Nonlinear excitations in dusty plasma (Debye) crystals: a new test-bed for nonlinear theories

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1. Introduction

A number of recent theoretical studies have been devoted to collective processes in *strongly coupled* dusty plasmas (DP), motivated by recent experiments. Dust (quasi-)lattices (DL) are typically formed in the sheath region above the negative electrode in discharge experiments, horizontally suspended at a levitated equilibrium position, at $z = z_0$, where gravity and electric (and/or magnetic) forces balance. The linear regime of low-frequency oscillations in DP crystals, in the longitudinal (acoustic mode) and transverse (in-plane, shear acoustic mode and vertical, off-plane optical mode) direction(s), is now quite well understood. However, the *nonlinear* (NL) behaviour of DP crystals is little explored, and has lately attracted experimental [1-3] and theoretical [1-8] interest. Similar nonlinear studies are being carried out in ultra-cold plasmas (UCPs), i.e. strongly-coupled micro-plasma configuration formed in magnetic traps [9].

Recently, we considered the coupling among the horizontal ($\sim \hat{x}$) and vertical (off-plane, $\sim \hat{z}$) degrees of freedom in dust mono-layers; a set of NL equations for longitudinal and transverse dust lattice waves (LDLWs, TDLWs) was thus rigorously derived [4].

Here, we review the nofnlinear dust grain excitations which may occur in a DP crystal (assumed quasi-one-dimensional and infinite, composed from identical grains, of equilibrium charge q and mass M, located at $x_n = n\,r_0,\ n \in \mathcal{N}$). Ion-wake and ion-neutral interactions (collisions) are omitted, for simplicity.

2. Transverse envelope structures (continuum)

Taking into account the intrinsic nonlinearity of the sheath electric (and/or magnetic) potential, the vertical (off-plane) n−th grain displacement $\delta z_n = z_n - z_0$ in a dust crystal (where $n = ..., -1, 0, 1, 2, ...$), obeys the equation

$$\frac{d^2\delta z_n}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} + \omega_{T,0}^2\,(\delta z_{n+1} + \delta z_{n-1} - 2\,\delta z_n) + \omega_g^2\,\delta z_n + \alpha\,(\delta z_n)^2 + \beta\,(\delta z_n)^3 = 0\,. \quad (1)$$

(where coupling anharmonicity and second+ neighbor interactions are omitted)
The characteristic frequency

$$\omega_{T,0} = \left[-qU'(r_0)/(Mr_0)\right]^{1/2}$$

is related to the (electrostatic) interaction potential; for a Debye-Hückel potential: $U_D(r) = (q/r)\,e^{-r/\lambda_D}$, one has

$$\omega_{0,D}^2 = \omega_{DL}^2\,\exp(-\kappa)\,(1+\kappa)/\kappa^3$$

$\omega_{DL} = [q^2/(M\lambda_D^3)]^{1/2}$ is the characteristic dust-lattice frequency; λ_D is the Debye length;
$\kappa = r_0/\lambda_D$ is the DP lattice parameter. $U(r)$.
The *gap frequency* ω_g and the nonlinearity coefficients α, β are defined via the potential $\Phi(z) \approx \Phi(z_0) + M[\omega_g^2\delta z_n^2/2 + \alpha\,(\delta z_n)^3/3 + \beta\,(\delta z_n)^4/4] + \mathcal{O}[(\delta z_n)^5]$ (expanded near z_0, in account of the electric and/or magnetic field inhomogeneity and charge variations), which is related to the overall vertical force

$$F(z) = F_{el/m}(z) - Mg \equiv -\partial\Phi(z)/\partial z$$

[recall that $F(z_0) = 0$].
Linear excitations, viz. $\delta z_n \sim \cos\phi_n$ (here $\phi_n = nkr_0 - \omega t$; k and ω are the wavenumber and frequency; damping is neglected) obey the *optic*-like *discrete* dispersion relation

$$\omega^2 = \omega_g^2 - 4\omega_{T,0}^2\sin^2\!\big(kr_0/2\big) \equiv \omega_T^2\,. \quad (2)$$

