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Nonlinear Field Line Random Walk and Generalized Compound Diffusion of Charged Particles in Turbulent Magnetized Plasmas

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Statistical theory for magnetic field line random walk in turbulent magnetized plasmas: two-component slab/2D vs. isotropic turbulence model(s) *

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These Lecture Notes are devoted to a presentation of a statistical theory for magnetic field line random walk in magnetized turbulent plasmas. The random displacement of magnetic field lines in the presence of magnetic turbulence in plasmas is investigated from first principles. Two different models for the turbulence spectrum are employed and critically discussed, namely a two-component (slab/two-dimensional composite) model and an isotropic turbulence model . An analytical investigation of the asymptotic behavior of the field-line mean square displacement (FL-MSD) is carried out. It is shown that the magnetic field lines behave superdiffusively in the composite model, since the FL-MSD σ varies as $\sigma \sim z^{4/3}$. This superdiffusive result, which is confirmed numerically via an iterative algorithm, is in disagreement with earlier results obtained via quasilinear theory. Contrary to the composite model, quasi-linear theory is shown to provide an adequate description of FLRW for isotropic turbulence. In the latter case, asymptotic FL-MSD behavior is shown to be superdiffusive, as $\sigma \sim z(\ln z + \text{cst.})$. Contrary to the composite model, quasi-linear theory is shown to provide an adequate description of FLRW for isotropic turbulence. In the latter case, asymptotic FL-MSD behavior is shown to be superdiffusive, as $\sigma \sim z(\ln z + \text{cst.})$. The relevance to previous results is discussed.

I. INTRODUCTION

Elucidating the physical mechanisms underlying the transport of particles and energy in magnetized plasmas is a long standing problem. Although it traces its roots in cosmic astrophysics [1], transport theory has received new impulse due to investigations of relevance with man-made magnetically confined systems (nuclear fusion plasmas) [2]. Plasma transport theory initially relied on the picture of *collisional* transport, i.e., scattering of trajectories in position and in velocity space due to mutual charged particle interactions. This transport mechanism is possible even in a plasma embedded in a uniform magnetic field [2, 3] and, in fact, also for unmagnetized plasma (typically relying on a binary collision term in Landau kinetic theory) [2]. In contrast with (and complementary to) this paradigm, *collisionless* plasma transport occurs due to magnetic field turbulence, which results in particle trajectories becoming erratic, yet on average confined to the (helicoidal) motion imposed by the strong mean magnetic field.

The random behavior of magnetic field lines in magnetized plasmas embedded in turbulent fields plays a major role in particle diffusion, as intuitively expected, from a physical point of view. Charged particle trajectories (cosmic rays) are confined by magnetic lines, to a first approximation, essentially performing uniform motion along the field lines, in addition to Larmor gyration in the plane perpendicular to them. In general, the magnetic field lines are defined as the solutions, at a given point in space-time, of the equations $dx/B_x = dy/B_y = dz/B_z$, thus plainly prescribing rectilinear field lines for a uniform magnetic field. Due to inevitable random fluctuations, however, neighboring field lines tend to walk away from each other, in an intrinsically stochastic manner. This *random walk* of magnetic field lines can be modelled statistically, as shown in a classic series of papers by Jokipii and co-workers [4, 5], who followed the general statistical-mechanical framework of random processes in order to formulate a stochastic theory for magnetic field line behavior. The erratic transport of charged particles in turbulent plasmas [6] was later associated to the stochastic spatial field line topology via the notion of *compound diffusion* (see e.g. in Refs. [7–9]; also Refs. therein), which essentially describes a convolution of the two intertwined random processes. Recent extensions of this formalism account for effects like non-axisymmetry and magnetic field line mutual separation [10, 11].

Initial investigations of charged particle transport across a large scale magnetic field (essentially the solar magnetic field, in our case of interest) have relied on an *ad hoc* diffusive assumption for the field-line displacement; e.g. [8, 9, 12].

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Subsequent approaches have adopted various degrees of sophistication, in search of a description which accounts for the intrinsic nonlinearity of the complex problem of particle transport [13–16], and have thus extended the theoretical toolbox by relaxing restricting (e.g. quasilinear [4]) analytical assumptions. Open problems in cosmic ray transport theory include the subdiffusive behavior of perpendicular transport in certain slab models and the recovery of diffusion for non-slab geometry [17], as observed in test-particle simulations; e.g. [18]. Although certain progress was marked by the extended nonlinear guiding center theory [15] (which explains subdiffusion in the slab model, as well as the recovery of diffusion in slab/two-dimensional (2D) composite geometry), there is no satisfactory definite physical explanation allowing one to understand the forementioned diffusion regimes. An attempt to fill in this gap was carried out in [19] by relating field-line transport coefficients to particle transport parameters. A generalized compound diffusion model has recently attempted to relate particle distribution along the field to lateral transport perpendicular to it, thus confirming results obtained by numerical simulations [20].

