; = \; 0 \tag{1}$$

$$\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{u}_j \cdot \nabla \mathbf{u}_j \; = \; \frac{q_j}{m_j} \left(\mathbf{E} + \frac{1}{c} \, \mathbf{u}_j \times \mathbf{B} \right), \tag{2}$$

where n_j and $\mathbf{u}_j$ respectively denote the density and the mean (fluid) velocity of species j $(= 1, 2)$. The (total) electric and magnetic fields, denoted by $\mathbf{E}$ and $\mathbf{B}$ respectively, obey Maxwell's laws:

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \; = \; -\nabla \times \mathbf{E} \, , \tag{3}$$

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \; = \; \nabla \times \mathbf{B} - \frac{4\pi}{c} \sum_j n_j q_j \mathbf{u}_j \, , \tag{4}$$

The electric field $\mathbf{E} = -\nabla \, \phi$ (deriving from an electric potential ϕ), obeys Poisson's equation

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi \; = \; 4\pi \sum_{j=1,2,3} q_j \, n_j \, , \tag{5}$$

where the index "0" denotes the (fixed) equilibrium densities, and the magnetic field satisfies Gauss' law

$$\nabla \cdot \mathbf{B} = 0 \, . \tag{6}$$

We may note that, although valid, the latter two equations are here treated as constraints, rather than part of the system's evolution law, since Eqs. (1) - (4) form a closed system of (14) scalar evolution equations, for the (14 scalar) [45] elements of the state vector

$$\mathbf{S} = (n_1, u_{1,x/y/z}; n_2, u_{2,x/y/z}; E_{x/y/z}; B_{x/y/z}) \; \in \Re^{14} \, . \tag{7}$$

At equilibrium, overall charge neutrality is assumed, so the densities satisfy

$$\sum_j q_j \, n_{j,0} = \sum_j s_j Z_j n_{j,0} = 0 \, , \tag{8}$$

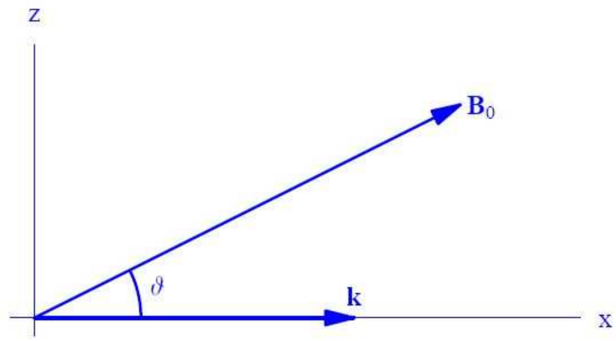

FIG. 4: The reference frame: EM wave propagation takes place along the $x-$axis, while the ambient magnetic field lies in the $xz-$plane.

viz. $Zn_{+,0} - Zn_{-,0} + s_i Z_i n_{i,0} = 0$ in doped pair plasmas, or in e-p-i plasmas (where $Z = s_i = 1$). Note that the existence of the third species, which otherwise appears nowhere in the plasma fluid-dynamical equations (1) - (4), affects the dynamics via the charge balance; for instance, the ratio $n_{1,0}/n_{2,0} = n_{+,0}/n_{-,0}$ ($= 1$ in pair plasmas) now becomes $n_{1,0}/n_{2,0} = 1 - s_i(n_{i,0}/n_{2,0})Z_i/Z\ < 1$ (or > 1) for positive (respectively, negative) fixed ions, i.e. $n_+/n_- = 1 - n_{i,0}Z_i/n_{-,0}\ < 1$ in e-p-i plasmas. Furthermore, a quiescent plasma is considered, so $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ at equilibrium. Finally, although no ambient electric field exists, viz. $\mathbf{E_0} = \mathbf{0}$, a uniform external magnetic field induction $\mathbf{B_0}$ is considered.

The tedious analytical calculation is simplified by introducing an appropriate reference frame. We may assume that the direction of wave propagation defines the axis x, e.g. implying a wave number $\mathbf{k} = k\hat{x}$ in the linear case ($\hat{x}, \hat{y}, \hat{z}$ here denote the unit vectors along the respective directions). Furthermore, we shall assume that the external magnetic field $\mathbf{B_0}$ lies on the $x - z$ plane, i.e. $\mathbf{B_0} = B_{0,x}\hat{x} + B_{0,z}\hat{z} = B_0(\cos\theta\hat{x} + \sin\theta\hat{z})$. All quantities are assumed to vary along the direction of propagation, i.e. $\nabla \to \partial/\partial x$ (see that the operator $\nabla\times\cdot$, in Maxwell's Eqs., thus becomes $\hat{x}\times\partial\cdot/\partial x$). Eqs. (6) and (the $x-$component of) (3) thus immediately imply a static magnetic field component along propagation, i.e. $B_x = B_{x,0} = B_0\cos\theta =$cst. [38].

The analytical model described here is generic, and agrees with the model adopted in Refs. [31, 35, 36], for oblique EM wave propagation, as well as in Ref. [14], for parallel propagation (see that a three-fluid model was employed in Refs. [14, 36], though). The analysis of linear wave characteristics carried out in these Refs. is therefore herein confirmed, and is understood as a prerequisite to the nonlinear investigation proposed in the following.

It may be noted, for rigor, that an alternative description may be adopted (also in view of reducing the number of dynamical variables), by defining the vector potential $\mathbf{A}$, viz. $\mathbf{B} = \nabla \times \mathbf{A}$ (thus $\mathbf{B_0} = \nabla \times \mathbf{A_0}$); cf. in Ref. [14]. Defining the transverse component of the vector potential perturbation $\mathbf{A} - \mathbf{A_0}$, as

$$A_\perp = A_y \pm i A_z \,, \tag{9}$$

and denoting the transverse velocity, electric and magnetic field corrections accordingly, via complex variables

$$u_{j,\perp} = u_{j,y} \pm i u_{j,z} \,, \qquad E_\perp = E_y \pm i E_z \,, \qquad B_\perp = B_y \pm i B_z \,, \tag{10}$$

the latter two evolve according to

$$E_\perp = -\frac{1}{c}\frac{\partial A_\perp}{\partial t} \,, \qquad B_\perp = \pm i \frac{\partial A_\perp}{\partial x} \,. \tag{11}$$

Here (and below), the upper/lower (here +/-) sign refers to the L/R, i.e. left-/right-hand polarized wave(s), respectively. Eq. (3) is identically satisfied now, while (4) thus takes a closed form, in terms of $\mathbf{A}$. This notation is useful in the parallel propagation case, yet the apparent “symmetry” among the dynamics of the transverse components of the magnetic field is broken in the oblique propagation case, and physical transparency (by introducing $\mathbf{B}$) is lost. We may nevertheless keep definitions (9) - (11) here, for reference.

III. PERTURBATIVE ANALYSIS: THE ANALYTICAL FRAMEWORK

Equations (1) – (4) describe the evolution of the state vector $\mathbf{S}$ (defined above), which accepts a harmonic (electrostatic) wave solution in the linear (weak amplitude) approximation, in the form $\mathbf{S} = \mathbf{S_0} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] + \text{c.c.}$ (the carrier phase is essentially $\phi = kx - \omega t$, in our frame; ”c.c.” denotes the complex conjugate quatity). Once the wave amplitude becomes important (small but finite), nonlinearity enters into play, due to self-interaction of the carrier wave. This is first manifested as self- (auto-)modulation of the amplitude, i.e. a weak variation of the wave’s envelope in space and time, and the creation of sidebands in Fourier space. The evolution profile then includes generation of secondary phase harmonics and, as the amplitude grows locally, energy localization via modulational instability of the wave envelope.

What follows is essentially an implementation of the long known *reductive perturbation* technique [10], which was first applied in the study of electron plasma waves [10] and electron-cyclotron waves [13], more than three decades ago. In order to study the nonlinear (amplitude) modulational stability profile of these electrostatic waves, we consider small deviations from the equilibrium state

$$\mathbf{S}^{(0)} = (n_{1,0}, \mathbf{0}; n_{2,0}, \mathbf{0}; \mathbf{0}; \mathbf{B_0})^T \,,$$

viz.

$$\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}^{(1)} + \epsilon^2 \mathbf{S}^{(2)} + ... \,,$$

where $\epsilon \ll 1$ is a (real) smallness parameter. We assume that

$$S_j^{(n)} \; = \sum_{l=-\infty}^{\infty} S_j^{(n,l)}(X,\, T)\, \exp\left[il(kx - \omega t)\right]$$

(for $j = 1, 2, ..., 14$ here; see above); the condition $S_j^{(n,-l)} = S_j^{(n,l)\,*}$ holds, for reality [46]. The wave amplitude is thus allowed to depend on the stretched (*slow*) coordinates of space $X = \{\epsilon^n x,\, n = 1, 2, ...\} = \{X_1, X_2, ...\}$ (viz. $X_1 = \epsilon x$, $X_2 = \epsilon^2 x$, and so forth) and time $T = \{\epsilon^n t,\, n = 1, 2, ...\} = \{T_1, T_2, ...\}$ (i.e. $T_1 = \epsilon t$, $T_2 = \epsilon^2 t$, ...), to be distinguished from the (fast) carrier variables x ($\equiv X_0$) and t ($\equiv T_0$).

It may be appropriate to note, here, that this perturbation scheme is exactly equivalent to (yet rather more physically transparent than) the alternative assumption – often encountered in literature – which consists in defining the slow parameters $X = \epsilon(x - \lambda\, t)$ and $T = \epsilon^2\, t$. The real variable λ, is thus interpreted at a later stage as the wave's *group velocity*, i.e. $\lambda = v_g = \omega'(k)$.

According to the above considerations, we set:

$$\partial/\partial t \rightarrow \partial/\partial T_0 + \epsilon\partial/\partial T_1 + \epsilon^2\partial/\partial T_2 + ... \qquad \text{and} \qquad \nabla \rightarrow \partial/\partial X_0 + \epsilon\partial/\partial X_1 + \epsilon^2\partial/\partial X_2 + ... \,,$$

so that

$$\begin{aligned} \frac{\partial}{\partial t}\, \Psi_l^{(n)}\, e^{il\phi} \; &= \; \biggl(- il\omega\, \Psi_l^{(n)} \; + \epsilon \frac{\partial \Psi_l^{(n)}}{\partial T_1} + \epsilon^2\, \frac{\partial \Psi_l^{(n)}}{\partial T_2} \biggr)\, e^{il\phi} + \mathcal{O}(\epsilon^3) \,, \\ \nabla\, \Psi_l^{(n)}\, e^{il\phi} \; &= \; \biggl(+ ilk\, \Psi_l^{(n)} \; + \epsilon \frac{\partial \Psi_l^{(n)}}{\partial X_1} + \epsilon^2\, \frac{\partial \Psi_l^{(n)}}{\partial X_2} \biggr)\, e^{il\phi} + \mathcal{O}(\epsilon^3) \,, \end{aligned} \qquad (12)$$

for any $l-$th phase harmonic amplitude $\Psi_l^{(n)}$ among the components of $\mathbf{S}^{(n)}$. Recall that ϕ denotes the carrier (1st harmonic) phase $\phi \equiv kx - \omega t$.

By inserting the above ansatz into Eqs. (1) to (4), one obtains a set of (coupled) reduced evolution equations for the state variable harmonic amplitudes $S_j^{(n,l)}$ (here $j = 1, 2, ..., 13$ [38]), which should be solved in each perturbation order $\sim \epsilon^n$ for the $l-$th phase harmonic amplitudes ($l = -n, -n+1, ..., n-1, n$). Although particularly lengthy, the calculation is perfectly straightforward, so unnecessary details will be omitted in the following.

