# Nonlinear electromagnetic wavepackets in plasmas

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## 1. Introduction

*Amplitude modulation* (AM) (related to *modulational instability*, MI) is a well-known mechanism of energy localization dominating wave propagation in nonlinear dispersive media.

The purpose of this study is to provide a *generic* methodological framework for the study of the nonlinear (self-)modulation of the amplitude of electromagnetic modes, a mechanism known to be associated with *harmonic generation* and the formation of *localized envelope modulated wave packets*, such as the ones abundantly observed during laboratory experiments and satellite observations, e.g. *in the Earth's magnetosphere*:

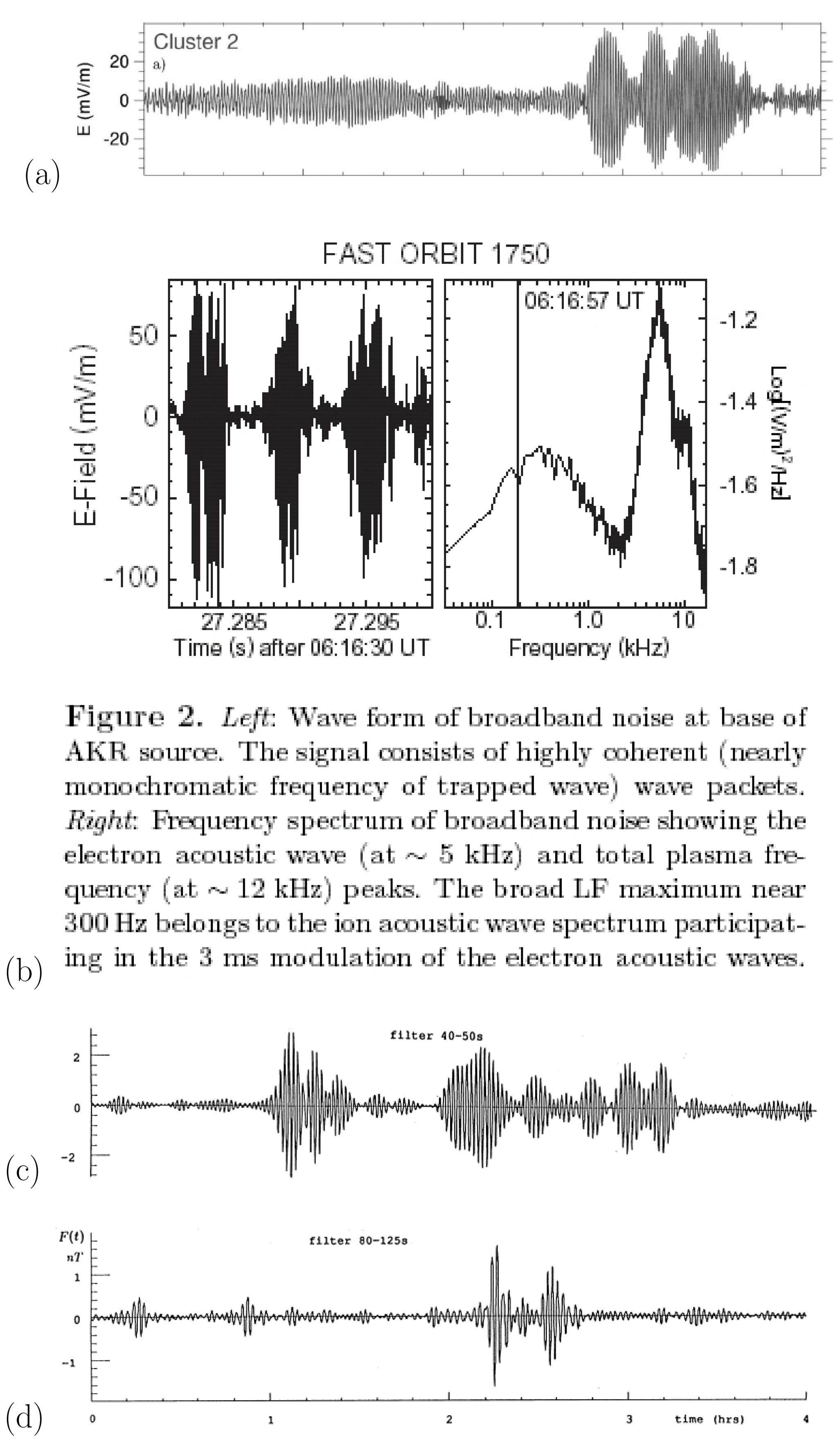


Figure 1. Satellite observations of modulation phenomena: (a) Cluster data, from O. Santolik *et al.*, *J. Geophys. Res.* **108**, 1278 (2003); (b) FAST data, from R. Pottelette *et al.*, *Geophys. Res. Lett.* **26** (16) 2629 (1999); (c), (d) from Ya. Alpert, *Phys. Reports* 339, 323 (2001).

