Nonlinear Field Line Random Walk and Generalized Compound Diffusion of Charged Particles in Turbulent Magnetized Plasmas

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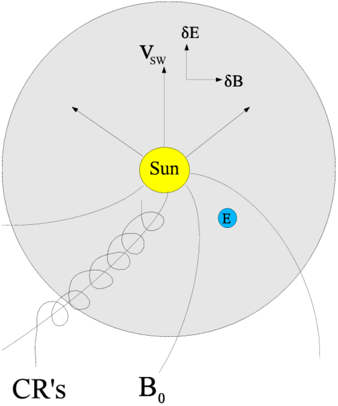
www.tp4.rub.de/∼ioannis/conf/2007-ICTP-oral1.pdf

Content:

- Introduction
- Part I: *Field Line Random Walk (FLRW)*: Modelling the random topology of turbulent magnetic field lines
- Part II: Generalized Compound Diffusion (GCD) model for particle transport across the magnetic field (perpendicular scattering of cosmic rays)
- Summary

Introduction: the physical picture



- Cosmic Rays (CR)
- Magnetic field: $\vec{B}_0 + \delta\vec{B}(\vec{r})$
- Mean magnetic field: $\vec{B}_0 = B_0\vec{e}_z$
- Turbulent magnetic field component $\delta\vec{B}(\vec{r})$
- Turbulent electric field $\delta\vec{E}$ neglected

Aim:

- Describe FLRW via a statistical-mechanical model;
- Identify the asymptotic FL wandering ($\perp \mathbf{B_0}$) regime, viz.

$$\left\langle (\Delta x(z))^2 \right\rangle_{FL} \sim z^{\beta_{FL}}$$

where:

  - $\beta_{FL} = 1$: Diffusion
  - $\beta_{FL} < 1$: Subdiffusion
  - $\beta_{FL} > 1$: Superdiffusion

- Associate FLRW with particle random walk in space;
- Identify the random particle ($\perp \mathbf{B_0}$) motion regime, viz.

$$\left\langle (\Delta x(t))^2 \right\rangle_{P} \sim t^{\beta_{P}}$$

- $\rightarrow$ *Anomalous particle transport* in turbulent plasmas (cf. cosmic plasmas, fusion plasmas, ...).

Part I: Field Line Random Walk (FLRW) - prerequisites and modelling

Field line equation (for $\delta B_z \ll B_0$):

$$dx = \frac{\delta B_x(\vec{x}(z))}{B_0} dz$$

$\longrightarrow$ cf. *Green-Kubo formalism for random processes.*

Field line Mean Square Deviation (MSD):

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{1}{B_0^2} Re \int_0^z dz' \int_0^z dz'' \, R_{xx}(z', z'')$$

with

$$R_{xx}(z', z'') = \left\langle \delta B_x(\vec{x}(z')) \delta B_x^*(\vec{x}(z'')) \right\rangle$$

$\longrightarrow$ *Field correlation function: key element in the theory.*

Theoretical modelling:

- Fourier transformation for the turbulent field;
- Assumption: *homogeneous* and *axisymmetric* turbulence;
- Corrsin's independence hypothesis:

$$\left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}) e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle \approx \left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}) \right\rangle \left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle$$

- Assumption: Gaussian field line d.f. $f_{\|}(z)$;
- *Hybrid (composite slab/2D)* turbulence model:

$$\delta\vec{B}(\vec{r}) = \delta\vec{B}^{(slab)}(z) + \delta\vec{B}^{(2D)}(x,y)$$

  - Cf. Slab turbulence assumption: $\delta\vec{B}(\vec{r}) = \delta\vec{B}(z)$
  - Cf. 2D turbulence assumption: $\delta\vec{B}(\vec{r}) = \delta\vec{B}(x,y)$