IV. ANALYTICAL MANIPULATION OF THE EVOLUTION EQUATIONS: THE METHODOLOGY

In every order $\sim \epsilon$, one needs to cope with the tedious task of solving the large system of evolution equations, namely Eqs. (1, 2) for each fluid j (=1, 2), Faraday's law (3) and Ampère's law (4) (as expressed in the specific order). For clarity, the scalar equations for the state variables are explicitly reproduced in the following.

A. The complete system of scalar evolution equations for the state variables

In our reference frame, the equations for the 2 fluid densities and velocity components read:

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial t}(n_j \mathbf{u}_j) = 0 \tag{13}$$

$$\frac{\partial u_{j,x}}{\partial t} + u_{j,x}\frac{\partial u_{j,x}}{\partial x} = \frac{q_j}{m_j}\left[E_x + \frac{1}{c}\left(u_{j,y}B_z - u_{j,z}B_y\right)\right], \tag{14}$$

$$\frac{\partial u_{j,y}}{\partial t} + u_{j,x}\frac{\partial u_{j,y}}{\partial x} = \frac{q_j}{m_j}\left[E_y + \frac{1}{c}\left(u_{j,z}B_x - u_{j,x}B_z\right)\right], \tag{15}$$

$$\frac{\partial u_{j,z}}{\partial t} + u_{j,x}\frac{\partial u_{j,z}}{\partial x} = \frac{q_j}{m_j}\left[E_z + \frac{1}{c}\left(u_{j,x}B_y - u_{j,y}B_x\right)\right]. \tag{16}$$

Recall that $m_1 = m_2 = m$ and $q_1 = -q_2 = +Ze$, here. Faraday's law relating the E/M fields reads:

$$\frac{\partial B_x}{\partial t} = 0\,, \tag{17}$$

$$\frac{\partial B_y}{\partial t} = +c\frac{\partial E_z}{\partial x}\,, \tag{18}$$

$$\frac{\partial B_z}{\partial t} = -c\frac{\partial E_y}{\partial x}\,. \tag{19}$$

Remember that Eq. (17) (together with $\nabla \cdot \mathbf{B} = \partial B_x/\partial x = 0$) is consistent with $B_x =$ cst., as stated above, implying $B_x^{(n')} = 0$ at every order n'. Ampère's law, which couples the dynamics of the separate fluids, reads

$$
\begin{aligned}
\frac{\partial E_x}{\partial t} &= -4\pi \sum_j n_j q_j u_{x,y} \\
&= -4\pi Z e \left(n_1 u_{1,x} - n_2 u_{2,x} \right) , \qquad (20) \\
\frac{\partial E_y}{\partial t} &= -c \frac{\partial B_z}{\partial x} - 4\pi \sum_j n_j q_j u_{j,y} \\
&= -c \frac{\partial B_z}{\partial x} - 4\pi Z e \left(n_1 u_{1,y} - n_2 u_{2,y} \right) , \qquad (21) \\
\frac{\partial E_z}{\partial t} &= +c \frac{\partial B_y}{\partial x} - 4\pi \sum_j n_j q_j u_{j,z} \\
&= +c \frac{\partial B_y}{\partial x} - 4\pi Z e \left(n_1 u_{1,z} - n_2 u_{2,z} \right) . \qquad (22)
\end{aligned}
$$

B. Method of analytical treatment

The method employed to disentangle the large system above consists in eliminating E_y and E_z everywhere, by making use of Eqs. (18, 19), and then separately considering the (two - formally analogous - systems of) fluid equations for the fluid variables, on one hand, and Ampère's law (20-22) for E_x, B_y and B_z, on the other. Further eliminating the parallel electric field E_x everywhere, via (the $x-$component of Ampère's law (20), provides a set of expressions for the state variables in terms of the transverse magnetic field components, namely B_y and B_z.

In specific:

(i) The first subset of equations is obtained by combining Eqs. (13 - 16) and (18, 19) into a system for the $j-$th fluid variables, in terms of E_x, B_y and B_z, namely

$$
\begin{aligned}
\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial t}(n_j \mathbf{u}_j) &= 0 \qquad (23) \\
\frac{\partial u_{j,x}}{\partial t} + u_{j,x} \frac{\partial u_{j,x}}{\partial x} &= s_j \Omega_j \left(c E'_x + u_{j,y} B'_z - u_{j,z} B'_y \right) , \qquad (24) \\
\frac{\partial}{\partial x} \left(\frac{\partial u_{j,y}}{\partial t} + u_{j,x} \frac{\partial u_{j,y}}{\partial x} \right) &= s_j \Omega_j \left(- \frac{\partial B'_z}{\partial t} + u_{j,z} B'_x - u_{j,x} B'_z \right) , \qquad (25) \\
\frac{\partial}{\partial x} \left(\frac{\partial u_{j,z}}{\partial t} + u_{j,x} \frac{\partial u_{j,z}}{\partial x} \right) &= s_j \Omega_j \left(+ \frac{\partial B'_y}{\partial t} + u_{j,x} B'_y - u_{j,y} B'_x \right) . \qquad (26)
\end{aligned}
$$

For reference, let us write down the *linearized* version of this system (to be made extensive use of, below), which can be cast in the form

$$
\mathbf{L}_{0,j} \, \mathbf{f}_j = \mathbf{F} ,
$$

viz.

$$\begin{pmatrix} \partial/\partial t & n_{j,0}\partial/\partial x & 0 & 0 \\ 0 & \partial/\partial t & -s_j\Omega_j \sin\theta & 0 \\ 0 & s_j\Omega_j \sin\theta\partial/\partial x & \partial^2/\partial t\partial x & -s_j\Omega_j \cos\theta\partial/\partial x \\ 0 & 0 & s_j\Omega_j \cos\theta\partial/\partial x & \partial^2/\partial t\partial x \end{pmatrix} \begin{pmatrix} n_j^{(n,1)} \\ u_{j,x}^{(n,1)} \\ u_{j,y}^{(n,1)} \\ u_{j,z}^{(n,1)} \end{pmatrix} = \begin{pmatrix} 0 \\ cE_x^{(n,1)} \\ -\partial B_z^{(n,1)}/\partial t \\ +\partial B_y^{(n,1)}/\partial t \end{pmatrix}, \tag{27}$$

where the definitions of the linear matrix operator $\mathbf{L}_{\mathbf{0},j}(\partial/\partial t, \partial/\partial x)$ and of the vectors $\mathbf{f}_j$ and $\mathbf{F}$ are obvious. We have defined the $j-$th species cyclotron frequency $\Omega_j = |q_j|B_0/(m_j c)$, viz. $q_j B_0/(m_j c) = s_j\Omega_j$, i.e. $\Omega_1 = \Omega_2 = ZeB_0/(mc)$ and $s_1 = -s_2 = +1$. The primed quantities denote field components scaled by B_0, i.e. the dimensionless quantities $E'_x = E_x/B_0$, $B'_y = B_y/B_0$ and $B'_z = B_z/B_0$. One thus has two distinct (yet of formally identical structure) sets of equations, to be solved for the two fluid variables, in terms of E'_x, B'_y and B'_z.

(ii) One more subset of equations is obtained by combining Eqs. (20 - 22), respectively, via (18, 19), into

$$\begin{aligned} \frac{\partial E'_x}{\partial t} &= -\frac{4\pi}{B_0}\sum_j n_j q_j u_{x,y} \\ &= -4\pi Ze\,(n_1 u_{1,x} - n_2 u_{2,x})\,, \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial^2 B'_z}{\partial t^2} - c^2\frac{\partial^2 B'_z}{\partial x^2} &= +\frac{4\pi c}{B_0}\frac{\partial}{\partial x}\sum_j n_j q_j u_{j,y} \\ &= +\frac{4\pi Zec}{B_0}\frac{\partial}{\partial x}\,(n_1 u_{1,y} - n_2 u_{2,y})\,, \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial^2 B'_y}{\partial t^2} - c^2\frac{\partial^2 B'_y}{\partial x^2} &= -\frac{4\pi c}{B_0}\frac{\partial}{\partial x}\sum_j n_j q_j u_{j,z} \\ &= -\frac{4\pi Zec}{B_0}\frac{\partial}{\partial x}\,(n_1 u_{1,z} - n_2 u_{2,z})\,. \end{aligned} \tag{30}$$

(iii) Now,

– solving Eqs. (13-26) for $\{n_j, u_{j,x/y/z}\}$, in terms of E_x, B_y and B_z (for $j = 1, 2$: the expressions will be analogous), and then

– substituting into (28-30) (now explicitly taking $j = 1, 2$ *separately*),

one obtains a system for E'_x, B'_y and B'_z; in principle, this (linear) system is homogeneous to 1st order, and should thus be imposed to bear a vanishing determinant $D = 0$ (otherwise only the trivial, zero-vector solution exists); apart from a linear dispersion relation (which thus naturally arises as a compatibility constraint), one obtains a solution for E'_x in terms of, say, B'_y and B'_z (any other variable combination could be chosen at this stage).

– The latter may now be substituted back into $\{n_j, u_{j,x/y/z}\}$ (obtained above), to provide the *final* expressions for the fluid variables (in terms of B_y and B_z). The remaining transverse electric field variables E_y and E_z are then readily obtained from Eqs. (18, 19).

By iterating this procedure at every order n (= 1, 2, ...) and for each harmonic l (= $0, 1, ..., n$), we obtain the solution for the harmonic amplitudes and a number of relations in the form of compatibility conditions, which provide the linear EM wave dispersion relation, the wave envelope's (group) velocity and the amplitude evolution equation(s). A symbolic computation software package (Maple, Mathematica) may be used to facilitate this tedious task.

V. 1ST-ORDER DYNAMICS ($n = 1$): LINEAR EM WAVES, HARMONIC AMPLITUDES, DISPERSION RELATION

The 1st-order equations describe the dynamics of a *linear* solution of the system of Eqs. (1) - (4) which, for $n = l = 1$, lead to the system of equations

$$-\omega n_j^{(1,1)} + n_{j,0}\, k u_{j,x}^{(1,1)} = 0 \tag{31}$$

$$\omega \mathbf{u}_j^{(1,1)} = i\frac{q_j}{m_j}\left(\mathbf{E}^{(1,1)} + \frac{1}{c}\,\mathbf{u}_j^{(1,1)} \times \mathbf{B_0}\right), \tag{32}$$

$$\frac{\omega}{c}\,\mathbf{B}^{(1,1)} = \mathbf{k} \times \mathbf{E}^{(1,1)}, \tag{33}$$

$$\frac{\omega}{c}\,\mathbf{E}^{(1,1)} = -\mathbf{k} \times \mathbf{B}^{(1,1)} - \frac{4\pi i}{c}\sum_j n_{j,0} q_j \mathbf{u}_j^{(1,1)}, \tag{34}$$

where $j = 1, 2$.

The homogeneous linear system obtained for $n = l = 1$ (1st-order, 1st harmonics) possesses a non-trivial (non-zero) solution only if its determinant, say D_0, vanishes. This requirement provides the harmonic wave dispersion law. Therefore, the remaining part of this Section is dedicated to different distinct tasks. First, the linear dispersion relation (which ensures the compatibility condition $D_0 = 0$) is derived and briefly discussed. Then, the system is manipulated in order to obtain a set of explicit relations among various state variables; these may serve in interpreting e.g. EM wave Space or laboratory measurements of electric/magnetic field components. The evolution law is then solved for the dynamical variables involved (recall that 13 variables evolve here, since B_x =cst.), in terms of a subset of free evolving variables: these will be chosen as the transverse magnetic field components, B_y and B_z, as we shall see below. A final note is added, about the zeroth harmonic contribution, for rigor.