## 2. A (2+1) component plasma fluid model

We consider a multi-component collisionless plasma embedded in a uniform magnetic field $\mathbf{B_0}$. The plasma is composed of:

* (species 1) positive ions (mass $m_1$, charge $q_1 = s_1 Z_1 e$) and
* (species 2) negative ions, or electrons (mass $m_2 = m$, charge $q_2 = s_2 Z_2 e$).
* (species 3) massive, immobile particles (e.g. dust, or ions in e-p-i plasmas), charge $q_3 = s_3 Z_3 e$ (here $s_3 \pm 1$), mass $m_3 \gg m_{1/2}$.
* We have defined the charge state(s) $Z_j$ ($j = 1, 2$), the charge sign $s_j = q_j/|q_j| = \pm 1$ and the absolute electron charge $e$; we shall denote the respective equilibrium number densities by $n_{j,0}$.
* Application 1: *Dusty e-i plasmas*: $q_2 = -e$ ($Z_2 = 1$, $s_2 = -1$);
* Application 2: *Pair- (or e-p-i) plasmas*: $Z_1 = Z_2$, $m_1 = m_2$.

We consider the (two-) fluid density and momentum equations:

$$\frac{\partial n_j}{\partial t} + \nabla\cdot(n_j\,\mathbf{u}_j) = 0 \qquad (1)$$

$$\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{u}_j\cdot\nabla\mathbf{u}_j = \frac{q_j}{m_j}\left(\mathbf{E} + \mathbf{u}_j\times\mathbf{B}\right), \qquad (2)$$

where $n_j$ and $\mathbf{u}_j$ denote the density and the mean (fluid) velocity of species $j$ ($= 1, 2$). The (total) electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ respectively, obey Maxwell's laws:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\,, \qquad (3)$$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mu_0\sum_j n_j q_j \mathbf{u}_j\,, \qquad (4)$$

The electric field $\mathbf{E}$ obeys Poisson's equation

$$\epsilon_0\,\nabla\cdot\mathbf{E} = e(Z_1 n_1 - Z_2 n_2 + s_3 n_3 Z_3) \qquad (5)$$

while the magnetic field satisfies Gauss' law

$$\nabla\cdot\mathbf{B} = 0\,. \qquad (6)$$

The RHS of Poisson's Eq. (5) cancels at equilibrium (*only*):

$$Z_1 n_{1,0} - Z_2 n_{2,0} + s_3 n_3 Z_3 = 0\,, \qquad (7)$$

We underline the fact that no *a priori* assumption is made on the (conservation of) charge neutrality (or density balance) during dynamical evolution in time (off equilibrium).

The system of Eqs. (1) - (4) form a closed system of scalar evolution equations, for the elements of the state vector $\mathbf{S} = (n_1, u_{1,x/y/z}; n_2, u_{2,x/y/z}; E_{x/y/z}; B_{x/y/z})$. Our aim is to use Eqs. (1) - (4) as an analytical basis for a perturbative description of the evolution of the system's state; at every stage, Eqs. (5) and (6) are satisfied, if initially valid.

To simplify the calculation, we shall assume that the direction of wave propagation defines the axis $x$, implying a wave number $\mathbf{k} = k\hat{x}$ for linear waves, and that the external magnetic field $\mathbf{B_0}$ determines the $z-$ axis, i.e. $\mathbf{B_0} = B_0\hat{z}$ (here $\hat{x}, \hat{y}, \hat{z}$ denote the unit vectors along the respective directions). All quantities are assumed to vary along the direction of propagation, i.e. $\nabla \to \partial/\partial x$ (thus $\nabla\times\cdot$ is $\hat{x}\times\partial\cdot/\partial x$ here). Notice that a static magnetic field component along the direction of propagation is prescribed by Eqs. (6) and (the $x-$component of) (3), so that $B_x = 0$ is satisfied here, at all times. The analytical model (and frame) adopted here agrees (for $\theta = \pi/2$ therein) with the oblique propagation picture described in Refs. [1, 2, 3], and also in Ref. [4], for parallel propagation in multi-component plasmas (i.e. for $\theta = 0$).