These Lecture Notes are devoted to an investigation of the random displacement of magnetic field lines, from first principles. A turbulent magnetic field is assumed to be present, in addition to a dominant uniform ambient field. A hybrid two-component (slab/two-dimensional) description of the stochastic random field spatial behavior is employed. No approximation or any restricting assumption of any kind is made, other than the Corrsin hypothesis [6, 21] and assuming stationary turbulence statistics, as well as a Gaussian distribution of magnetic field lines. Part of the material presented here relies on the published Ref. [20] cited below, while part of it will be included in a future publication, currently under preparation. An interested reader should consider reading this text in combination with the generalized compound diffusion model introduced in Ref. [19]a.

The layout of this text includes a detailed exhibition of the model framework, in Section II. The two-component slab/2D model is introduced in Section III, followed by an asymptotic analysis and an analytical identification of characteristic super-diffusive regimes, in Section IV. The analytical findings are confirmed by a numerical analysis in Sec. V. Isotropic turbulence is then considered in Sec. VI, where (contrary to the composite model), quasi-linear theory is shown to provide an adequate description of FLRW. Finally, our results are summarized in the concluding Section.

II. DESCRIPTION OF THE PHYSICAL PROBLEM & ANALYTICAL FORMALISM

This Section is devoted to a brief review of the analytical framework employed in a statistical description of FLRW as a random process. We shall focus on statistically homogeneous, axisymmetric two-component magnetic turbulence. We consider a plasma immersed in a magnetic field $\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \delta\mathbf{B}(x, y, z)$. Here $\mathbf{B}_0 = B_0\hat{e}_z$ denotes the uniform ambient field ($B_0 = $ cst.), while $\delta\mathbf{B}$ is a turbulent (random) magnetic field, here assumed to lie in the $xy-$ plane, i.e. $\delta\mathbf{B} \cdot \mathbf{B}_0 = 0$ (or $\delta B_z = 0$). The stochastic magnetic field function $\delta\mathbf{B}$ is assumed to be essentially time-independent for time scales of interest to us. Electric field turbulence is neglected for simplicity.

We shall focus on a two-component turbulence model, by explicitly assuming that the fluctuating magnetic field is given by

$$\delta\mathbf{B}(x, y, z) = \delta\mathbf{B}_{2D}(x, y) + \delta\mathbf{B}_{slab}(z)\,, \tag{1}$$

where both slab and two-dimensional (2D) contributions (i.e. the latter and the former terms, respectively) are assumed to be of zero mean and statistically uncorrelated. In contrast, a standard isotropic turbulence model consists in assuming

$$\delta\mathbf{B}(x, y, z) = \delta\mathbf{B}(r)\,, \tag{2}$$

where r denotes the radial position coordinate $(x^2 + y^2 + z^2)^{1/2}$ (where no distinction is made among different degrees of freedom). These two assumptions will be critically compared in the following.

The field-line equation for the $x-$ component reads $dx/dz = \delta B_x/B_0$. The solution of this equation provides the field-lines in the form $x = x(z)$. A similar equation can be formulated for the $y-$ component.

Following the standard field-line random walk formalism (see e.g. in Ref. [5]), the field-line displacement $\Delta x = x(z) - x(0)$ in the x-direction takes the form

$$\Delta x(z) = \frac{1}{B_0}\int_0^z dz'\ \delta B_x(\mathbf{x}(z')). \tag{3}$$

The field-line (FL) mean square displacement (MSD) is thus given by

$$\begin{aligned}\left\langle (\Delta x(z))^2 \right\rangle &= \frac{1}{B_0^2} \int_0^z dz' \int_0^z dz'' \ \langle \delta B_x(\mathbf{x}(z'))\, \delta B_x^*(\mathbf{x}(z''))\rangle \\ &= \frac{1}{B_0^2} \int_0^z dz' \int_0^z dz'' \ R_{xx}(z', z'') \end{aligned} \tag{4}$$

where the brackets in the first step denote an average over a statistical ensemble. In the second step we have employed the $xx-$component of the magnetic correlation tensor $R_{xx}(z', z'') = \langle \delta B_x(\mathbf{x}(z'))\delta B_x^*(\mathbf{x}(z''))\rangle$. The real part of the integral(s) in the right-hand side (*rhs*) is henceforth understood everywhere. Since the correlation tensor is itself dependent on $\mathbf{x}(z) = (x(z), y(z), z)$, this is an implicitly nonlinear integral equation for the magnetic field-line space topology.