- Magnetic turbulence correlation tensor:

$$P_{xx}(\vec{k},t) = < \delta B_x(\vec{k},t)\delta B_x^*(\vec{k},0) >$$

- Magnetostatic turbulence: $P_{xx}(\vec{k},t) = P_{xx}(\vec{k})$
- Slab/2D composite geometry:

$$P_{xx}(\vec{k}) = P_{xx}^{slab}(\vec{k}) + P_{xx}^{2D}(\vec{k})$$

where

$$P_{xx}^{slab}(\vec{k}) = g^{slab}(k_{\parallel})\frac{\delta(k_{\perp})}{k_{\perp}}$$

$$P_{xx}^{2D}(\vec{k}) = g^{2D}(k_{\perp})\delta(k_{\parallel})\frac{k_y^2}{k_{\perp}^3}$$

- Standard form of the wave spectrum:

$$g^{slab}(k_{\parallel}) = \frac{C(\nu)}{2\pi} l_{slab} \delta B_{slab}^2 \left(1 + k_{\parallel}^2 l_{slab}^2\right)^{-\nu}$$

and

$$g^{2D}(k_{\perp}) = \frac{2C(\nu)}{\pi} l_{2D} \delta B_{2D}^2 \left(1 + k_{\perp}^2 l_{2D}^2\right)^{-\nu}$$

Defs.:

the normalization constant $C(\nu)$,

the characteristic bendover (box) scales l_{slab} and l_{2D},

the strength of the turbulent fields δB_{slab} and δB_{2D},

the inertial range spectral index 2ν;

The tedious calculation leads to:

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int d^3\vec{k}\, P_{xx}(\vec{k}) \times \int_0^z dz'\, (z - z') \cos(k_\parallel z')\, e^{-\frac{1}{2}\left\langle (\Delta x(z'))^2 \right\rangle k_\perp^2}$$

Alternatively, applying the operator d^2/dz^2 we get the ODE:

$$\frac{d^2}{dz^2} \left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int d^3\vec{k}\, P_{xx}(\vec{k}) \times \cos(k_\parallel z)\, e^{-\frac{1}{2}\left\langle (\Delta x(z))^2 \right\rangle k_\perp^2}.$$

Analytical results for the slab/2D composite model

- For pure slab geometry we have (for $z \gg l_{slab}$):

$$\left\langle (\Delta x(z))^2 \right\rangle = 2\,\kappa_{FL}\,|z|$$

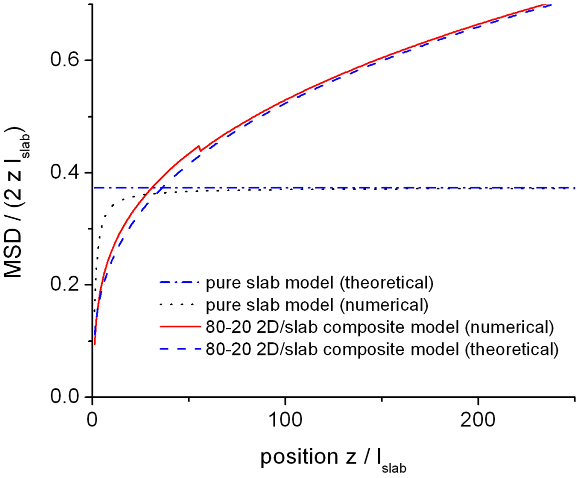
$\Rightarrow$ (Markovian, classical) diffusion of field lines

- For slab/2D composite geometry we find (for $z \gg l_{slab}$):

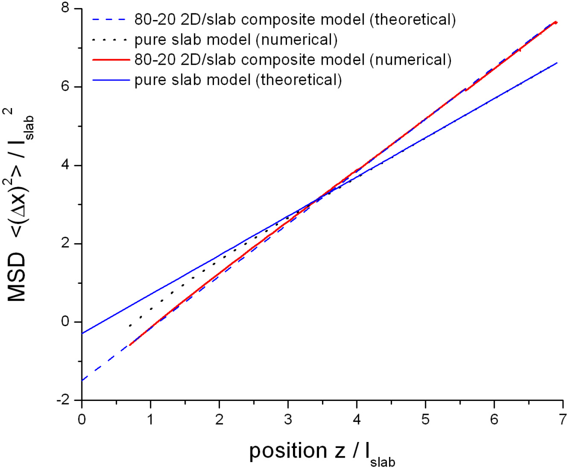
$$\left\langle (\Delta x)^2 \right\rangle = \left[9C(\nu)\sqrt{\frac{\pi}{2}} l_{2D} \frac{\delta B_{2D}^2}{B_0^2} \right]^{2/3} |z|^{4/3}$$