A. Dispersion relation

Evaluating the determinant of the (linear) system of Eqs. (31)-(34), for e-p-i (or "doped" pair-ion) plasma, and taking it to vanish (for a non-trivial solution to exist), one is led to a dispersion relation in the form

$$D(\omega, k; \theta) = d_0(\omega, k) + d_1(\omega, k) \sin^2\theta = 0 \,, \tag{35}$$

where d_0 and d_1 are polynomial expressions given by

$$\begin{aligned} d_0(\omega, k) &\equiv D(\omega, k; \theta = 0) \\ &= (\omega^2 - \omega_{p,eff}^2) \\ &\quad \times \left\{ \left[(\omega^2 - c^2k^2)(\omega^2 - \Omega^2) - \omega^2 \omega_{p,eff}^2 \right]^2 - \omega^2 \Omega^2 (\omega_{p,1}^2 - \omega_{p,2}^2)^2 \right\} \\ &= (\omega^2 - \omega_{p,eff}^2) \\ &\quad \times \left\{ (\omega + \Omega) \left[-(\omega^2 - c^2k^2)(\omega - \Omega) + \omega \omega_{p,1}^2 \right] + \omega(\omega - \Omega)\omega_{p,2}^2 \right\} \\ &\quad \times \left\{ (\omega - \Omega) \left[-(\omega^2 - c^2k^2)(\omega + \Omega) + \omega \omega_{p,1}^2 \right] + \omega(\omega + \Omega)\omega_{p,2}^2 \right\} \,, \end{aligned} \tag{36}$$

and

$$d_1(\omega, k; \theta) = -c^2k^2\Omega^2 \left\{ c^2k^2\omega_{p,eff}^2(\omega^2 - \Omega^2) + \omega^2 [4\omega_{p,1}^2\omega_{p,2}^2 - (\omega^2 - \Omega^2)\omega_{p,eff}^2] \right\} \,, \tag{37}$$

where we have defined the plasma frequency $\omega_{p,j} = (4\pi n_{j,0} Z^2 e^2 / m_j)^{1/2}$ of the $j-$th (i.e. $1 = +$ or $2 = -$) species, the effective (total) plasma frequency $\omega_{p,eff} = (\omega_{p,1}^2 + \omega_{p,2}^2)^{1/2}$ and the cyclotron frequency $\Omega = ZeB_0/(m_j c)$. Note that d_0 is a 10th order polynomial in the frequency ω, while d_1 is a 4th order polynomial in ω; in both quantities, only even terms are present, so that d_0 (d_1) is essentially a quintic (quartic) polynomial in ω^2. Therefore, upto 5 different solutions for ω^2 may exist, hence 5 distinct even modes for the (real part of the) frequency ω are possible to propagate, depending on the angle θ and the values of the physical parameters involved.

Dimensionless form of the dispersion relation. The general dispersion relation (35) may be reduced, for future manipulation, into a dimensionless form, by defining appropriate scales. Following the notation in [37], we may define the density mismatch parameter

$$\eta = \frac{n_{+,0} - n_{-,0}}{n_{+,0} + n_{-,0}} \tag{38}$$

(see that $\omega_{p,1}^2 - \omega_{p,2}^2 = \eta \omega_{p,eff}^2$), which measures deviation from pair-ion neutrality (the "pure" pair plasma case is recovered for $\eta \to 0$; the case $\eta \neq 0$ thus indicates the existence of a third species, or an overall neutrality violation in the plasma composition, at equilibrium). We shall also define the reduced wave frequency, wavenumber and reduced plasma frequency

$$f = \omega/\Omega\,, \qquad \kappa = ck/\Omega\,, \qquad h = \omega_{p,eff}^2/\Omega^2 = (\omega_{p,1}^2 + \omega_{p,2}^2)/\Omega^2\,, \tag{39}$$

respectively; see that $\omega_{p,1/2}^2/\Omega^2 \to (1 \pm \eta)h/2$. Eqs. (36) and (37) thus become

$$\begin{aligned} \hat{d}_0(\omega, k) &\equiv d_0/\Omega^{10} \\ &= (f^2 - h^2)\left\{\left[(f^2 - \kappa^2)(f^2 - 1) - f^2 h^2\right]^2 - f^2 \eta^2 h^2\right\} \\ &= (f^2 - h^2) \\ &\quad \times\left\{(f+1)\left[-(f^2 - \kappa^2)(f-1) + f(1+\eta)h/2\right] + f(f-1)(1-\eta)h/2\right\} \\ &\quad \times\left\{(f-1)\left[-(f^2 - \kappa^2)(f+1) + f(1+\eta)h/2\right] + f(f+1)(1-\eta)h/2\right\}\,, \end{aligned} \tag{40}$$

and

$$\hat{d}_1(\omega, k; \theta) \equiv d_1/\Omega^4 = -\kappa^2\left\{\kappa^2 h(f^2 - 1) + f^2[(1 - \eta^2)h^2 - (f^2 - 1)h]\right\}\,, \tag{41}$$

Parallel EM wave propagation. For parallel propagation, i.e. for $\theta = 0$, expression (35) reduces to $d_0 = 0$ [recall (37)], implying the existence of a number of distinct non-trivial (parallel) propagation modes, which are described by the dispersion relations:

$$\omega^2 = \omega_{p,eff}^2 \tag{42}$$

$$\omega^4 - \omega^2(\omega_{p,eff}^2 + \Omega^2 + c^2k^2) - \omega\Omega(\omega_{p,1}^2 - \omega_{p,2}^2) + c^2k^2\Omega^2 = 0 \tag{43}$$

and

$$\omega^4 - \omega^2(\omega_{p,eff}^2 + \Omega^2 + c^2k^2) + \omega\Omega(\omega_{p,1}^2 - \omega_{p,2}^2) + c^2k^2\Omega^2 = 0 \tag{44}$$

We see that a number of distinct modes are present, i.e. solutions of (43, 44) for ω. Note that the latter two relations are only different due to the deviation from incompressibility (i.e.

$n_{1,0} \neq n_{2,0}$), due to the existence of the third (fixed ion) species, e.g. ions in e-p-i plasmas, or "dust" defects in pair-ion (eg. fullerene) plasmas; indeed, $\omega_{p,1}^2 - \omega_{p,2}^2 \sim n_{1,0} - n_{2,0}$, so that these relation merge into one another in the p.p. limit (see below).

The modes described by Eqs. (43, 44) have been briefly discussed in Ref. [37] [see that the relations are identical [39] to Eqs. (5, 6) therein], relying upon the results in [14]. These equations may be cast in the form [37]

$$(f^2 - 1)(f^2 - \kappa^2) - f^2 h \pm \eta h f = 0 \tag{45}$$

where the reduced frequency f and all (dimensionless) parameters were defined above.

Interestingly, in the pure pair-plasma case (i.e. for $\eta = 0$), (45) can be solved exactly for f^2, leading (apart from $f = \pm h$) to

$$f^2 = \frac{1}{2}(1 + \kappa^2 + h)\left\{1 \pm \left[1 - 4\kappa^2/(1 + \kappa^2 + h)^2\right]^{1/2}\right\}, \tag{46}$$

i.e.

$$f^2 \approx \frac{1}{2}(1 + \kappa^2 + h)\left\{1 \pm \left[1 - 2\kappa^2(1 - 2h)\right]\right\} \tag{47}$$

for small wavenumber κ (and, say, plasma frequency h). One thus obtains (lower branch) acoustic mode:

$$f_-^2 \approx (1 + \kappa^2 + h)\kappa^2(1 - 2h) \approx (1 - 2h)\kappa^2 \; + \mathcal{O}(\kappa^2), \tag{48}$$

and an (upper branch) optic-type mode:

$$f_+^2 \approx (1 + \kappa^2 + h)\left[1 - \kappa^2(1 - 2h)\right]. \tag{49}$$

Now, switching back to $\eta \neq 0$, the effect of the density mismatch, which results e.g. from the existence of a third (fixed ion) species, is to split the two linearly polarized EM modes (present in p.p. [35]) to four distinct circularly polarized modes, as pointed out in [37]. Focusing on the behavior near $k = 0$, one finds that three out of these modes present a frequency cutoff, i.e. $\omega(k = 0) \neq 0$, below which no wave propagates. For instance, in the vicinity of $f \approx 0$, and for small η and h, one finds analytically that the Alfvén type wave which occurs for $\eta = 0$, splits into two modes, one of which presents a cutoff at $f_0 = |\eta| h/(1 + h)$.

The pair-plasma limit. In the absence of the 3rd (fixed ion) species, one recovers the special case of a pair plasma; setting $\omega_{p,1} = \omega_{p,2} = \omega_p$) in (35), one obtains:

$$D|_{p.p.}(\omega, \theta) = \left[(\omega^2 - c^2k^2)(\omega^2 - \Omega^2) - 2\omega^2\omega_p^2\right] \times \left[\omega^2(\omega^2 - c^2k^2 - 2\omega_p^2)(\omega^2 - \Omega^2 - 2\omega_p^2) - 2c^2k^2\Omega^2\omega_p^2\cos^2\theta\right]. \tag{50}$$

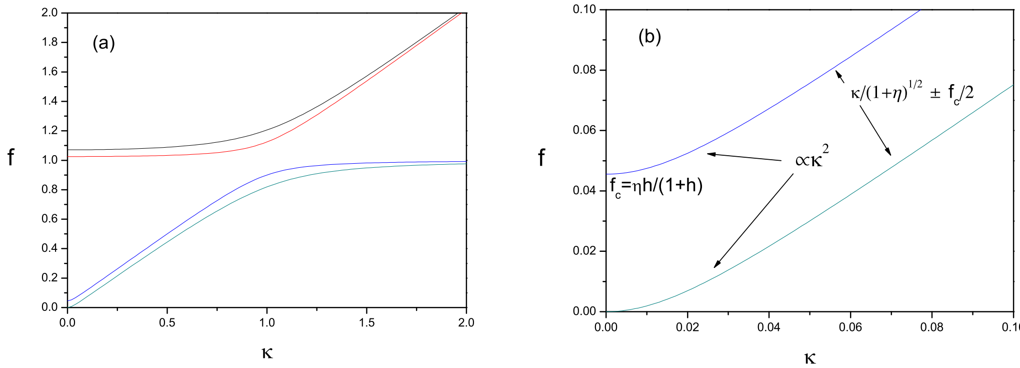

FIG. 5: Linear dispersion relation for EM waves in a three component pair-ion (or *epi*) plasma: the reduced frequency $f = \omega/\Omega$ is depicted against the reduced wavenumber $\kappa = ck/\Omega$; (a) full frequency range; (b) focusing near the origin. Here $\eta = 0.5$, $h = 0.1$ (definitions in the text).

Upon setting the first quantity within brackets, in the right-hand side (rhs), to zero, one recovers the dispersion relation:

$$\omega^4 - (c^2k^2 + \Omega^2 + 2\omega_p^2)\,\omega^2 + c^2k^2\Omega^2 \,=\, 0\,, \tag{51}$$

which coincides [39] with Eq. (9) in [35]; also see (24)-(26) in [31]. As shown in Ref. [35], Eq. (51) represents the dispersion relation of an EM wave which propagates for any value of the pitch angle θ, and whose only non-vanishing electric field component E_y is perpendicular to the plane spanned by the magnetic field $\mathbf{B_0}$ and the wave vector $\mathbf{k}$ (i.e. $E_x = E_z = 0$); this mode is always characterized by charge neutrality ($n_i = n_e = n \neq n_0$, off equilibrium), for $\theta \neq 0$. For parallel propagation ($\theta = 0$, or $B_z = 0$), this mode corresponds to a splitting of the incompressible ($n_i = n_e = n_0$), circularly polarized EM waves (present in e-i plasmas) into two orthogonal, linearly polarized EM waves, both obeying Eq. (51). For perpendicular propagation ($\theta = \pi/2$, or $B_x = 0$), this mode is part of the extraordinary (X) mode [29]; also see (21)-(22) in [31], and the accompanying discussion therein. See that both relations (43) and (44) merge into (51) for $\omega_{p,1} = \omega_{p,2}$; vice versa, in the presence of a third species (e.g. in e-p-i plasmas), this mode splits into 2 parts, given by (43) and (44).