## 3. Perturbative analysis

*Reductive perturbation* technique: consider small deviations from the equilibrium state

$$\mathbf{S}^{(0)} = (n_{1,0}, \mathbf{0}; n_{2,0}, \mathbf{0}; \mathbf{0}; \mathbf{B_0})^T\,,$$

i.e.

$$\mathbf{S} = \mathbf{S}^{(0)} + \epsilon\mathbf{S}^{(1)} + \epsilon^2\mathbf{S}^{(2)} + ...$$

where $\epsilon \ll 1$ is a (real) smallness parameter. We assume that

$$S_j^{(n)} = \sum_{l=-\infty}^{\infty} S_j^{(n,l)}(X,\,T)\,\exp\left[il(kx-\omega t)\right],$$

where the condition $S_j^{(n,-l)} = S_j^{(n,l)\,*}$ holds, for reality. The wave amplitude is thus allowed to depend on the stretched (*slow*) coordinates of space

$$X = \{\epsilon^n x,\, n = 1, 2, ...\} = \{X_1, X_2, ...\}$$

and time

$$T = \{\epsilon^n t,\, n = 1, 2, ...\} = \{T_1, T_2, ...\}$$

(viz. $X_1 = \epsilon x$, $X_2 = \epsilon^2 x$, and so forth; same for time), distinguished from the (fast) carrier variables $x$ ($\equiv X_0$) and $t$ ($\equiv T_0$). According to the above considerations, we set:

$$\frac{\partial}{\partial t}\Psi_l^{(n)}\,e^{il\phi} = \left(-il\omega\,\Psi_l^{(n)} + \epsilon\frac{\partial\Psi_l^{(n)}}{\partial T_1} + \epsilon^2\frac{\partial\Psi_l^{(n)}}{\partial T_2}\right)e^{il\phi} + \mathcal{O}(\epsilon^3)\,,$$

$$\nabla\,\Psi_l^{(n)}\,e^{il\phi} = \left(+ilk\,\Psi_l^{(n)} + \epsilon\frac{\partial\Psi_l^{(n)}}{\partial X_1} + \epsilon^2\frac{\partial\Psi_l^{(n)}}{\partial X_2}\right)e^{il\phi} + \mathcal{O}(\epsilon^3)\,, \qquad (8)$$

for any $l-$th phase harmonic amplitude $\Psi_l^{(n)}$ among the components of $\mathbf{S}^{(n)}$. The carrier (fundamental) phase is $\phi \equiv kx - \omega t$.

By inserting the above ansatz into Eqs. (1) to (4), one obtains a set of (coupled) reduced evolution equations, which must be solved in each perturbation order $\sim\epsilon^n$ for the $l-$th harmonic amplitudes $S_j^{(n,l)}$ of the state variables (here, $l = -n, -n+1, ..., n-1, n$).

## 4. Multi-harmonic solution up to order $\sim\epsilon^2$

The results *for the ordinary (O-) mode* (→ simplicity) read:

$$n_j = n_{j,0} + \epsilon\, c_j^{(11)} B'_y\, e^{i\phi_c} + \epsilon^2[c_j^{(22)} B'^2_y e^{i2\phi_c} + n_j^{(20)}]$$

$$\mathbf{u_j} = \mathbf{0} + \epsilon\, c_{j,z}^{(11)} B'_y e^{i\phi_c}\,\hat{z} + \epsilon^2\left\{c_{j,z}^{(21)}\frac{\partial B'_y}{\partial X_1}e^{i\phi_c}\hat{z} + B'^2_y e^{i2\phi_c}[c_{j,x}^{(22)}\hat{x} + c_{j,y}^{(22)}\hat{y}] + \mathbf{u_j}^{(20)}\right\}$$