In principle, one is interested in determining the MSD $\left\langle (\Delta x)^2 \right\rangle$ for large values of z. Anticipating an asymptotic behavior in the form

$$\left\langle (\Delta x)^2 \right\rangle |_{|z|\to\infty} = a\, |z|^b \,, \tag{5}$$

one identifies distinct regimes, depending on the value of the exponent: $b = 1$ corresponds to classical ("Markovian") diffusion, while $b < 1$ ($b > 1$) denotes a subdiffusive (superdiffusive, respectively) "anomalous" regime.

Applying a Fourier representation for the magnetic field, the magnetic correlation tensor in Eq. (4) takes the form

$$\begin{aligned} R_{xx}(z', z'') &= \int d^3\mathbf{k} \int d^3\mathbf{k}' \left\langle \delta B_x(\mathbf{k})\delta B_x^*(\mathbf{k}')\, e^{i[\mathbf{k}\cdot\mathbf{x}(z') - \mathbf{k}'\cdot\mathbf{x}(z'')]} \right\rangle \\ &\simeq \int d^3\mathbf{k} \int d^3\mathbf{k}' \ \langle \delta B_x(\mathbf{k})\delta B_x^*(\mathbf{k}')\rangle \left\langle e^{i[\mathbf{k}\cdot\mathbf{x}(z') - \mathbf{k}'\cdot\mathbf{x}(z'')]} \right\rangle \,, \end{aligned} \tag{6}$$

where Corrsin's independence hypothesis [6, 13, 21, 22] was applied in the last step, for the sake of analytical tractability. Assuming stationary turbulence statistics, viz. $\langle \delta B_x(\mathbf{k})\delta B_x^*(\mathbf{k}')\rangle = P_{xx}(\mathbf{k})\, \delta(\mathbf{k} - \mathbf{k}')$, one finds

$$R_{xx}(z', z'') = \int d^3\mathbf{k}\ P_{xx}(\mathbf{k}) \left\langle e^{i\mathbf{k}\cdot[\mathbf{x}(z') - \mathbf{x}(z'')]} \right\rangle \,. \tag{7}$$

This expression allows for an exact evaluation of the FL-MSD, once one has adopted an analytical assumption for the magnetic field correlation P_{lm} $(l, m = x, y, z)$ and for the characteristic function $< e^{i\vec{k}\cdot\Delta\vec{x}(z)} >$. This is exactly the task remaining to be undertaken in the following.

In order to evaluate Eq. (7), one needs to determine the correlation tensor R_{xx}. For the sake of analytical tractability, in order to evaluate the characteristic function $< e^{i\vec{k}\cdot\Delta\vec{x}(z)} >$, we will assume Gaussian statistics for the field-lines, thus

$$\begin{aligned}\left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle &= e^{-\frac{1}{2}\left\langle (\Delta x(z))^2\right\rangle k_x^2 - \frac{1}{2}\left\langle (\Delta y(z))^2\right\rangle k_y^2 + ik_\parallel z} \\ &= e^{-\frac{1}{2}\left\langle (\Delta x(z))^2\right\rangle k_\perp^2 + ik_\parallel z}, \end{aligned} \tag{8}$$

where axisymmetric turbulence ($\left\langle (\Delta x)^2 \right\rangle = \left\langle (\Delta y)^2 \right\rangle$) led to the last step ($k_\perp^2 = k_x^2 + k_y^2$).

Combining Eqs. (8) and (7) into (4), one obtains an expression for the FL-MSD, in the form of a multiple integral (to be evaluated once an exact form for $P_{xx}(\mathbf{k})$ is specified). This is the final result of this Section.

An ordinary differential equation for the FL-MSD. Although we may directly refer to the following Section, at this stage, we wish to add some clarifying information on our analytical toolbox, for the sake of comparison to previous works, and for future reference.

Considering Eq. (4) and assuming homogeneous turbulence $R_{xx}(z', z'') = R_{xx}(|z' - z''|)$, one obtains, via a trivial transformation [23]

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int_0^z dz' \ (z - z') R_{xx}(z') \,. \tag{9}$$

Upon simple differentiation with respect to z, we find for the field-line MSD

$$\frac{d}{dz} \left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int_0^z dz' \ R_{xx}(z') \,. \tag{10}$$

A second differentiation leads to

$$\frac{d^2}{dz^2}\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} R_{xx}(z)\,,$$
$$= \frac{2}{B_0^2}\int d^3k\, P_{xx}(\vec{k})\cos(k_\parallel z) e^{-\frac{1}{2}\left\langle (\Delta x(z))^2\right\rangle k_\perp^2} \tag{11}$$

where (7) and (8) were taken into account in the last step. Interestingly, Eq. (11) has also been previously derived by Lerche [24], for purely 2D turbulence (also see Refs. [16]); however, no further investigation was carried out in Ref. [24].