On the other hand, upon setting the second quantity within brackets, in rhs(50) to zero, one recovers exactly [39] Eq. (10) in [35], representing an EM mode propagating in pair plasmas, for which $E_y = 0$ (i.e. no electric field is generated perpendicular to the plane defined by $\mathbf{B_0}$ and $\mathbf{k}$). Interestingly, for $\theta = \pi/2$, one obtains a pair of (decoupled) dispersion relations, namely $\omega^2 = 2\omega_p^2 + c^2k^2$ (corresponding to an incompressibly, linearly polarized ordinary (O) mode [29], with $E_z \neq 0$) and $\omega^2 = 2\omega_p^2 + \Omega^2$ (representing a fixed frequency, pure upper-hybrid mode, with $E_x \neq 0$) [35].

The case of parallel EM wave propagation in p.p. is obtained either by setting $\theta = 0$ in (50), or by setting $\omega_{p,1} = \omega_{p,2} = \omega_p$ in (36); one thus obtains

$$D_0|_{p.p.}(\omega, \theta = 0) = -i\,\omega^3\,(\omega^2 - 2\omega_p^2)\,[\omega^4 - (c^2k^2 + \Omega^2 + 2\omega_p^2)\,\omega^2 + c^2k^2\Omega^2]^2\,, \tag{52}$$

thus recovering the E_y-mode discussed above, plus trivial (non-propagating, since pressure effects are neglected) plasma oscillations at $\omega = \sqrt{2}\omega_p$.

B. Algebraic manipulation of the 1st order 1st harmonic ($n = 1$, $l = 1$) amplitudes: fluid velocities vs. E/M fields

We shall now attempt to clarify the analytical dependence of the first harmonic amplitudes on various parameters, as well as their relation to one another.

First, the density perturbation amplitudes $n_j^{(1,1)}$ are readily determined for given fluid velocity amplitudes along propagation $u_{j,x}^{(1,1)}$, from (31), as

$$n_j^{(1,1)} = n_{j,0}\,\frac{k u_{j,x}^{(1,1)}}{\omega}\,, \tag{53}$$

(see that the density amplitudes $n_j^{(1,1)}$ appear nowhere else in the 1st harmonic equations).

The velocity and magnetic field amplitudes are determined by (32) and (33), respectively, in terms of the electric field perturbation, and may thus be eliminated in (34).

In our reference frame, the system of momentum equation(s) (32) takes a linear matrix form $\mathbf{M}_j\mathbf{u}_j^{(1,1)} = \mathbf{E}^{(1,1)}$, viz.

$$\frac{m_j}{q_j}\begin{pmatrix} -i\omega & -s_j\Omega_j\sin\theta & 0 \\ s_j\Omega_j\sin\theta & -i\omega & -s_j\Omega_j\cos\theta \\ 0 & s_j\Omega_j\cos\theta & -i\omega \end{pmatrix}\begin{pmatrix} u_{j,x}^{(1,1)} \\ u_{j,y}^{(1,1)} \\ u_{j,z}^{(1,1)} \end{pmatrix} = \begin{pmatrix} E_x^{(1,1)} \\ E_y^{(1,1)} \\ E_z^{(1,1)} \end{pmatrix}\,, \tag{54}$$

(the definition of the matrix $\mathbf{M}_j$ is obvious) which may be solved formally, for $\omega \neq \pm\Omega_j$, viz. $\mathbf{u}_j^{(1,1)} = \mathbf{M}_j^{-1}\mathbf{E}^{(1,1)}$, i.e.

$$\begin{pmatrix} u_{j,x}^{(1,1)} \\ u_{j,y}^{(1,1)} \\ u_{j,z}^{(1,1)} \end{pmatrix} =$$

$$i\frac{q_j}{m_j}\frac{1}{\omega(-\omega^2+\Omega_j^2)}\begin{pmatrix} -\omega^2+\Omega_j^2\cos^2\theta & -is_j\Omega_j\omega\sin\theta & \Omega_j^2\sin\theta\cos\theta \\ is_j\Omega_j\omega\sin\theta & -\omega^2 & -is_j\Omega_j\omega\cos\theta \\ \Omega_j^2\sin\theta\cos\theta & is_j\Omega_j\omega\cos\theta & -\omega^2+\Omega_j^2\sin^2\theta \end{pmatrix}\begin{pmatrix} E_x^{(1,1)} \\ E_y^{(1,1)} \\ E_z^{(1,1)} \end{pmatrix}. \quad (55)$$

See that the parallel ($\sim \hat{x}$) component is decoupled from the perpendicular ones ($\sim \hat{y}, \hat{z}$) in the case of parallel propagation, i.e. for $\theta = 0$ (only). It is straightforward to show that the latter system of equations then amounts to:

$$\begin{aligned}
\mathbf{u}_j^{(1,1)} &= \frac{Ze}{m(\omega^2-\Omega^2)}\left[+s_j i\omega\mathbf{E}^{(1,1)} - s_j i\frac{Z^2e^2}{m^2c^2\omega}(\mathbf{E}^{(1,1)}\cdot\mathbf{B_0})\mathbf{B_0} + \frac{Ze}{mc}\mathbf{B_0}\times\mathbf{E}^{(1,1)}\right] \\
&= \frac{q_j}{m(\omega^2-\Omega^2)}\left[+i\omega\mathbf{E}^{(1,1)} - i\frac{q_j^2}{m^2c^2\omega}(\mathbf{E}^{(1,1)}\cdot\mathbf{B_0})\mathbf{B_0} + \frac{q_j}{mc}\mathbf{B_0}\times\mathbf{E}^{(1,1)}\right] \\
&= \frac{q_j}{m(\omega^2-\Omega^2)}\left[+i\omega\mathbf{E}^{(1,1)} - i\frac{\Omega^2}{\omega}(\mathbf{E}^{(1,1)}\cdot\mathbf{b_0})\mathbf{b_0} + s_j\Omega\mathbf{b_0}\times\mathbf{E}^{(1,1)}\right],
\end{aligned} \quad (56)$$

where $q_j = s_j Ze$ is the pair-ion species charge ($s_j = \pm 1$) and

$$\mathbf{b_0} = \mathbf{B_0}/B_0 = (\cos\theta, 0, \sin\theta)^T$$

is the unit vector along the magnetic field. The index j on the cyclotron frequency Ω_j will be dropped in the following, since $\Omega_1 = \Omega_2$ in the plasmas of interest here. Eq. (56) is identical to (6) in [35] (upon a trivial change in notation).

C. A closed equation for the electric field $\mathbf{E}^{(11)}$ (component amplitudes)

A closed equation may be obtained for the field $\mathbf{E}^{(1,1)}$ in a concise vector form. Eliminating the wave magnetic field $\mathbf{B}^{(1,1)}$ between Eqs. (33) and (34) yields

$$(\omega^2 - c^2k^2)\mathbf{E}^{(1,1)} + c^2(\mathbf{E}^{(1,1)}\cdot\mathbf{k})\mathbf{k} + 4\pi i Ze\omega(n_{1,0}\mathbf{u}_1^{(1,1)} - n_{2,0}\mathbf{u}_2^{(1,1)}) = \mathbf{0}\,. \quad (57)$$

Inserting the pair ion species' velocities from (56) leads to the relation

$$\begin{aligned}
&[(\omega^2-c^2k^2)(\omega^2-\Omega^2) - \omega^2\omega_{p,eff}^2]\mathbf{E}^{(1,1)} + c^2(\omega^2-\Omega^2)(\mathbf{E}^{(1,1)}\cdot\mathbf{k})\mathbf{k} \\
&+\omega_{p,eff}^2\frac{(Ze)^2}{m^2c^2}(\mathbf{E}^{(1,1)}\cdot\mathbf{B_0})\mathbf{B_0} + i(\omega_{p,1}^2-\omega_{p,2}^2)\frac{\omega Ze}{mc}\mathbf{B_0}\times\mathbf{E}^{(1,1)} = \mathbf{0}\,,
\end{aligned} \quad (58)$$

or

$$[(\omega^2 - c^2k^2)(\omega^2 - \Omega^2) - \omega^2\omega_{p,eff}^2]\mathbf{E}^{(1,1)} + c^2(\omega^2 - \Omega^2)(\mathbf{E}^{(1,1)} \cdot \mathbf{k})\mathbf{k}$$
$$+\omega_{p,eff}^2\Omega^2(\mathbf{E}^{(1,1)} \cdot \mathbf{b_0})\mathbf{b_0} + i(\omega_{p,1}^2 - \omega_{p,2}^2)\omega\Omega\,(\mathbf{b_0} \times \mathbf{E}^{(1,1)}) = \mathbf{0}\,. \tag{59}$$

Note that the last term in the *rhs* vanishes for pure pair plasmas (where, unlike our case here, $n_{1,0} = n_{2,0}$ implies $\omega_{p,1} = \omega_{p,2}$), thus recovering exactly relation (8) in [35] (upon a difference in notation and system of units). The latter equation is therefore generalized by Eq. (59), in e-p-i (or asymmetic pair-ion) plasmas.

Alternatively, an equation for the electric field $\mathbf{E}^{(11)}$ may be obtained by eliminating $\mathbf{u}_j$ and $\mathbf{B}$ in Ampère's law (34). Substituting (33) and (54) into Eq. (34), one obtains:

$$\begin{aligned}\mathbf{k} \times (\mathbf{k} \times \mathbf{E}^{(1,1)}) &= -\frac{\omega}{c^2}\left(\omega\mathbf{I} + 4\pi i \sum_j n_{j,0}q_j\mathbf{M}_j^{-1}\right)\mathbf{E}^{(1,1)}\,,\\ &= -\frac{\omega}{c^2}\left[\omega\mathbf{I} + 4\pi i Ze\,(n_{1,0}\mathbf{M}_1^{-1} - n_{2,0}\mathbf{M}_2^{-1})\right]\mathbf{E}^{(1,1)}\,. \end{aligned} \tag{60}$$

Here, $\mathbf{I}$ is the unit matrix; the matrices $\mathbf{M}_j^{-1}$ were defined in (54). Note that the *lhs* is

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}^{(1,1)}) = -k^2\mathbf{E}^{(1,1)} + (\mathbf{E}^{(1,1)} \cdot \mathbf{k})\mathbf{k} = -k^2\left(0, E_y^{(1,1)}, E_z^{(1,1)}\right)^T.$$

Eq. (60) may be viewed as a closed algebraic system of (3) equations for the 3 components of the electric field. It is straightforward to verify that this is tantamount to the concise relation (58), and that the determinant quantity defined by the linear system (60) explicitly coincides with the one derived above.