We see that transverse vibrations propagate as a *backward wave* [see that $v_{g,T} = \omega_T'(k) < 0$], in fact regardless of the for any form of $U(r)$: the group velocity $v_g = \omega'(k)$ and the phase speed $v_{ph} = \omega/k$ have opposite directions (this is in agreement with recent experiments [2].
Notice the *gap frequency* ω_g, as well as the lower cutoff $\omega_{T,min} = (\omega_g^2 - 4\omega_{T,0}^2)^{1/2}$ (at the edge of the Brillouin zone, at $k = \pi/r_0$), which is *absent in the continuum limit*, viz. $\omega^2 \approx \omega_g^2 - \omega_0^2 k^2 r_0^2$ (for $k \ll r_0^{-1}$).
Assuming a weakly nonlinear *continuum* amplitude, one obtains, via a multiple scale technique [5]:

$$\delta z_n \approx \epsilon\,(A\,e^{i\phi_n} + \text{c.c.}) + \epsilon^2\left[w_0^{(2)} + (w_2^{(2)}\,e^{2i\phi_n} + \text{c.c.})\right] + ...$$

where $w_0^{(2)} \sim |A|^2$, $w_2^{(2)} \sim A^2$; the amplitude A obeys the *nonlinear Schrödinger equation* (NLSE):

$$i\,\frac{\partial A}{\partial T} + P\,\frac{\partial^2 A}{\partial X^2} + Q\,|A|^2\,A = 0\,, \quad (3)$$

where $\{X, T\}$ are the *slow* variables $\{\epsilon(x - v_g t), \epsilon^2 t\}$.

The *dispersion coefficient* $P_T = \omega_T''(k)/2$ takes negative (positive) values for low (high) k.
The *nonlinearity coefficient* $Q = \left[10\alpha^2/(3\omega_g^2) - 3\,\beta\right]/2\omega_T$ is positive for *all* known experimental values of α, β [3].
For small wavenumbers k (where $PQ < 0$), TDLWs will be modulationally stable, and may propagate in the form of dark/grey envelope excitations (*hole* solitons or *voids* [5].
For larger k, *modulational instability* may lead to the formation of bright (*pulse*) envelope solitons.
Exact expressions for these excitations can be found in [5].

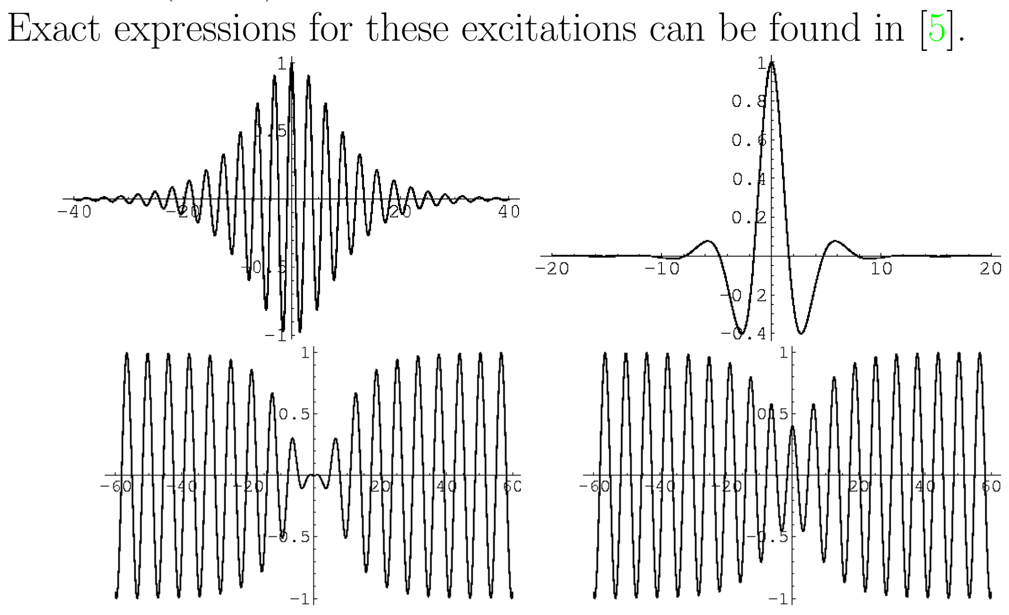

Fig. Envelope solitons of the (a, b) bright type; (c, d) dark (black/grey) type.

3. Intrinsic transverse Localized Modes (ILMs) – Discrete Breathers (DBs)

ILMs, i.e. highly localized *Discrete Breather* (DB) and *multi-breather*-type few-site vibrations, were also shown to occur in transverse DL motion [6, 7], from first principles. These excitations have recently received increased interest among researchers in solid state physics, due to their omnipresence in periodic lattices and remarkable physical properties [8]. The existence of such DB structures at a frequency ω_{DB}) generally requires the *non-resonance condition*

$$n\omega_{DB} \neq \omega(k) \qquad \forall n \in \mathcal{N}$$

which *is*, remarkably, satisfied in all known TDLW experiments [2].