The ordinary differential equation (ODE) (11) was obtained exactly, relying on no other assumptions than Corrsin's hypothesis and Gaussian FL statistics. Once a turbulence model is specified, it provides a general basis for the determination of the FL-MSD, which is thus directly related to the turbulence statistics. Also note that the "running FL diffusion coefficient" is thus related to the turbulence characteristics in an elegant manner, as

$$d_{xx} \equiv \frac{1}{2}\frac{d}{dz}\left\langle (\Delta x(z))^2 \right\rangle\Big|_{z\to\infty} = \frac{1}{B_0^2}\int_0^{z\to\infty} R_{xx}(z')dz'\,. \tag{12}$$

The relations in this Section form the basic toolbox of our transport-theoretical study. One needs to specify the correlation tensor $P_{xx}(\mathbf{k})$ of the magnetic fluctuations, and also model the characteristic function $\left\langle e^{i\mathbf{k}\cdot\left[\mathbf{x}(z')-\mathbf{x}(z'')\right]}\right\rangle$. In the following, we shall consider and compare two different hypotheses for the magnetic turbulence statistics, namely a hybrid (composite) slab/2D and an isotropic turbulence model.

III. THE COMPOSITE SLAB/2D TURBULENCE MODEL

We shall now adopt a hybrid (composite) slab/2D model (cf. Refs. [7, 10, 11, 25]), by explicitly assuming

$$P_{xx}(\mathbf{k}) = P_{xx}^{slab}(\mathbf{k}) + P_{xx}^{2D}(\mathbf{k})\,. \tag{13}$$

The slab and 2D contributions to the correlation tensor are given by

$$P_{lm}^{slab}(\mathbf{k}) = g^{slab}(k_\parallel)\frac{\delta(k_\perp)}{k_\perp}\begin{cases}\delta_{lm} & \text{if } l,m = x,y\\ 0 & \text{if } l \text{ or } m = z\end{cases} \tag{14}$$

and

$$P_{lm}^{2D}(\mathbf{k}) = g^{2D}(k_\perp)\frac{\delta(k_\parallel)}{k_\perp}\begin{cases}\delta_{lm} - \frac{k_l k_m}{k^2} & \text{if } l,m = x,y\\ 0 & \text{if } l \text{ or } m = z\end{cases} \tag{15}$$

respectively, i.e., here $P_{xx}^{slab}(\mathbf{k}) = g^{slab}(k_\parallel)\delta(k_\perp)/k_\perp$ and $P_{xx}^{(2D)}(\mathbf{k}) = g^{2D}(k_\perp)\delta(k_\parallel)k_y^2/k_\perp^3$. It is straightforward to calculate, by substituting the latter expressions into (7) above, the FL MSD $\left\langle (\Delta x(z))^2\right\rangle_{(comp)}$ in the composite slab/2D model. Since slab and 2D fluctuations are uncorrelated, they provide two distinct additive contributions; cf. (13) above. Let us evaluate these two distinct parts, separately.

The first (slab) contribution to the FL-MSD, which is due to the first term in the *rhs* of (13), reads

$$\left\langle (\Delta x(z))^2\right\rangle_{(slab)} = \frac{2\pi}{B_0^2}\int_{-\infty}^{+\infty} dk_\parallel\, g^{slab}(k_\parallel)\int_0^z dz' \int_0^z dz''\, e^{ik_\parallel(z'-z'')}$$
$$= \frac{4\pi}{B_0^2}\int_{-\infty}^{+\infty} dk_\parallel\, g^{slab}(k_\parallel)\frac{1-\cos(k_\parallel z)}{k_\parallel^2}$$
$$= 4\,C(nu)\, l_{slab}^2\left(\frac{\delta B_{slab}}{B_0}\right)^2\int_0^\infty d\zeta\,(1+\zeta^2)^{-\nu}\frac{1-\cos(\zeta z/l_{slab})}{\zeta^2}\,, \tag{16}$$

where $\zeta = k_\parallel l_{slab}$ [23]. In the last step, we have adopted the spectrum [25]

$$g^{slab}(k_\parallel) = \frac{C(\nu)}{2\pi}\, l_{slab}\,\delta B_{slab}^2\,(1+k_\parallel^2 l_{slab}^2)^{-\nu}\,, \tag{17}$$