D. Exact solution for the 1st-order harmonic amplitudes

Relying on the above relations, and following the methodology described in Section IV, we may now obtain exact expressions for the first harmonic amplitudes of all state variables [47]. For convenience in the physical description, we shall express all quantities in terms of the components of the (reduced) transverse magnetic field (perturbation) components $B'_y = B'^{(11)}_y/B_0$ and $B'_z = B'^{(11)}_z/B_0$. Since the long procedure is perfectly straightforward, only the final outcome will be provided below, thus omitting unnecessary details.

a. Preliminaries. First, recall that the densities are given, once the velocities are determined, from (53). Eq. (33) gives

$$B_x^{(1,1)} = 0\,, \qquad B_y^{(1,1)} = -\frac{ck}{\omega}E_z^{(1,1)}\,, \qquad B_z^{(1,1)} = \frac{ck}{\omega}E_y^{(1,1)}\,. \tag{61}$$

The electric field components perpendicular to the wave's propagation direction are thus readily given from Faraday's law (61), which is here equivalent to

$$E_{\perp}^{(1,1)} = \mp i \frac{\omega}{ck} B_{\perp}^{(1,1)} , \tag{62}$$

recalling the definitions (10) above. Also retain that the parallel (to propagation) electric field component $E_{\parallel}^{(1,1)} = E_x^{(1,1)}$ is determined by the $x-$ component of Ampère's law (34) as

$$E_x^{(1,1)} = -i \frac{4\pi e}{\omega} \sum_{j=1,2} s_j Z_j n_{j,0} u_{j,x}^{(1,1)} = -i \frac{4\pi Z e}{\omega} (n_{1,0} u_{1,x}^{(1,1)} - n_{2,0} u_{2,x}^{(1,1)}) . \tag{63}$$

b. Exact solution. The linearized (1st order, 1st harmonic) problem for the state variables may be solved in terms of two freely evolving variables, among the 14 involved. For reasons of physical interpretation, we have chosen these to be the transverse magnetic field components, i.e. B_y and B_z.

The final solution the fluid variables takes the form

$$\begin{aligned} n_j^{(11)} &= n_{j,0} \frac{k}{\omega} u_{j,x}^{(11)} = c_{j,n,y}^{(11)} B'_y + c_{j,n,z}^{(11)} B'_z , \\ u_{j,i}^{(11)} &= c_{j,i,y}^{(11)} B'_y + c_{j,i,z}^{(11)} B'_z \qquad (\text{for } j = 1,2 \text{ and } i = x,y,z) . \end{aligned} \tag{64}$$

The electric field components also take a similar form:

$$E_i^{(11)} = c_{el,i,y}^{(11)} B'_y + c_{el,i,z}^{(11)} B'_z \qquad (\text{for } i = x,y,z) . \tag{65}$$

As previously said, B_x is stationary.

The analytical expressions for all of the coefficients $c_{\cdot,\cdot,y/z}$ can be found in the Appendix.

c. Brief discussion. Note that these expressions bear a number of interesting properties, reflecting inherent symmetries in the plasma constituents. For instance, it is straightforward to verify that, *for parallel propagation*, i.e. upon setting $\theta = 0$ in the above relations, one finds that no parallel fluid velocity perturbation occurs for parallel EM wave propagation; the density and electric field perturbations also cancel, as a consequence, i.e. $n_{1/2}^{(11)} = u_{1/2,x}^{(11)} = E_x^{(11)} = 0$, for $\theta = 0$. This is in agreement with the results in [14]; see (B.6-8) therein.

On the other hand, still for *parallel* propagation, one finds for the perpendicular components ($\sim \hat{y}, \hat{z}$) that

$$u_{1,\perp}^{(11)} \sim \left(1 \pm \frac{\omega}{\Omega}\right) B_{\perp}^{(11)} \qquad \text{and} \qquad u_{2,\perp}^{(11)} \sim \left(1 \mp \frac{\omega}{\Omega}\right) B_{\perp}^{(11)} \tag{66}$$

(for $\theta = 0$ only), where the upper (lower) signs throughout this text correspond to left- (right-) hand polarized waves. This nice property, which is due to the identical mass and absolute charge of the two pair-ion species, is nevertheless destroyed in the oblique propagation case. In a more general manner, for a "pure" pair plasma, i.e. for $n_{1,0} = n_{2,0}$ (implying $\omega_{p,1} = \omega_{p,2}$), the circular polarization of parallel propagating EM waves degenerates into a combination of orthogonal linear polarizations, as shown in Refs. [31, 35]). This result is recovered here.

E. On the zeroth harmonic amplitude ($n = 1$, $l = 0$)

The solution (for the state variables) anticipated above takes the form of a double series, in order of perturbation ($\sim \epsilon^n$) and also in the phase harmonic l ($= 0, 1, ..., n$). A fact which is often neglected in perturbative studies of this kind, is the possibility of existence of a zeroth-harmonic ("direct current", DC) term, accounting for a non-oscillatory weak perturbation from equilibrium.

For $n = 1$ and $l = 0$, from the Eq. of motion (2) we obtain:

$$\mathbf{E}^{(1,0)} = -\frac{1}{c}\,\mathbf{u}_j^{(1,0)} \times \mathbf{B_0}\,, \tag{67}$$

for $j =$ either 1 or 2. On the other hand, Eq. (4) provide the current neutrality condition $\sum_j n_{j,0} q_j \mathbf{u}_j^{(1,0)} = 0$, i.e. for the type of plasmas of interest to us ($q_2 = -q_1$):

$$n_{1,0}\mathbf{u}_1^{(1,0)} = n_{2,0}\mathbf{u}_2^{(1,0)}\,. \tag{68}$$

In "pure" pair plasmas (no third species), charge neutrality at equilibrium ($n_{1,0} = n_{2,0}$) imposes a common fluid velocity $\mathbf{u}_1^{(1,0)} = \mathbf{u}_2^{(1,0)} = \mathbf{u}^{(1,0)}$, prescribing the occurrence of a finite constant electric field given by (67). The field components are given by $E_i^{(1,0)} = -\frac{1}{c}\,\epsilon_{ijk} u_j^{(1,0)} B_{0,k}$, where ϵ_{ijk} is the Levi-Civita tensor elements ($\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ and zero otherwise), i.e. in our reference frame: $E_x^{(1,0)} = -\frac{B_0}{c} u^{(1,0)} \sin\theta$, $E_z^{(1,0)} = \frac{B_0}{c} u^{(1,0)} \cos\theta$, and $E_y^{(1,0)} = -E_x^{(1,0)} - E_z^{(1,0)}$. See that overall charge neutrality (even off equilibrium) is ensured by Poisson's law (5).

In three-component e-p-i plasmas, where $n_{1,0} \neq n_{2,0}$, Eqs. (67) and (68) are only satisfied if the zeroth harmonic vanishes.

In conclusion, the existence (however weak) of a streaming ion velocity $\mathbf{u}_j^{(1,0)}$ in "pure" (2 species) pair-ion plasmas may generate a zeroth-harmonic (non-propagating) electric field;

this is true for any direction of EM wave propagation. On the other hand, *no zeroth-harmonic term arises in e-p-i (or asymmetric pair) plasmas.*

VI. 2ND ORDER DYNAMICS ($n = 2$): HARMONIC GENERATION, SECONDARY HARMONIC AMPLITUDES, GROUP VELOCITY

The 2nd-order system bears a solution in the form:

$$S_i^{(2)} = S_i^{(20)} + (S_i^{(21)} \exp i(kx - \omega t) + \text{c.c.}) + (S_i^{(22)} \exp 2i(kx - \omega t) + \text{c.c.})\,, \tag{69}$$

where S_i (for $i = 1, 2, ..., 14$, here) denotes any of the dynamical variables (components of the state vector $\mathbf{S}$) in play. We note the generation of secondary Fourier phase harmonics, which is the "signature" of the self-modulation nonlinear mechanism, in addition to the generation of a (weak, to order $\sim \epsilon^2$) zeroth-harmonic (non-oscillating) contribution.

A. 2nd order first harmonics ($n = 2$, $l = 1$) & group velocity

Solution for the amplitudes. The equations obtained for $n = 2$ and $l = 1$ lead to a solution in the form

$$\begin{aligned} n_j^{(21)} &= c_{j,n,TBy}^{(21)} \frac{\partial B'_y}{\partial T_1} + c_{j,n,XBy}^{(21)} \frac{\partial B'_y}{\partial X_1} + c_{j,n,TBz}^{(21)} \frac{\partial B'_z}{\partial T_1} + c_{j,n,XBz}^{(21)} \frac{\partial B'_z}{\partial X_1}\,, \\ u_{j,i}^{(21)} &= c_{j,i,TBy}^{(21)} \frac{\partial B'_y}{\partial T_1} + c_{j,i,XBy}^{(21)} \frac{\partial B'_y}{\partial X_1} + c_{j,i,TBz}^{(21)} \frac{\partial B'_z}{\partial T_1} + c_{j,i,XBz}^{(21)} \frac{\partial B'_z}{\partial X_1}\,, \end{aligned} \tag{70}$$

where j $(= 1, 2)$ denotes the $j-$th fluid and i $(= x, y, z)$ the respective velocity component, here and below (the subscript notation, yet rather complicated, is obvious). The E/M field components $E_i^{(22)}$ (for $i = x, y, z$) and $B_i^{(22)}$ (for $i = y, z$) also take a similar form:

$$\begin{aligned} E_i^{(21)} &= c_{el,i,TBy}^{(21)} \frac{\partial B'_y}{\partial T_1} + c_{el,i,XBy}^{(21)} \frac{\partial B'_y}{\partial X_1} + c_{el,i,TBz}^{(21)} \frac{\partial B'_z}{\partial T_1} + c_{el,i,XBz}^{(21)} \frac{\partial B'_z}{\partial X_1}\,, \\ B_i^{(21)} &= c_{B,i,TBy}^{(21)} \frac{\partial B'_y}{\partial T_1} + c_{B,i,XBy}^{(21)} \frac{\partial B'_y}{\partial X_1} + c_{B,i,TBz}^{(21)} \frac{\partial B'_z}{\partial T_1} + c_{B,i,XBz}^{(21)} \frac{\partial B'_z}{\partial X_1}\,. \end{aligned} \tag{71}$$

As previously mentioned, B_x is stationary, so $B_x^{(21)} = 0$. Furthemore, we assume that $E_x^{(21)} = B_y^{(21)} = B_z^{(21)} = 0$, with no loss of generality (these variables are left arbitrary by the algebra).

The (lengthy) expressions for the coefficients are omitted here.

On the group velocity. The system of equations obtained for $n = 2$ and $l = 1$ yields a compatibility condition, in order for secular terms to be annihilated. This requirement takes the exact form:

$$\frac{\partial B_z'}{\partial T_1} + v_g \frac{\partial B_z'}{\partial X_1} + C\left(\frac{\partial B_y'}{\partial T_1} + v_g \frac{\partial B_y'}{\partial X_1}\right) = 0\,, \tag{72}$$

where v_g denotes the group velocity $\omega'(k) = -(\partial D/\partial k)/(\partial D/\partial \omega)$ [48], as it results from the dispersion relation $D(\omega, k) = 0$ obtained previously; the (dimensionless, complex) quantity C will be defined below. The latter relation may be re-arranged as:

$$\left(\frac{\partial}{\partial T_1} + v_g \frac{\partial}{\partial X_1}\right)\left(B_z' + C B_y'\right) = 0\,. \tag{73}$$

We will shall henceforth assume that both quantities within parentheses in (72) cancel, simultaneously. The physical meaning of this assumption is obvious: the amplitude(s) of both transverse components of the magnetic field $\mathbf{B}$ propagate at the group velocity $v_g = \omega'(k)$, as physically expected. In other words, (all of) the slowly varying variable amplitudes will be functions of a single traveling-wave variable, namely $\xi = X_1 - v_g T_1 \equiv \epsilon(x - v_g t)$.