$\Rightarrow$ Superdiffusion of field lines

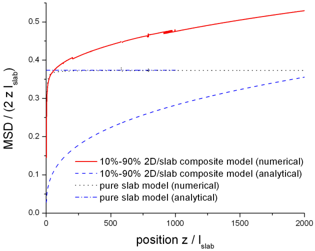
Parameter set 1: 80% 2D, 20% slab, $l_{2D}/l_{slab} = 0.1$:

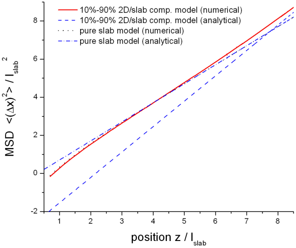


Parameter set 1: 80% 2D, 20% slab, $l_{2D}/l_{slab} = 0.1$:



Parameter set 2: 10% 2D, 90% slab, $l_{2D}/l_{slab} = 0.1$:





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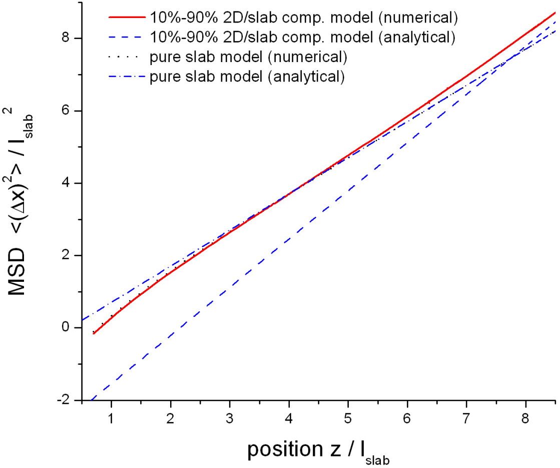

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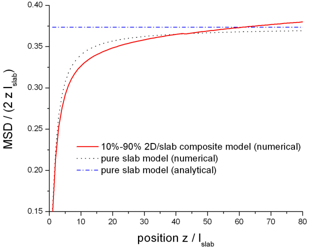

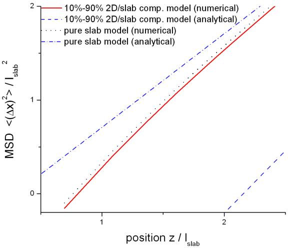

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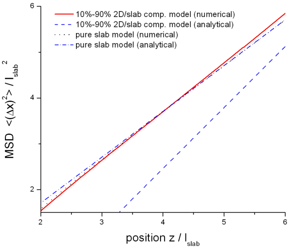
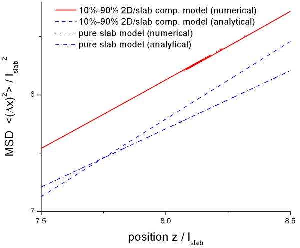




Part II: Perpendicular scattering of charged particles - prerequisites

Previous approaches for $\perp \mathbf{B}_0$ particle random walk:

- Quasilinear theory of particle transport (Jokipii 1966)
- Nonlinear closure approximation (Owens 1974)
- The BAM model (Bieber & Matthaeus 1997)
- The compound transport model (Kota & Jokipii 2000)
- Non-Gaussian statistics (Zimbardo *et al.* 2000)
- The NL guiding center theory (Matthaeus *et al.* 2003)
- The weakly NL theory (WNLT, Shalchi *et al.* 2004)
- The extended NLGC-theory (ENLGCT, Shalchi 2006)
- Chapman-Kolmogorov description (Webb *et al.* 2006)
- $+$ Ruffolo *et al.* 2004, Ragot 2006, ...