where we have defined the slab bendover length scale l_{slab}, the strength of the turbulent field δB_{slab}, and the inertial-range spectral index 2ν. The normalization constant is $C(\nu) = \Gamma(\nu)/[2\pi^{1/2}\Gamma(\nu - 1/2)]$, where Γ denotes the Euler gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ [26, 27]. As shown in Ref. [19], the slab contribution behaves as $\left\langle(\Delta x(z))^2\right\rangle_{FL} \approx (1/2)(\delta B_{slab}/B_0)^2 z^2$ for small $|z|$ ($\ll l_{slab}$), while $\left\langle(\Delta x(z))^2\right\rangle_{FL} \approx 2\kappa_{FL}|z|$ for large $|z|$ ($\gg l_{slab}$). The field-line diffusion coefficient κ_{FL} is given by [19]

$$\kappa_{FL} = \pi\, C(\nu)\, l_{slab}\, (\delta B_{slab}/B_0)^2 \tag{18}$$

It should be stressed that this diffusive behaviour of field-line separation is only correct if the wave-spectrum, which controls the MSD, is constant at the large turbulence scales of the energy-range. Otherwise, field-lines may present a superdiffusive behavior, for a descreasing spectrum in the energy-range [19].

The second (2D) contribution to the FL-MSD is due to the second term in *rhs*(13); it reads

$$\begin{aligned} \left\langle(\Delta x(z))^2\right\rangle_{(2D)} &= \frac{1}{B_0^2}\int d^3\mathbf{k}\, g^{2D}(k_\perp)\frac{\delta(k_\parallel)}{k_\perp}\sin^2\Psi \int_0^z dz' \int_0^z dz'' \left\langle e^{i\mathbf{k}\cdot[\mathbf{x}(z')-\mathbf{x}(z'')]}\right\rangle \\ &= \frac{\pi}{B_0^2}\int_0^\infty dk_\perp\, g^{2D}(k_\perp)\int_0^z dz' \int_0^z dz'' \left\langle e^{i\mathbf{k}\cdot[\mathbf{x}(z')-\mathbf{x}(z'')]}\right\rangle\Big|_{k_\parallel=0}\,, \end{aligned} \tag{19}$$

where polar coordinates $\{k_\perp, \Psi, k_\parallel\}$ were used in Fourier space. Different approximations have been proposed in the past for the characteristic function $\left\langle e^{i\mathbf{k}\cdot[\mathbf{x}(z')-\mathbf{x}(z'')]}\right\rangle$. We shall here assume that the field-line distribution is described by a Gaussian distribution, so that

$$\left\langle e^{i\mathbf{k}\cdot[\mathbf{x}(z')-\mathbf{x}(z'')]}\right\rangle = e^{-\frac{\left\langle(\Delta x)^2\right\rangle}{2}k_\perp^2} \tag{20}$$

(a factor $e^{ik_\parallel z}$ was omitted due to the δ function in $P^{2D}_{lm}(\mathbf{k})$). Assuming this to be a function of $|z' - z''|$ [23], and substituting into Eq. (19), one finds

$$\begin{aligned} \left\langle(\Delta x(z))^2\right\rangle_{(2D)} &= \frac{2\pi}{B_0^2}\int_0^\infty dk_\perp\, g^{2D}(k_\perp)\int_0^z dz'\,(z-z')\, e^{-\left\langle(\Delta x(z'))^2\right\rangle k_\perp^2/2} \\ &= 4C(\nu)\, l_{2D}\left(\frac{\delta B_{2D}}{B_0}\right)^2 \int_0^\infty dk_\perp\, \frac{1}{(1+k_\perp^2 l_{2D}^2)^\nu}\int_0^z dz'\,(z-z')\, e^{-\left\langle(\Delta x(z'))^2\right\rangle k_\perp^2/2}\,. \end{aligned} \tag{21}$$

In the last step, we have adopted the spectrum [14]

$$g^{2D}(k_\perp) = \frac{2C(\nu)}{\pi}\, l_{2D}\, (\delta B_{2D})^2\, (1 + k_\perp^2 l_{slab}^2)^{-\nu} \tag{22}$$

Here, we have defined the 2D bendover length scale l_{2D} and the strength of the turbulent field δB_{2D}, while the constant $C(\nu)$ was defined above. We stress that Eq. (21) is a transcendental integral equation: note the appearance of the MSD in the left-hand side (*lhs*), and also in the exponent in the integrand, in the *rhs*.