It may be appropriate here to discuss the form of the complex quantity C, which "ponderates" the relative contribution of the two transverse components to the travelling modulated envelope (or, rather, determines their relative phase shift); cf. (73). The expression for C reads:

$$C(\omega, k; \theta) = -2\, i\, \omega\, \Omega\, (\omega_{p,1}^2 - \omega_{p,2}^2)\, \cos\theta\, \frac{C_1}{C_2} = 0\,, \tag{74}$$

where

$$C_1 = 4c^2k^2(\omega^2 - \Omega^2)(\omega^2 - \omega_{p,eff}^2) + 2\Omega^2\omega_{p,eff}^2(\omega^2 - \omega_{p,eff}^2 - c^2k^2)\sin^2\theta$$

and

$$\begin{aligned} C_2 \;=\;& 8c^2k^2(\omega^2 - \Omega^2)(\omega^2 - \omega_{p,eff}^2)[\omega^2(\omega^2 - \Omega^2 - \omega_{p,eff}^2) - c^2k^2(\omega^2 - \Omega^2)] \\ & + 4\Omega^2\omega_{p,eff}^2 \sin^2\theta\, [2c^4k^4(\omega^2 - \Omega^2) - i\,\omega\Omega(\omega_{p,1}^2 - \omega_{p,2}^2)(\omega^2 - \omega_{p,eff}^2 - c^2k^2)\, \cos\theta]\,. \end{aligned}$$

Interestingly, for *parallel* EM wave propagation (i.e. for $\theta = 0$), $C \to \pm i = e^{\pm i\pi/2}$, suggesting a phase difference of $\pm\pi/2$ among B_y and B_z, specifically when the frequency ω obeys

$$(\omega \mp \Omega)\left[-(\omega^2 - c^2k^2)(\omega - \Omega) + \omega\omega_{p,1}^2\right] + \omega(\omega \pm \Omega)\omega_{p,2}^2 = 0$$

(combining the upper/lower signs, respectively); cf. the (parallel EM wave) dispersion relation (36). The slowly evolving transverse magnetic field component is then $B_z \pm iB_y =$

$\pm i(B_y \mp iB_z) \equiv \pm iB_\perp^*$ [recall definition (10c) above]. The anticipated circular polarization encountered for modulated EM wavepackets propagating parallel to $\mathbf{B_0}$ (see e.g. in Ref. [14]) in multi-component plasmas is thus recovered.

Note, for rigor, that the quantity C vanishes for *perpendicular* EM wave propagation (i.e. for $\theta = \pi/2$), for $\omega_{p,1} \neq \omega_{p,2}$, and also in the *pure* pair-ion plasma case (i.e. for $\omega_{p,1} = \omega_{p,2}$, $\forall\,\theta$).

B. 2nd order 2nd-harmonics ($n = 2$, $l = 2$)

The system of equations obtained for $n = 2$ and $l = 2$ provides the amplitudes of the second-harmonics. The solution reads:

$$
\begin{aligned}
n_j^{(22)} &= c^{(22)}_{n_j,yy} B'^2_y + c^{(22)}_{n_j,zz} B'^2_z + c^{(22)}_{n_j,yz} B'_y B'_z \\
&\qquad + c^{(22)}_{n_j,y,T} \frac{\partial B'_y}{\partial T_1} + c^{(22)}_{n_j,y,X} \frac{\partial B'_y}{\partial X_1} + c^{(22)}_{n_j,z,T} \frac{\partial B'_z}{\partial T_1} + c^{(22)}_{n_j,z,X} \frac{\partial B'_z}{\partial X_1} \, , \\
u_{j,i}^{(22)} &= c^{(22)}_{u_{j,x},yy} B'^2_y + c^{(22)}_{u_{j,x},zz} B'^2_z + c^{(22)}_{u_{j,x},yz} B'_y B'_z \\
&\qquad + c^{(22)}_{u_{j,x},y,T} \frac{\partial B'_y}{\partial T_1} + c^{(22)}_{u_{j,x},y,X} \frac{\partial B'_y}{\partial X_1} + c^{(22)}_{u_{j,x},z,T} \frac{\partial B'_z}{\partial T_1} + c^{(22)}_{u_{j,x},z,X} \frac{\partial B'_z}{\partial X_1} \, ,
\end{aligned} \tag{75}
$$

where j $(= 1, 2)$ denotes the $j-$th fluid and i $(= x, y, z)$ the respective velocity component, here and below (again, the subscript notation is obvious). The E/M field components $E_i^{(22)}$ (for $i = x, y, z$) and $B_i^{(22)}$ (for $i = y, z$) take a similar form:

$$
\begin{aligned}
E_i^{(22)} &= c^{(22)}_{E_i,yy} B'^2_y + c^{(22)}_{E_i,zz} B'^2_z + c^{(22)}_{E_i,yz} B'_y B'_z \\
&\qquad + c^{(22)}_{E_i,y,T} \frac{\partial B'_y}{\partial T_1} + c^{(22)}_{E_i,y,X} \frac{\partial B'_y}{\partial X_1} + c^{(22)}_{E_i,z,T} \frac{\partial B'_z}{\partial T_1} + c^{(22)}_{E_i,z,X} \frac{\partial B'_z}{\partial X_1} \, , \\
B_i^{(22)} &= c^{(22)}_{B_i,yy} B'^2_y + c^{(22)}_{B_i,zz} B'^2_z + c^{(22)}_{B_i,yz} B'_y B'_z \\
&\qquad + c^{(22)}_{B_i,y,T} \frac{\partial B'_y}{\partial T_1} + c^{(22)}_{B_i,y,X} \frac{\partial B'_y}{\partial X_1} + c^{(22)}_{B_i,z,T} \frac{\partial B'_z}{\partial T_1} + c^{(22)}_{B_i,z,X} \frac{\partial B'_z}{\partial X_1} \, ,
\end{aligned} \tag{76}
$$

Recall that B_x is stationary, thus $B_x^{(22)} = 0$.

The (particularly long) expressions for the coefficients in the above relations are omitted here, for brevity.

C. 2nd order zeroth-harmonics ($n = 2$, $l = 0$)

The system of equations obtained for $n = 2$ and $l = 0$ provides the zeroth-harmonic contributions to the variable amplitudes. Note that the density equations bear no zeroth-harmonic contribution, and neither does Faraday's law; therefore, the density and magnetic field perturbations (for $n = 2$, $l = 0$), viz. $n_j^{(20)}$ and $B_i^{(20)}$ (for $j = 1, 2$ and $i = y, z$; $B_x^{(20)} = 0$ is assumed), remain undetermined. The solution for the fluid velocities is of the form:

$$u_{2,i}^{(20)} = \frac{\omega_{p,2}^2}{\omega_{p,1}^2} u_{2,i}^{(20)} + c_{u_{j,i},yy}^{(20)} |B'_y|^2 + c_{u_{j,i},zz}^{(20)} |B'_z|^2 + c_{u_{j,i},yz}^{(20)} B_y'^* B'_z + c_{u_{j,i},zy}^{(20)} B'_y B_z'^* \tag{77}$$

(for $i = x, y, z$). The electric field components are:

$$\begin{aligned} E_x^{(20)} &= -\frac{\omega_{p,2}^2}{\omega_{p,1}^2} \frac{u_{2,y}^{(20)}}{c} \sin\theta + c_{E_x,yy}^{(20)} |B'_y|^2 + c_{E_x,zz}^{(20)} |B'_z|^2 + c_{E_x,yz}^{(20)} B_y'^* B'_z + c_{E_x,zy}^{(20)} B'_y B_z'^* \,, \\ E_y^{(20)} &= \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \frac{1}{c} (u_{2,x}^{(20)} \sin\theta - u_{2,z}^{(20)} \cos\theta) \\ &\qquad + c_{E_y,yy}^{(20)} |B'_y|^2 + c_{E_y,zz}^{(20)} |B'_z|^2 + c_{E_y,yz}^{(20)} B_y'^* B'_z + c_{E_y,zy}^{(20)} B'_y B_z'^* \\ E_z^{(20)} &= \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \frac{u_{2,y}^{(20)}}{c} \cos\theta + c_{E_z,yy}^{(20)} |B'_y|^2 + c_{E_z,zz}^{(20)} |B'_z|^2 + c_{E_z,yz}^{(20)} B_y'^* B'_z + c_{E_z,zy}^{(20)} B'_y B_z'^* \,, \end{aligned} \tag{78}$$

The long expressions for the coefficients $c_{u_{j,i},zy}^{(20)}$ and $c_{E_i,yz}^{(20)}$ appearing in the above relations are omitted in this text. We may limit ourselves to pointing out, for rigor, that known limits are recovered from the omitted expressions. For instance, for wave propagation parallel to the magnetic field (i.e. for $\theta = 0$), we find

$$\mathbf{u_1}^{(20)} = \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \mathbf{u_2}^{(20)} \tag{79}$$

and

$$E_x^{(20)} = 0 \,, \qquad E_y^{(20)} = -\frac{\omega_{p,2}^2}{\omega_{p,1}^2} \frac{u_{2,z}^{(20)}}{c} \,, \qquad E_z^{(20)} = \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \frac{u_{2,y}^{(20)}}{c} \,. \tag{80}$$

Recall that $\omega_{p,2}^2/\omega_{p,1}^2 = n_{2,0}/n_{1,0}$, i.e. unity in the pure p.p. limit.

VII. AMPLITUDE EVOLUTION EQUATION(S)

So far, we have obtained explicit expressions for the perturbative solution upto $\sim \epsilon^2$, in terms of the (arbitrary-valued) principal harmonic amplitude(s) appearing in ϵ^1. Considering the system of equations for $n = 3$ and $l = 1$, one needs to ensure that secular terms (i.e. terms $\sim e^{i(kx-\omega t)}$ in the *rhs*, entering in resonance with the null space – here exactly determined by the linear dispersion obtained above) annihilate exactly, otherwise no long-lived analytical solution is possible. In a generic manner, this gives a compatibility condition to be imposed on the 1st-harmonic amplitudes.

In general, the compatibility condition obtained for $(n, l) = (3, = 1$ takes the form of a set of coupled nonlinear Schrödinger-type equations (CNLSEs):

$$
\begin{aligned}
i\frac{\partial B_y}{\partial \tau} + P\frac{\partial^2 B_y}{\partial \xi^2} + Q_{11}\,|B_y|^2 B_y + Q_{12}\,|B_y|^2 B_z \; &= 0\,, \\
i\frac{\partial B_z}{\partial \tau} + P\frac{\partial^2 B_z}{\partial \xi^2} + Q_{22}\,|B_z|^2 B_z + Q_{21}\,|B_z|^2 B_y \; &= 0\,,
\end{aligned}
\tag{81}
$$

where the slow time scale is $\tau = \epsilon^2 t$ and the moving envelope space coordinate is $\xi = \epsilon(x - v_g t)$. The dispersion coefficient is $P = \omega''(k)/2$, while the nonlinearity and coupling coefficients, Q_{ii} and Q_{ij} (here $i, j = 1, 2$, and $j \neq 1$ is understood). All coefficients in Eqs. (81) are perplex functions of the wavenumber k and the angle θ, and also depend on intrinsic plasma parameters ($\omega_{p,1/2}$, Ω). The long expressions are omitted here.

Interestingly, for certain values of θ, one may show after a tedious calculation that Eqs. (81) reduce to a single NLS equation

$$
i\frac{\partial \tilde{B}_\perp}{\partial \tau} + P\frac{\partial^2 \tilde{B}_\perp}{\partial \xi^2} + Q\,|\tilde{B}_\perp|^2 \tilde{B}_\perp \; = 0\,, \tag{82}
$$

where $\tilde{B}_\perp = B_z + CB_y$ (remember that C is a complex quantity, defined in (74) above) and $Q = Q_{22}$. Admittedly, the passing from Eqs. (81) to Eq. (82) (which requires a particularly tedious calculation, and should be confirmed specifically for a given value of interest for the angle θ), although certainly simplifies the algebra, adds nothing to the physics of the EM wave modulation phenomenon we aim at describing here. It may be stated, for rigor, that in other physical contexts, the set of coupled NLS Eqs. (81) provides a rich dynamics and, potentially, a higher instability growth rate than the single NLS Eq. (81); see e.g. in [40] for details.