$$\mathbf{E} = \mathbf{0} + \epsilon\, c_{el,z}^{(11)} B'_y e^{i\phi_c}\hat{z} + \epsilon^2\left\{c_{el,z}^{(21)}\frac{\partial B'_y}{\partial X_1}e^{i\phi_c}\hat{z} + B'^2_y e^{i2\phi_c}[c_{el,x}^{(22)}\hat{x} + c_{el,y}^{(22)}\hat{y}] + \mathbf{E}^{(20)}\right\}$$

$$\mathbf{B} = B_0\hat{z} + \epsilon\, B'_y e^{i\phi_c}\hat{y} + \epsilon^2\left[c_{B,y}^{(21)}\frac{\partial B'_y}{\partial X_1}e^{i\phi_c}\hat{y} + c_{B,z}^{(22)} B'^2_y e^{i2\phi_c}\hat{z} + \mathbf{B}^{(20)}\right]$$

$+\mathcal{O}(\epsilon^3)$ everywhere

($j = 1, 2 \equiv +, -$), where $B'_y = B_y^{(11)}/B_0$ and $\phi_c = kx - \omega t$; $S_i^{(20)}$ are *arbitrary* state variable corrections satisfying

$$u_{1,x}^{(20)} = -u_{2,x}^{(20)} = cE'^{(20)}_y, \qquad u_{1,y}^{(20)} = -u_{2,y}^{(20)} = -cE'^{(20)}_x\,.$$

The amplitudes in the latter expressions are:

$$u_{j,z}^{(11)} = (-1)^j\, i\frac{\Omega_j}{k}B'_y\,, \qquad E'^{(11)}_z = -\frac{\omega}{ck}B'_y\,,$$

$$n_j^{(11)} = u_{j,x}^{(11)} = u_{j,y}^{(11)} = E_x^{(11)} = E_y^{(11)} = 0\,,$$

$$u_{j,z}^{(21)} = (-1)^j\,\frac{c^2\omega_{p,eff}^2\Omega_j}{\omega^2 k^2}\frac{\partial B'_y}{\partial X_1}\,,$$

$$E_z^{(21)} = i\,\frac{\omega}{ck^2}\frac{\partial B'_y}{\partial X_1}\,, \qquad B_y^{(21)} = -i\,\frac{\omega^2 + \omega_{p,eff}^2}{\omega^2 k}\frac{\partial B'_y}{\partial X_1}\,,$$

$$n_j^{(21)} = u_{j,x/y}^{(21)} = E_{x/y}^{(21)} = B_{x/z}^{(21)} = 0\,,$$

$$u_{j,x}^{(22)} = \frac{\omega n_j^{(22)}}{k n_{j,0}} = \frac{D_{j,x}^{(22)}}{D_0^{(22)}}B'^{\,2}_y\,, \qquad u_{j,y}^{(22)} = \frac{D_{j,y}^{(22)}}{D_0^{(22)}}B'^{\,2}_y\,, \qquad u_{j,z}^{(22)} = 0$$

$$E_x^{(22)} = \frac{D_{el,x}^{(22)}}{D_0^{(22)}}B'^{\,2}_y\,, \qquad E_y^{(22)} = \frac{\omega}{ck}B_z^{(22)} = \frac{D_{el,y}^{(22)}}{D_0^{(22)}}B'^{\,2}_y\,,$$

$$E_z^{(22)} = = B_y^{(22)} = 0\,,$$

where $j = 1, 2 \equiv +, -$; $D_{*,\dagger}^{(nl)}$ are omitted here.

## 5. Nonlinear Schrödinger (NLS) equation for $B_y^{(11)}$:

$$i\frac{\partial\psi}{\partial\tau} + P\frac{\partial^2\psi}{\partial\zeta^2} + Q\,|\psi|^2\,\psi = 0\,. \qquad (9)$$

where

- $\psi \equiv B_y^{(11)}(\zeta, \tau)$.
- $\zeta = X_1 - v_g T_1$, $\tau = T_2$ ($v_g = \omega'(k)$ is the *group velocity*).
- Dispersion coefficient: $P = \frac{1}{2}\omega''(k) = c^2\omega_{p,eff}^2/(2\omega^3)$.
- Nonlinearity coefficient: $Q = Q(\{\omega_j, \Omega_j, n_{j,0}\}) = ...$ (→ a lengthy expression, omitted here).