It is straightforward to show that Eq. (21) for the 2D contribution to the MSD is tantamount to

$$\frac{d^2}{dz^2}\left\langle(\Delta x(z))^2\right\rangle_{(2D)} = 4C(\nu)\, l_{2D}\left(\frac{\delta B_{2D}}{B_0}\right)^2 \int_0^\infty dk_\perp\, (1+k_\perp^2 l_{2D}^2)^{-\nu}\, e^{-\left\langle(\Delta x(z'))^2\right\rangle k_\perp^2/2}\,. \tag{23}$$

As shown in the Appendix, the 2D contribution to the MSD behaves as $\left\langle(\Delta x(z))^2\right\rangle_{2D} \approx (1/2)(\delta B_{2D}/B_0)^2 z^2$ for small $\left\langle(\Delta x(z))^2\right\rangle$ ($\ll l_{2D}^2$), so that the well known *superdiffusive* (ballistic) previous result is thus recovered in this limit.

We have seen that the MSD $\left\langle(\Delta x(z))^2\right\rangle_{comp}$ in the hybrid (composite slab/2D) model is expressed as the sum of two parts, which are exactly given by expressions (16) and (21). In order to obtain a working basis for our analytical purposes, it is appropriate to combine these expressions into a dimensionless formula for the FL-MSD. Scaling all lengths by the slab bendover length scale, i.e. setting $\left\langle(\Delta x(z))^2\right\rangle = \sigma l_{slab}^2$ and $\tilde{z} = z/l_{slab}$, one finds

$$\begin{aligned} \sigma &= 4C(\nu)\left(\frac{\delta B_{slab}}{B_0}\right)^2 \int_0^\infty dx\, (1+x^2)^{-\nu}\frac{1-\cos(x\tilde{z})}{x^2} \\ &+ 4C(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^2 \int_0^\infty dx'\, (1+x'^2)^{-\nu}\int_0^{\tilde{z}} d\tilde{z}'\,(\tilde{z}-\tilde{z}')\, e^{-\sigma x'^2/(2\xi^2)}\,, \end{aligned} \tag{24}$$

where we defined the (dimensionless) dummy parameters $x = k_\parallel l_{slab}$, $x' = k_\perp l_{2D}$, and the 2D-to-slab bendover scale ratio $\xi = l_{2D}/l_{slab}$.

It is straightforward to show that the latter relation is tantamount to the differential equation

$$\begin{aligned} \frac{d^2\sigma}{d\tilde{z}^2} &= 4C(\nu)\left(\frac{\delta B_{slab}}{B_0}\right)^2 \int_0^\infty dx\ (1+x^2)^{-\nu}\cos(x\tilde{z}) \\ &+4C(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^2 \int_0^\infty dx\ (1+x^2)^{-\nu}\, e^{-\sigma x^2/(2\xi^2)}\,, \end{aligned} \tag{25}$$

where we have dropped the primes in the dummy integration variable in the latter integral.

IV. ANALYTICAL EVALUATION OF THE MSD IN THE TWO-COMPONENT MODEL

A. Small field-line displacement behavior

First, let us consider the small displacement and MSD limit (i.e., $\tilde{z} \ll l_{slab}$, $\left\langle (\Delta x(z))^2 \right\rangle \ll l_{2D}^2$, viz. $\sigma \ll \xi^2$). In this approximate case, the two contributions in (24) can be evaluated along the considerations in the previous Section (cf. Appendix). It is straightforward to show that the FL-MSD behaves as $\left\langle (\Delta x(z))^2 \right\rangle \approx \frac{1}{2}\left(\frac{(\delta B_{slab})^2+(\delta B_{2D})^2}{B_0^2}\right) z^2$ [28]. In the limit of small $|z|$, therefore, the field lines separate superdiffusively with $\sim z^2$, as in the pure slab and/or 2D model(s). In this case, both (slab and 2D) contributions have the same form, bearing a free-streaming-like ballistic (parabolic MSD) result.

B. Infinitely large FL-MSD behavior: approximate analysis

In the large $|z|$-limit, the two contributions to the FL-MSD present a different behavior. The former (slab) contribution behaves diffusively, as shown in Ref. [19]. In the following paragraph we shall show that the latter (2D) contribution behaves superdiffusively, in fact as $\sim |z|^{4/3}$ (cf. Ref. [20]), in the infinite $|z|$ limit.