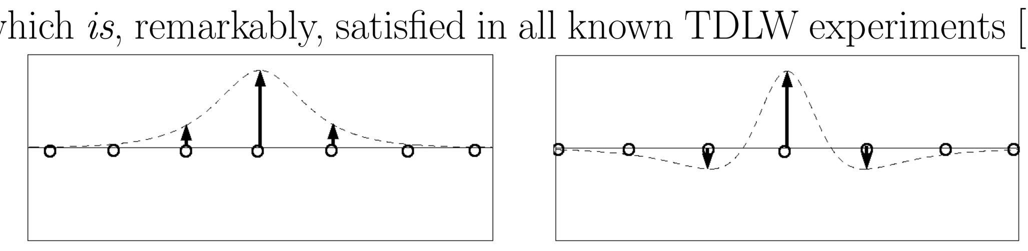

Fig. Discrete Breathers of even and odd parity.

4. Longitudinal envelope excitations

The *nonlinear* equation of motion

$$\frac{d^2(\delta x_n)}{dt^2} + \nu\,\frac{d(\delta x_n)}{dt} = \omega_{0,L}^2\,(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n) - a_{20}\left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2\right] + a_{30}\left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3\right], \quad (4)$$

where the characteristic frequency is given by

$$\omega_{0,L}^2 = [U''(r_0)/M)] = 2\,\omega_{DL}^2\,\exp(-\kappa)\,(1 + \kappa + \kappa^2/2)/\kappa^3$$

for Debye interactions,
describes the longitudinal dust grain displacements $\delta x_n = x_n - nr_0$.
The resulting *acoustic* linear mode[4] obeys

$$\omega^2 = 4\omega_{L,0}^2\,\sin^2\!\big(kr_0/2\big) \equiv \omega_L^2\,.$$

One now obtains (to lowest order $\sim \epsilon$)

$$\delta x_n \approx \epsilon\left[u_0^{(1)} + (u_1^{(1)}\,e^{i\phi_n} + \text{c.c.})\right] + \epsilon^2\,(u_2^{(2)}\,e^{2i\phi_n} + \text{c.c.}) + ...,$$

where $u_{1/0}^{(1)}$ obey [10]

$$i\frac{\partial u_1^{(1)}}{\partial T} + P_L\,\frac{\partial^2 u_1^{(1)}}{\partial X^2} + Q_0\,|u_1^{(1)}|^2 u_1^{(1)} + \frac{p_0 k^2}{2\omega_L}\,u_1^{(1)}\frac{\partial u_0^{(1)}}{\partial X} = 0\,, \quad (5)$$

$$\frac{\partial^2 u_0^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{g,L}^2 - \omega_{L,0}^2 r_0^2}\,\frac{\partial}{\partial X}\,|u_1^{(1)}|^2\,. \quad (6)$$

Here $v_{g,L} = \omega_L'(k)$, and $\{X, T\}$ are slow variables (as above).
We have defined:

$$p_0 = -r_0^3 U'''(r_0)/M \equiv 2a_{20}r_0^3\,, \quad q_0 = U''''(r_0)r_0^4/(2M) \equiv 3a_{30}r_0^4$$

(both positive, and similar in magnitude for Debye interactions [4, 11]); recall that U is the interaction potential.
Eqs. (5), (6) can be combined into an NLSE in the form of Eq. (3), for $A = u_1^{(1)}$ here, with $P = P_L = \omega_L''(k)/2 < 0$.
The exact form of $Q > 0$ (< 0) [10] prescribes stability (instability) at low (high) k.

Longitudinal envelope excitations are *asymmetric*: rarefactive bright or compressive dark envelope structures.

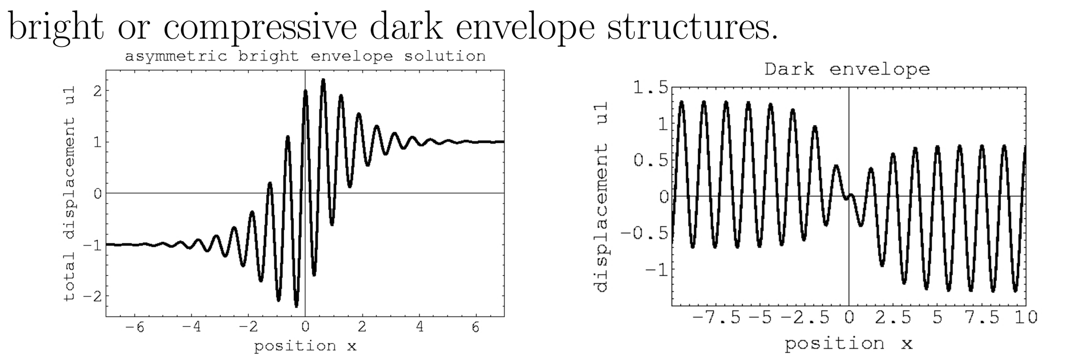

Fig. (a) Bright type; (b) dark type *asymmetric* envelope solitons.