## 6. Modulational (in)stability of EM wavepacket

- Plane wave solution of (9): $\psi = \psi_0\exp(iQ|\psi_0|^2\tau)$;
- Linear analysis: set $\hat{\psi} = \hat{\psi}_0 + \epsilon\,\hat{\psi}_{1,0}\cos(\tilde{k}\zeta - \tilde{\omega}\tau)$;
- (Perturbation) dispersion relation:

$$\tilde{\omega}^2 = P\,\tilde{k}^2(P\tilde{k}^2 - 2Q|\hat{\psi}_{1,0}|^2)\,; \qquad (10)$$

- If $PQ < 0$, the amplitude $\psi$ is stable;
- If $PQ > 0$, the amplitude $\psi$ is *unstable* for $\tilde{k} < \sqrt{2Q/P}|\hat{\psi}_{1,0}|$.

We conclude that the stability profile simply depends of the sign of the product $PQ$, which may be investigated in terms of the wavenumber $k$, in addition to intrinsic plasma parameters.

## 7. Envelope soliton solutions of the NLS Equation

Modulated wave-form:

$$\psi = \epsilon\hat{\psi}_0\,\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \Theta) + \mathcal{O}(\epsilon^2)\,.$$

The amplitude $\psi_0$ and phase correction $\Theta$ are functions of $\{\zeta, \tau\}$. These are given by exact expressions (here omitted).

The solutions thus obtained represent localized envelope excitations:

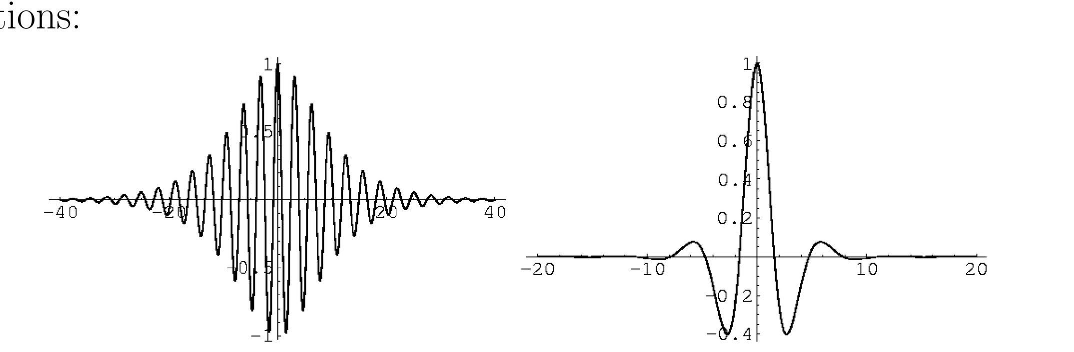

Figure 2. Bright-type envelope solitons (for $PQ > 0$).

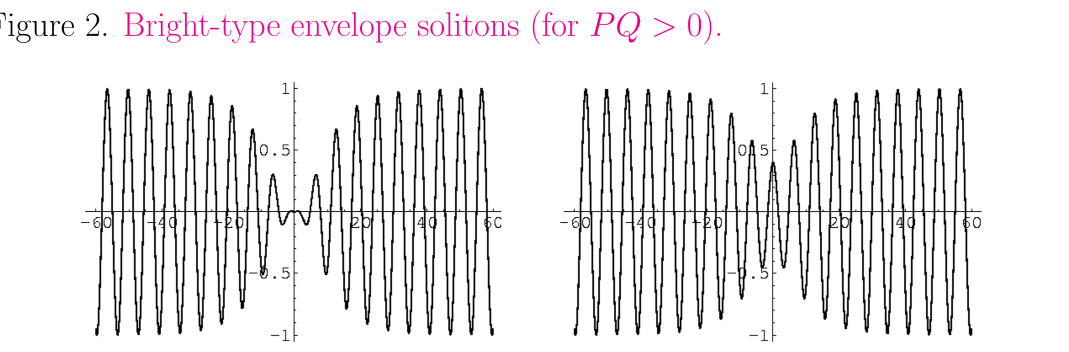

Figure 3. Dark-/grey-type envelope solitons (for $PQ < 0$).

→ excellent agreement with the observed structures!)

**References**

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- I. Kourakis, F. Verheest and N. Cramer, *Phys. Plasmas* **14** (2), 022306/1-10 (2007);
- *ibid, Nonlinear ordinary mode electromagnetic wave packets in space plasmas*, in preparation.