It can be shown that the slab contribution in (25) may be neglected, for an infinitely large distance $|z|$. To see this, note that the first integral in (25) varies as $\approx e^{-\tilde{z}}$, at large distance $\tilde{z}$, whence we deduce that it is negligible, with respect to the second one, for large $z \gg l_{slab}$ (see in Appendix A 1 for details). On the other hand, the second integral in (25) behaves (for large values of σ, i.e., $y \gg 1$) as $\approx \xi\sqrt{\pi}/\sqrt{2\sigma}$ (see in Appendix A 2). Omitting analytical details here (see in the Appendix), (25) now becomes

$$\sigma''(\tilde{z}) \approx \frac{4\pi C(\nu)}{2^\nu \Gamma(\nu)}\left(\frac{\delta B_{slab}}{B_0}\right)^2 \tilde{z}^{\nu-1}e^{-\tilde{z}} + \frac{4\xi\sqrt{\pi}C(\nu)}{\sqrt{2}}\left(\frac{\delta B_{2D}}{B_0}\right)^2 \sigma^{-1/2}\,. \tag{26}$$

We point out that this relation was obtained under the assumptions $z \gg l_{slab}$ and $\left\langle (\Delta x)^2 \right\rangle \gg l_{2D}^2$ (i.e., it is valid for large values of the reduced position $\tilde{z} \gg 1$ and MSD $\sigma \gg \xi$ variables). Note that the first term in the *rhs* (i.e., the slab contribution) vanishes for large $\tilde{z}$, and may thus be omitted in the asymptotic analysis. Multiplying both sides in (26) by $\sigma'(\tilde{z})/2$ and integrating, one easily obtains

$$[\sigma'(\tilde{z})]^2 \approx 4\alpha\sigma^{1/2} + \beta\,, \tag{27}$$

where

$$\alpha = 4C(\nu)\,\xi\sqrt{\frac{\pi}{2}}\left(\frac{\delta B_{2D}}{B_0}\right)^2 \tag{28}$$

, and β is an integration constant, to be determined below [29].

Eq. (27) can be cast (for $y' > 0$) in the form

$$\frac{d\sigma}{(4\alpha\sigma^{1/2}+\beta)^{1/2}} = d\tilde{z}\,, \tag{29}$$

so both sides can now be integrated to obtain the explicit (asymptotic) dependence of σ on $\tilde{z}$. Different special cases for the coefficients α and β may be singled out, as we shall see below.

C. The pure slab turbulence limit

It is clear from (27) that the real constant β physically corresponds to the (square) limit value of σ' for vanishing α, that is, cancelling the 2D component of turbulence stochasticity. Indeed, assuming $\alpha = 0$ (purely slab MSD), (27) can be easily integrated to give $\sigma = \sqrt{\beta}\,\tilde{z}$, which implies a diffusive behavior of the FL-MSD. For consistence, the value of β should therefore be determined in agreement with the well known (diffusive) result in this limit, which reads

$$\left\langle (\Delta x(z))^2 \right\rangle_{slab} = 2\kappa_{FL}|z| \tag{30}$$

(see above; details in Ref. [19]), hence, in dimensionless form,

$$\sigma'_{slab}(\tilde{z}) = 2\kappa_{FL}/l_{slab}\,.$$

Therefore, we shall set

$$\beta = \left(2\frac{\kappa_{FL}}{l_{slab}}\right)^2 = 4\pi^2\,C(\nu)^2\left(\frac{\delta B_{slab}}{B_0}\right)^4, \tag{31}$$

where the slab model diffusion coefficient κ_{FL} was defined in (18) above.

D. The pure 2D turbulence limit

We shall now consider the vanishing β limit. For $\beta = 0$, Eq. (27) becomes $\sigma''(\tilde{z}) = \alpha\sigma^{-1/2}$. Anticipating a solution in the form $\sigma(\tilde{z}) = c\,\tilde{z}^d$ [and assuming $\sigma'(0) = 0$], one is easily led to $c = (3\sqrt{\alpha}/2)^{4/3}$ and $d = 4/3$. We therefore obtain

$$\sigma(\tilde{z}) \approx 2^{-1/3}3^{4/3}[C(\nu)^{2/3}]\xi^{2/3}\pi^{1/3}\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3}\tilde{z}^{4/3} = \left(\frac{81\pi\xi^2}{2}\right)^{1/3}[C(\nu)]^{2/3}\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3}\tilde{z}^{4/3}\,. \tag{32}$$

The same result is obtained upon integrating Eq. (29). Scaling back to the original variables, we have

$$\left\langle (\Delta x(z))^2 \right\rangle \approx \left(\frac{9\sqrt{\pi}C(\nu)}{\sqrt{2}}\right)^{2/3}\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^2\left(\frac{z}{l_{2D}}\right)^{4/3}. \tag{33}$$

We draw the conclusion that the asymptotic behavior of the MSD is *superdiffusive*, in fact bearing a characteristic exponent 4/3, in addition to an asymptotic value which is related to intrinsic turbulence parameters via relation (33). This result was here obtained via an approximate treatment of (25), and adopting simple qualitative arguments. We shall show below that this prediction is indeed confirmed by a direct numerical evaluation of (24).