5. Longitudinal solitons

Equation (4) is essentially the equation of atomic motion in a chain with anharmonic springs, i.e. in the celebrated **FPU** (*Fermi-Pasta-Ulam*) problem. At a first step, one may adopt a continuum description, viz. $\delta x_n(t) \to u(x, t)$. This leads to different nonlinear evolution equations (depending on the simplifying hypotheses adopted), some of which are critically discussed in [11]. What follows is a summary of the lengthy analysis therein.
Keeping lowest order nonlinear and dispersive terms, $u(x, t)$ obeys

$$\ddot{u} + \nu\,\dot{u} - c_L^2\,u_{xx} - \frac{c_L^2}{12}\,r_0^2\,u_{xxxx} = -\,p_0\,u_x\,u_{xx} + q_0\,(u_x)^2\,u_{xx}\,, \quad (7)$$

where $(\cdot)_x \equiv \partial(\cdot)/\partial x$; $c_L = \omega_{L,0}\,r_0$; p_0 and q_0 were defined above.
Assuming *near-sonic propagation* (i.e. $v \approx c_L$), and defining the relative displacement $w = u_x$, one has

$$w_\tau - a\,w\,w_\zeta + \hat{a}\,w^2\,w_\zeta + b\,w_{\zeta\zeta\zeta} = 0 \quad (8)$$

(for $\nu = 0$), where
$a = p_0/(2c_L) > 0$, $\hat{a} = q_0/(2c_L) > 0$, and $b = c_L r_0^2/24 > 0$.
Following Melandsø [12], various studies have relied on the *Korteweg - deVries* (KdV) equation, i.e. Eq. (8) for $\hat{a} = 0$, to gain analytical insight in the *compressive* structures observed in experiments [1]. Indeed, the KdV Eq. possesses *negative* (*only*, here, since $a > 0$) supersonic pulse soliton solutions for w, implying a compressive (anti-kink) excitation for u; the KdV soliton is thus interpreted as a density variation in the crystal, viz. $n(x, t)/n_0 \sim -\partial u/\partial x \equiv -w$. Also, the pulse width L_0 and height u_0 satisfy $u_0 L_0^2 = cst.$, a feature which is confirmed by experiments [1]. However, $\hat{a} \approx 2a$ in real Debye crystals (for $\kappa \approx 1$), which invalidates the KdV approximation $\hat{a} \approx 0$ [11]). Instead, one may employ the *extended KdV* Eq. (eKdV) (8), which accounts for *both* compressive *and* rarefactive lattice excitations (exact expressions in [11]). Alternatively, Eq. (7) can be reduced to a *Generalized Boussinesq* (GBq) Equation [11]; again, for $q_0 \sim \hat{a} \approx 0$, one recovers a *Boussinesq* (Bq) equation, widely studied in solid chains. The GBq (Bq) equation yields, like its eKdV (KdV) counterpart, both compressive and rarefactive (only compressive, respectively) solutions; however, the (supersonic) propagation speed v now does *not* have to be close to c_L. The lengthy analysis (see in [11] for details) is not reproduced here.

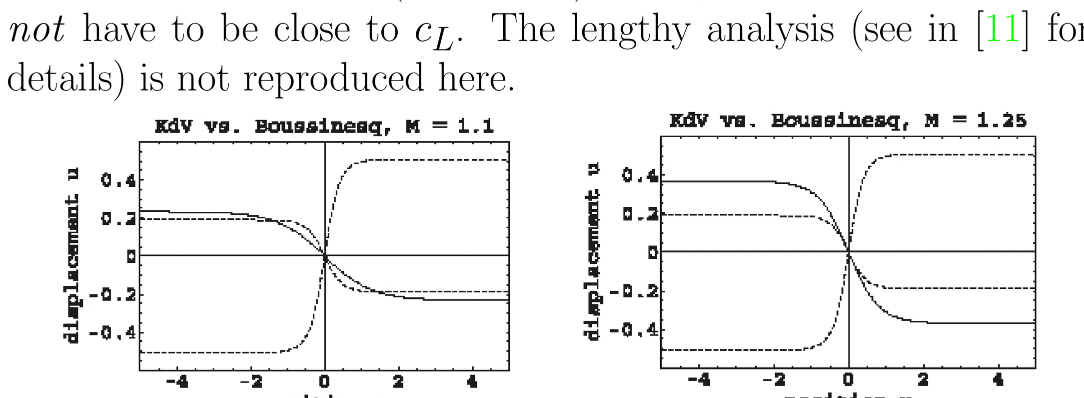

Fig. KdV vs. Boussinesq (displacement) solitons, varying Mach no. $M = v/c_L$.

5. Longitudinal Discrete Breathers

Following existing studies on *Discrete Breathers* (ILMs) *in FPU chains* [cf. (4) above], it is straightforward to show the existence of such localized excitations in the longitudinal direction. A detailed investigation, in terms of real experimental parameters, is on the way and will be reported soon.

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