For the sake of simplicity, we shall limit ourselves to Eq. (82) in the following, in view of an investigation of the modulational stability profile of the EM wave, and a study of the occurrence of wave localization via the formation of localized modulated envelope excitations (envelope solitons). A similar investigation may be carried on the basis of the set of Eqs. (81); the associated stability analysis and coupled solitary-wave solutions are reviewed e.g. in Refs. [40–42], and will be omitted here.

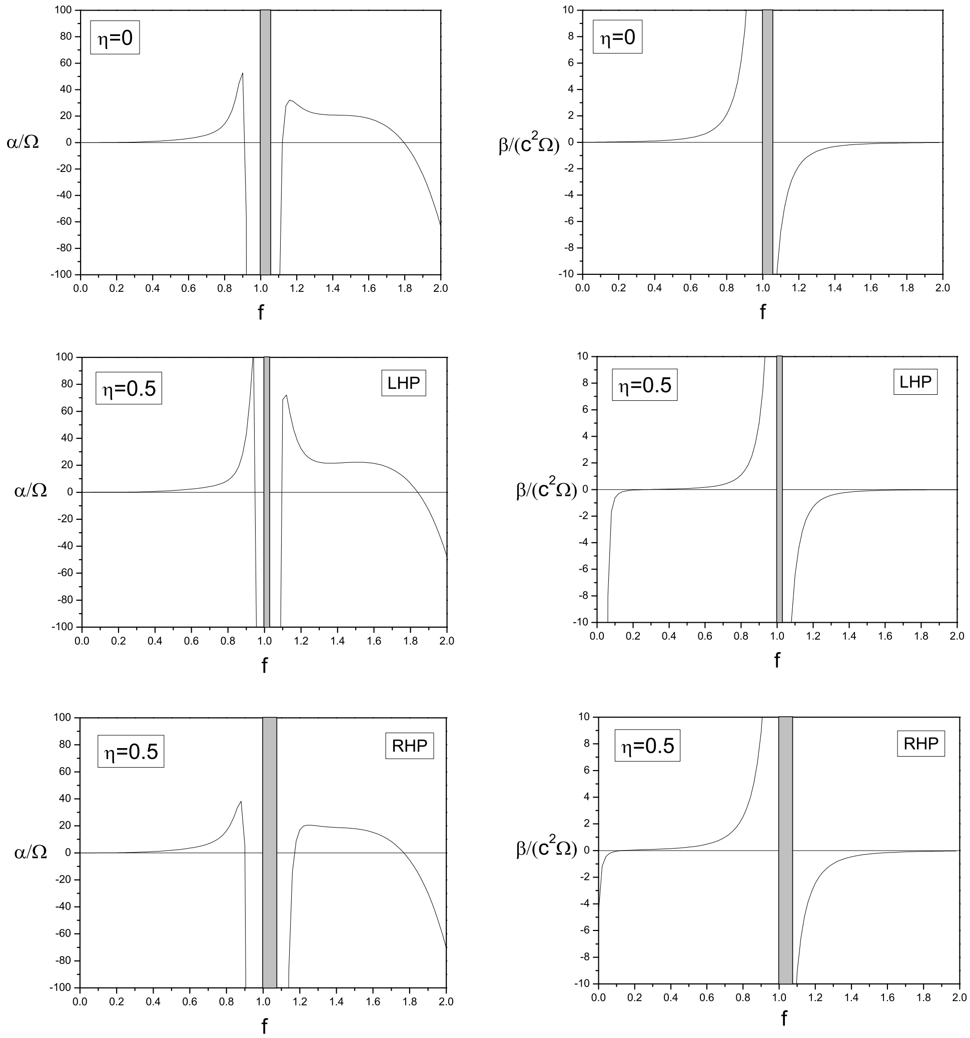

FIG. 6: The nonlinearity (left column) and dispersion (right column) coefficients in the NLS Eq. (82) for parallel EM wave propagation ($\theta = 0$) are plotted against the reduced frequency $f = \omega/\Omega$: (a) Pure pair plasma [$\eta = 0$; recall def. in (38)] - linear polarization (1st row); (b) Three-component pair plasma ($\eta = 0.5$) - left-hand polarization (2nd row); (c) Three-component pair plasma ($\eta = 0.5$) - right-hand polarization (3rd row). Note the frequency gap near $f = 1$, i.e. near $\omega = \Omega$; cf. Fig. 5.

VIII. MODULATIONAL (IN)STABILITY ANALYSIS

It is known (see e.g. in [4, 43]) that the evolution of a wave whose amplitude obeys Eq. (82) depends on the coefficient product PQ, which may be investigated in terms of the phys-

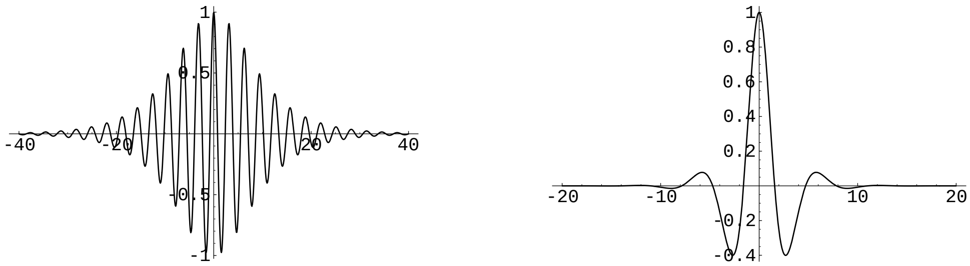

FIG. 7: *Bright* type modulated wavepackets (for $PQ > 0$), for two different (arbitrary) sets of parameter values.

ical parameters involved. To see this, first check that Eq. (82) supports the plane (Stokes') wave solution $\psi = \psi_0 \exp(iQ|\psi_0|^2 T)$; the standard linear analysis consists in perturbing the amplitude by setting: $\hat{\psi} = \hat{\psi}_0 + \epsilon\, \hat{\psi}_{1,0} \cos(\tilde{k}X - \tilde{\omega}T)$ (the perturbation wavenumber $\hat{k}$ and the frequency $\hat{\omega}$ should be distinguished from their carrier wave homologue quantities, denoted by k and ω). One thus obtains the (perturbation) dispersion relation:

$$\tilde{\omega}^2 = P\, \tilde{k}^2 \, (P\tilde{k}^2 - 2Q|\hat{\psi}_{1,0}|^2) \, . \tag{83}$$

One immediately sees that if $PQ > 0$, the amplitude ψ is *unstable* for $\tilde{k} < \sqrt{2Q/P}|\hat{\psi}_{1,0}|$; i.e. for perturbation wavelengths larger than a critical value. If $PQ < 0$, the amplitude ψ will be *stable* to external perturbations. This *modulational instability* mechanism is tantamount to the well-known *Benjamin-Feir* instability, in hydrodynamics, also long-known as an energy localization mechanism in solid state physics and nonlinear optics [4, 43].

This type of analysis allows for a numerical investigation of the stability profile in terms of intrinsic plasma parameters, e.g. wavenumber k, obliqueness angle θ, plasma and cyclotron frequencies $\omega_{p,1/2}$ and Ω, etc.

IX. ENVELOPE EXCITATIONS

The evolution equation (82) is known to be integrable [4, 43]. Localized solutions can be rigorously obtained via the tedious Inverse Scattering Transform method; these are, properly speaking, *solitons*, in the sense that they satisfy an infinity of conservation laws; they have been shown analytically (and confirmed numerically) to survive collisions between one another and also exhibit a robust behaviour against external perturbations.

The modulated (electrostatic potential) wave finally resulting from the above analysis is of the form[49]

$$\phi_1^{(1)} = \epsilon\hat{\psi}_0 \cos(\mathbf{kr} - \omega t + \Theta) + \mathcal{O}(\epsilon^2)\,.$$

The slowly varying amplitude $\psi_0(\xi,\tau)$ and phase correction $\Theta(\xi,\tau)$ (both real functions of $\{\xi,\tau\}$; see in [44] for details) are determined by (solving) Eq. (82) for $\psi = \psi_0 \exp(i\Theta)$. The different types of solution thus obtained are summarized in the following.

A. Bright-type envelope solitons

For *positive* PQ, the carrier wave is modulationally *unstable*; it may either *collapse*, due to (possibly random) external perturbations, or lead to the formation of *bright* envelope modulated wavepackets, i.e. localized envelope *pulses* confining the carrier (see Fig. 7), which are given by [44]

$$\psi_0 = \left(\frac{2P}{QL^2}\right)^{1/2} \mathrm{sech}\left(\frac{\xi - v_e\,\tau}{L}\right), \qquad \Theta = \frac{1}{2P}\left[v_e\xi + \left(\Omega - \frac{v_e^2}{2}\right)\tau\right], \tag{84}$$

where v_e is the envelope velocity; L and Ω represent the pulse's spatial width and oscillation frequency (at rest), respectively. We note that L and ψ_0 satisfy $L\psi_0 = (2P/Q)^{1/2} =$ constant (in contrast with KdV solitons [4], where $L^2\psi_0 =$ const. instead). Also, the amplitude ψ_0 is independent of the pulse (envelope) velocity v_e here.

B. Black-type envelope solitons

For $PQ < 0$, the carrier wave is modulationally *stable* and may propagate as a *dark* (*black* or *grey*) envelope wavepackets, i.e. a propagating localized *hole* (a *void*) amidst a uniform wave energy region. The exact expression for *dark* envelopes reads [44]:

$$\psi_0 = \psi'_0 \left|\tanh\left(\frac{\xi - v_e\,\tau}{L'}\right)\right|, \qquad \Theta = \frac{1}{2P}\big[v_e\xi + \big(2PQ{\psi'_0}^2 - \frac{v_e^2}{2}\big)\tau\big] \tag{85}$$

(see Fig. 8a); again, $L'\psi'_0 = (2|P/Q|)^{1/2}$ (=cst.).

C. Grey-type envelope solitons

The *grey*-type envelope (also obtained for $PQ < 0$) is given by [44]

$$\psi_0 \;=\; \psi''_0 \left[1 - d^2\,\mathrm{sech}^2\left(\frac{\xi - v_e\,\tau}{L''}\right)\right]^{1/2}$$

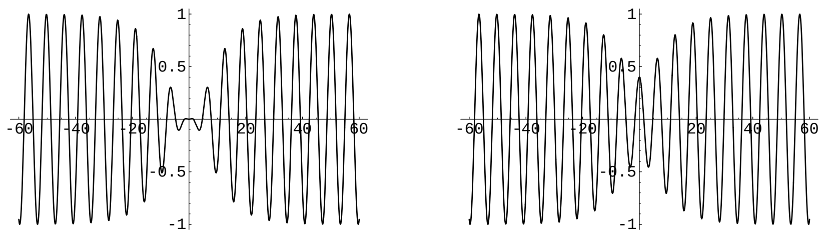

FIG. 8: *Dark*-type modulated wavepackets (for $PQ < 0$) of the *black* (left) and *grey* (right) kind. See that the amplitude never reaches zero in the latter case.

and

$$\Theta = \frac{1}{2P}\left[V_0\,\xi\,-\left(\frac{1}{2}V_0^2 - 2PQ{\psi''}_0^2\right)\tau + \Theta_0\right] - S\,\sin^{-1}\frac{d\,\tanh\left(\frac{\xi - v_e\,\tau}{L''}\right)}{\left[1 - d^2\,\mathrm{sech}^2\left(\frac{\xi - v_e\,\tau}{L''}\right)\right]^{1/2}}\,. \quad (86)$$

Here Θ_0 is a constant phase; S denotes the product $S = \mathrm{sign}(P)\,\times \mathrm{sign}(v_e - V_0)$. The pulse width $L'' = (|P/Q|)^{1/2}/(d{\psi''}_0)$ now also depends on the real parameter d, given by:

$$d^2\,=\,1\,+\,(v_e - V_0)^2/(2PQ{\psi''}_0^2)\,\leq\,1\,.$$

The (real) velocity parameter $V_0 = $ const. satisfies [44]:

$$V_0 - \sqrt{2|PQ|\,{\psi''}_0^2}\,\leq\,v_e\,\leq\,V_0 + \sqrt{2|PQ|\,{\psi''}_0^2}\,.$$

For $d = 1$ (thus $V_0 = v_e$), one recovers the *dark* envelope soliton.