Test-particle simulations:

- Slab geometry *:

$$\left\langle (\Delta x)^2 \right\rangle_P \sim \sqrt{t}$$

$\Rightarrow$ subdiffusion

- Slab/2D composite geometry **:

$$\left\langle (\Delta x)^2 \right\rangle_P \sim t$$

$\Rightarrow$ recovery of diffusion?

- No theoretical explanation
  ($\neq$ ENLGCT, but: crude approximations & assumptions);
- *Need for a generalized compound transport model.*

* Qin *et al.*, GRL **29** 1048, 2002; ** *ibid*, ApJ **578** L117, 2002.

Generalized Compound Diffusion (GCD)

Guiding center approximation

$$\left\langle (\Delta x)^2 \right\rangle_P (t) \approx \left\langle \left\langle (\Delta x)^2 \right\rangle_{FL} (z(t)) \right\rangle$$

and thus

$$\left\langle (\Delta x)^2 \right\rangle_P (t) = \int_{-\infty}^{+\infty} dz \, \left\langle (\Delta x)^2 \right\rangle_{FL} (z) f_P(z,t)$$

For $f_P(z,t)$ we assume a Gaussian particle distribution:

$$f_P(z,t) = \frac{1}{\sqrt{2\pi \left\langle (\Delta z)^2 \right\rangle_P}} \, e^{-\frac{z^2}{2\left\langle (\Delta z(t))^2 \right\rangle_P}}$$

For slab geometry and for the standard spectrum we have

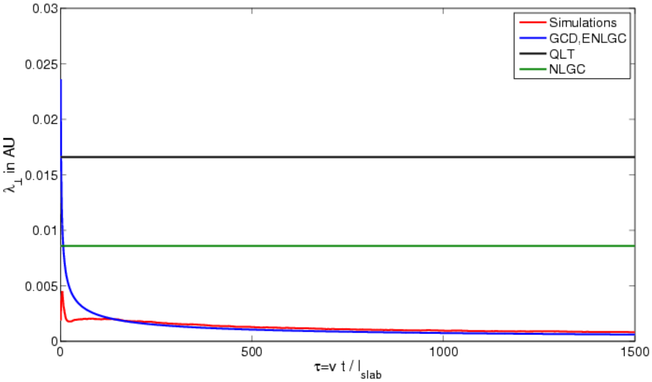
$$\begin{aligned} \left\langle (\Delta x)^2 \right\rangle_{FL}(z) &= 2\,\kappa_{FL}\,|z| \\ \left\langle (\Delta z)^2 \right\rangle_{P}(t) &= 2\kappa_{\parallel} t \end{aligned}$$

The GCD-model provides:

$$\left\langle (\Delta x)^2 \right\rangle_P (t) = 4\,\kappa_{FL} \sqrt{\frac{\kappa_{\parallel} t}{\pi}} \sim \sqrt{t}$$

$\Rightarrow$ Perpendicular particle transport behaves subdiffusively!

The time-dependent perpendicular mean free path $\lambda_\perp \sim \left\langle (\Delta x)^2 \right\rangle_P / (2t)$ for pure slab geometry:



[A. Shalchi, A & A, **453**, L43 (2006)].

For slab/2D composite turbulence: $\left\langle (\Delta x)^2 \right\rangle_{FL} \sim |z|^{4/3}$, so

$$\left\langle (\Delta x)^2 \right\rangle_P = \alpha(\nu) \left(\frac{\delta B_{2D}}{B_0} \right)^{4/3} \left[l_{2D} \left\langle (\Delta z)^2 \right\rangle_P \right]^{2/3}$$

with

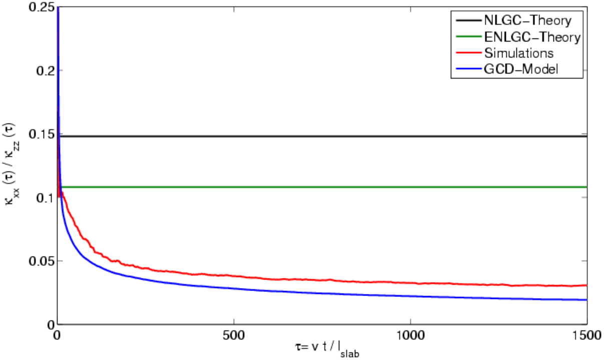
$$\alpha(\nu) = \frac{\Gamma(7/6)}{\sqrt{\pi}} \left(18 \sqrt{\frac{\pi}{2}} C(\nu) \right)^{2/3} \approx 0.5$$