X. ON MODULATIONAL (IN)STABILITY OF ES WAVES IN PAIR PLASMAS

The coefficients of the NLSE (82) are depicted in Fig. 6, for the case of $\theta = 0$, i.e. for EM wave propagation parallel to the external magnetic field. Note the forbidden frequency region (gap) near $f = 1$ (i.e. near $\omega = \Omega$); cf. Fig. 5. We see that, for pure p.p. (for $\eta = 0$, i.e. in the absence of a third species), the coefficient product PQ is positive at small frequencies (i.e. for the Alfvén type p.p. mode lying below the cyclotron frequency Ω), thus prescribing modulational instability and bright-type envelope excitations. However, PQ becomes negative as one approaches $f = 1$ from below, so high frequency waves will tend to be stable, and propagate as dark-type envelope solitons (envelope holes). A similar alternating (positive, then negative) behavior is obtained by gradually increasing ω (above Ω).

By "switching-on" the existence of the 3rd massive background species, the product PQ ($\alpha\beta$ in Fig. 6), becomes negative for low a frequency, thus apparently stabilizing the two sub-cyclotron modes (see in Fig. 5). This is due to a shift in sign of the dispersion coefficient (right column in Fig. 6, 2nd and 3rd rows) at low ω. We have seen (cf. Fig. 5) that the linearly polarized "pure" p.p. EM acoustic mode splits into two modes (one presenting a gap; see Fig. 5b) if a 3rd species is present. Both of these mode, namely a left-hand- and a right-hand-polarized one, exhibit the described behavior.

XI. SUMMARY AND CONCLUSIONS

To summarize, we have considered the propagation of nonlinear amplitude-modulated EM wavepackets in a multi-component plasma. By adopting a reductive-perturbation method, we have shown how secondary harmonic generation, modulational instability and envelope soliton formation may be modeled efficiently via a multiple scale analysis.

Focusing on EM modes propagating in pair plasmas, we have shown that the modulational stability and the type of excitation which may occur in such plasmas may be predicted by the perturbation theory presented above. The presence of a third massive species (in "doped" pair-ion plasmas, or e-p-i plasmas) may affect the stability profile of EM waves. For instance, it stabilizes parallel EM Alfvén-like waves, which now recover circular polarization (lost in ideal p.p., where the respective mode is linearly polarized).

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This manuscript is essentially an abbreviated (yet more pedagogically aimed) version of a (more concise) text, to appear as a published research article soon.

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Figure 4 was reprinted from [5].

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[38] Notice that, since B_x =cst. is prescribed, the number of dynamical state variables (state vector elements, see above) is thus reduced by one (i.e., one of the 14 equations in the dynamical system of evolution may be omitted, since trivially satisfied).

[39] Upon a trivial difference in notation, though: see that $\omega_{p,eff}^2 = \omega_{p,1}^2 + \omega_{p,2}^2$ here is denoted by ω_p^2 in Refs. [35] and [37].

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[45] By choosing an appropriate reference frame, the number of free state variables can be further reduced; see below.

[46] Essentially, one expects (and so do things usually come out to be) a finite harmonic contribution in the form:

$$S_j^{(n)} = \sum_{l=-n}^{n} S_j^{(n,l)}(X,\,T)\,\exp\left[il(kx-\omega t)\right],$$

i.e.

$$S_j^{(1)} = S_j^{(1,0)} + [S_j^{(1,1)}\,e^{i(kx-\omega t)} + c.c.]\,,$$

$$S_j^{(2)} = S_j^{(2,0)} + [S_j^{(2,1)}\,e^{i(kx-\omega t)} + c.c.] + [S_j^{(2,2)}\,e^{2i(kx-\omega t)} + c.c.]\,,$$

and so forth.

[47] A brief note may be added, about the Irie & Ohsawa paper (Ref. [14]). These authors make the *ad hoc* assumption [see in (B.4) - (B.8) therein] $E_x^{(1)} = u_x^{(1)} = 0$ (starting at ϵ^2), but $E_\perp^{(1)} = u_\perp^{(1)} \neq 0$ (starting at ϵ^1), without explaining why. We have chosen to make no limiting assumption here, and rather let the algebra determine the variable components which may be allowed in the system. However, as our analysis proceeds, it turns out that the assumptions by Irie and Ohsawa are indeed confirmed, yet *only* for EM wave propagation along the magnetic field, i.e. for $\theta = 0$.

[48] In specific, differentiating $D(\omega(k), k) = 0$ with respect to k gives:

$$\frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega}\frac{d\omega}{dk} = 0\,, \qquad \text{hence} \qquad \frac{d\omega}{dk} = -\frac{\frac{\partial D}{\partial k}}{\frac{\partial D}{\partial \omega}}\,.$$

[49] In fact, the potential correction amplitude here is $\hat{\psi}_0 = 2\psi_0$, from Euler's formula: $e^{ix} + e^{-ix} = 2\cos x \quad (x \in \Re)$.

APPENDIX A: SOLUTION FOR $n = 1$, $l = 1$.

The final solution of the linearized (1st order, 1st harmonic) problem for the fluid variables takes the form of Eqs. (64) and (65), in the text.

The coefficients $c_{j,i,y/z}$ for the ($i =$) x, y and z velocity components of the $j-$th fluid (1, 2) are defined as

$$\begin{aligned} c^{(11)}_{j,x,y} &= i(-1)^{j+1} \frac{\omega^2 \Omega^3 \sin\theta \cos\theta}{k\left[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta\right]} \\ &= \frac{\omega}{k n_{j,0}} c^{(11)}_{j,n,y} \,, \end{aligned} \tag{A1}$$

$$\begin{aligned} c^{(11)}_{j,x,z} &= \frac{\Omega^2 \sin\theta}{k\,(\omega^2 - \Omega^2)\left[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta\right]} \times \\ &\quad \left\{-\omega^3(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + i\Omega\omega^2_{p,eff} \cos\theta \left[(-1)^{j+1}\omega^2 + \Omega\cos\theta(i\omega + (-1)^j \Omega\cos\theta)\right]\right\} \\ &= \frac{\omega}{k n_{j,0}} c^{(11)}_{j,n,z} \end{aligned} \tag{A2}$$

($j, j' = 1, 2$ and $j' \neq j$ will be henceforth understood everywhere),

$$c^{(11)}_{j,y,y} = \frac{\omega \Omega^2 (\omega^2 - \omega^2_{p,eff}) \cos\theta}{k\left[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta\right]} \tag{A3}$$

$$c^{(11)}_{j,y,z} = \frac{\Omega\omega}{k\,(\omega^2 - \Omega^2)} \left[i(-1)^{j+1}\omega + \frac{\Omega^3 \omega^2_{p,eff} \cos\theta \sin^2\theta}{\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta}\right] \tag{A4}$$

and

$$\begin{aligned} c^{(11)}_{j,z,y} &= i\,(-1)^j \frac{\omega^2 \Omega (\omega^2 - \omega^2_{p,eff} - \Omega^2 \sin^2\theta)}{k\left[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta\right]} \\ c^{(11)}_{j,z,z} &= \frac{\Omega^2 \left\{\omega^3(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos\theta(\omega\cos\theta + i(-1)^j \Omega \sin^2\theta)\right\} \cos\theta}{k\,(\omega^2 - \Omega^2)\left[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta\right]} \,. \end{aligned} \tag{A5}$$

The electric field components bear a similar structure

$$E^{(11)}_i = c^{(11)}_{el,i,y} B'_y + c^{(11)}_{el,i,y} B'_z \,, \tag{A6}$$

for $i = x, y$ or z, where

$$\begin{aligned} c^{(11)}_{el,x,y} &= c^{(11)}_{el,x,z} = \frac{\omega \Omega^2 \omega^2_{p,eff} \sin\theta \cos\theta}{ck[\omega^2(\omega^2 - \Omega^2 - \omega^2_{p,eff}) + \Omega^2 \omega^2_{p,eff} \cos^2\theta]} \,, \\ c^{(11)}_{el,y,y} &= c^{(11)}_{el,z,z} = 0 \\ c^{(11)}_{el,y,z} &= -c^{(11)}_{el,z,y} = \frac{\omega}{ck} \,. \end{aligned} \tag{A7}$$

Finally, one may consider the obvious definitions $c^{(11)}_{B,x,y} = c^{(11)}_{B,x,z} = c^{(11)}_{B,y,z} = c^{(11)}_{B,z,y} = 0$ and $c^{(11)}_{B,y,y} = c^{(11)}_{B,z,z} = 1$.

As a by-product, a number of relations relating the fluid variables to the E/M field components are also obtained. These may be of use in comparing our theoretical findings to experimental or Space observations. In specific, we have

$$u^{(11)}_{j,i} = c_{j,i,el}E'_x + c_{j,i,by}B'_y + c_{j,i,bz}B'_z \,, \tag{A8}$$

for the $i-$th component $(= x, y, z)$ of the $j-$ $(= 1, 2)$ fluid velocity; the density is then given by

$$n^{(11)}_j = n_{j,0}\frac{\omega}{k}u^{(11)}_{j,x} \,,$$

so that the corresponding coefficients are obvious. The coefficients $c_{j,i,\dagger}$ (for $\dagger = el, by, bz$) read

$$\begin{aligned}
c_{j,x,el} &= i\,(-1)^{j+1}\,\frac{c\Omega\,(\omega^2-\Omega^2\cos^2\theta)}{\omega(\omega^2-\Omega^2)}\,,\\
c_{j,x,by} &= i\,(-1)^{j+1}\,\frac{\Omega^3}{k(\omega^2-\Omega^2)}\sin\theta\cos\theta\,,\\
c_{j,x,bz} &= -\frac{\omega\Omega^2\,\sin\theta}{k(\omega^2-\Omega^2)}\,,\\
c_{j,y,el} &= \frac{c\Omega^2\,\sin\theta}{\omega^2-\Omega^2}\,,\\
c_{j,y,by} &= \frac{\omega\Omega^2\,\cos\theta}{k(\omega^2-\Omega^2)}\,,\\
c_{j,y,bz} &= i\,(-1)^{j+1}\frac{\omega^2\Omega}{k(\omega^2-\Omega^2)}\,,\\
c_{j,z,el} &= i\,(-1)^{j}\,\frac{c\Omega^3}{\omega(\omega^2-\Omega^2)}\sin\theta\cos\theta\,,\\
c_{j,z,by} &= i\,(-1)^{j}\,\frac{\Omega\,(\omega^2-\Omega^2\sin^2\theta)}{k(\omega^2-\Omega^2)}\,,\\
c_{j,z,bz} &= \frac{\omega\Omega^2\,\cos\theta}{k(\omega^2-\Omega^2)}\,,
\end{aligned} \tag{A9}$$

where $j = 1$ or 2 is understood.