Assuming $\kappa_{xx}(t) = \frac{\left\langle (\Delta x)^2 \right\rangle_P}{2t} \sim t^{b_\perp}$, $\kappa_{zz}(t) = \frac{\left\langle (\Delta z)^2 \right\rangle_P}{2t} \sim t^{b_\parallel}$ we obtain:

$$b_\perp = \frac{2b_\parallel - 1}{3}$$

$\rightarrow \perp$ & $\parallel$ *t.p. motion related!*

The ratio of perpendicular and parallel diffusion coefficients for $R = R_L/l_{slab} = 0.001$:



⇒ Good agreement between the GCD-model & simulations!

[A. Shalchi & I. Kourakis, A & A, **470**, 405 (2007)].

Comparison with observations

Assume diffusion of parallel transport ($< (\Delta z(t))^2 >_P \approx 2\,\kappa_\parallel\, t$) and thus

$$\kappa_\perp(t) = \frac{\alpha(\nu)}{2^{1/3}} \left(\frac{\delta B_{2D}}{B_0} \right)^{4/3} \frac{\left(l_{2D}\kappa_\parallel\right)^{2/3}}{t^{1/3}}.$$

To proceed, we average over the scattering time

$$t_c = \lambda_\parallel / v$$

and we use $\lambda_\parallel = 3\kappa_\parallel / v$ and $\lambda_\perp = 3\kappa_\perp / v$ to find

$$\overline{\lambda}_\perp = \left(\frac{3}{2} \right)^{4/3} \alpha(\nu) \left(\frac{\delta B_{2D}}{B_0} \right)^{4/3} l_{2D}^{2/3} \lambda_\parallel^{1/3}.$$

For

$$
\begin{aligned}
\nu &= 5/6 \\
\delta B_{2D}^2 / B_0^2 &= 0.8 \\
l_{2D} &= 0.1 l_{slab} \approx 0.003 AU \\
\lambda_{\parallel, Palmer} &\approx 0.2 AU
\end{aligned}
$$

we find

$$\lambda_{\perp, GCD} \approx 0.009 AU$$

in agreement with observations (see e.g. Palmer (1982)) where we have

$$\lambda_{\perp, Palmer} \approx 0.007 AU$$

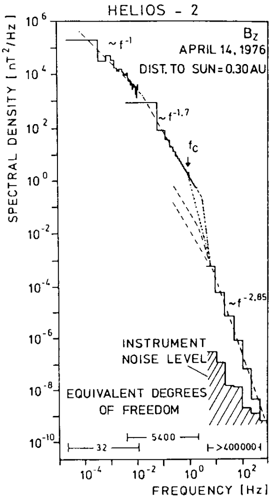
[A. Shalchi & I. Kourakis, A & A, **470**, 405 (2007)].

Summary and Conclusion

- In most cases: FLRW behaves superdiffusively
- The (generalized) compound transport model is a useful tool for describing perpendicular cosmic ray scattering analytically
- The GCD-model agrees with test-particle simulations for slab and slab/2D composite geometry
- By averaging the result for slab/2D turbulence we can explain observed perpendicular mean free path
- However, there is a weak subdiffusive behavior of perpendicular scattering for slab/2D composite geometry

Appendix 1

Observed turbulence spectrum (solar CR):



Appendix 2

⇒ Good agreement between the GCD-model & simulations!

Re-interpretation of the Qin *et al.* (2002) simulations (compatible with GCD) suggests that:

- Parallel transport is weakly superdiffusive

$$\left\langle (\Delta z)^2 \right\rangle_P \sim t^{1.2}$$

- Perpendicular transport is weakly subdiffusive

$$\left\langle (\Delta x)^2 \right\rangle_P \sim t^{0.8}$$