As a matter of fact, the purely 2D turbulence model obtained in this mathematical limit may, in fact, be somehow questioned physically (it implies $\delta B = \delta B(x, y)$, although the $z-$ behavior of FL is considered); however, we have considered it appropriate to study this limit, here, for the sake of comparison and future reference.

E. Two-component model

We shall henceforth consider the case $\alpha \neq 0$ and $\beta \neq 0$. The relation (29) can readily be integrated [32] to give

$$\frac{1}{6\alpha^2}\left[(2\alpha\sqrt{\sigma} - \beta)\,(4\alpha\sqrt{\sigma} + \beta)^{1/2} + \beta^{3/2}\right] = \tilde{z}\,, \tag{34}$$

where we have neglected a finite (arbitrary) real constant z_0 in the large z limit. Note that both limits considered in the latter two paragraphs are here properly recovered, for vanishing α or β, respectively.

The relation (34) can be rearranged as

$$(Y-1)\,(2Y+1)^{1/2} = Z - 1\,, \tag{35}$$

where

$$Z = \frac{6\alpha^2}{\beta^{3/2}}\tilde{z}\,, \qquad\qquad Y = \frac{2\alpha}{\beta}\sqrt{\sigma}\,. \tag{36}$$

Eq. (35) is an algebraic equation, which can be solved for Y in terms of Z [33]. Before doing so, however, it may be interesting to consider the asymptotic behavior of $Y(Z)$, as obtained from (35).

F. Asymptotic behavior and parameter regimes

Considering the large- and small-value regimes for Y, it is straightforward to obtain from (35) the asymptotic form(s):

$$Y^2 \approx 2Z/3 \quad \text{for} \quad Y \ll 1\,, \qquad \text{and} \qquad Y^2 \approx (Z/\sqrt{2})^{4/3} \quad \text{for} \quad Y \gg 1\,. \tag{37}$$

As expected, these forms correspond exactly to the diffusive (superdiffusive) results presented in the previous two paragraphs, in the pure-slab (pure-2D, respectively) turbulence limits. Let us consider the critical point where the two asymptotic regimes "cross into" each other, namely $Z_c = 2^5/3^3 \approx 1.185$, which implies [cf. definitions (36)] a critical value

$$\tilde{z} = \frac{2^4}{3^4}\frac{\beta^{3/2}}{\alpha^2} \approx 0.2\,\frac{\beta^{3/2}}{\alpha^2} \equiv \tilde{z}_{c,1} \tag{38}$$

for the reduced position variable $\tilde{z}$. On the other hand, our theoretical calculation relies on the validity of the assumptions $z \gg l_{slab}$ and $\langle(\Delta x)^2\rangle \gg l_{2D}^2$, implying $\tilde{z} \gg 1$ and $\sigma \gg \xi^2$, which are assumed to hold in any case of study. Combining with (36) and (37), in the respective regions, the latter requirements are seen to be tantamount to $\tilde{z} \gg \tilde{z}_{c,2}$ and $\tilde{z} \gg \tilde{z}_{c,3}$, respectively; here we defined two additional critical values, namely:

$$\tilde{z}_{c,2} = \xi/\sqrt{\beta}\,, \qquad \tilde{z}_{c,3} = 2(\alpha\xi)^{3/2}/3\,. \tag{39}$$

We thus finally find two distinct regimes, regarding the FL-MSD behavior,

$$\begin{array}{rcccc} \max\{1, z_{c,2}\} \ll \tilde{z} \ll \tilde{z}_{c,1} & \to & \sigma \sim \tilde{z} & \text{(diffusion)} & \text{(condition 1)} \\ \tilde{z} \gg \max\{1, z_{c,1}, z_{c,3}\} & \to & \sigma \sim \tilde{z}^{4/3} & \text{(superdiffusion)} & \text{(condition 2)}\,. \end{array}$$

Note that diffusion occurs only if $z_{c,1} \gg 1$, and is otherwise excluded. The above results provide a set of explicit criteria for the behavior of field-lines in the composite turbulence model. We recall that the turbulence spectrum-related constants α and β were defined in (28) and (31), respectively, so that the critical values $\tilde{z}_{cr,1/2/3}$ defined in (38) and (39) can be expressed in terms of relevant turbulent plasma parameters. We provide here, for future reference, the exact form of the critical values derived above:

$$\begin{aligned} \tilde{z}_{c,1} \equiv \frac{z_{c,1}}{l_{slab}} &= \left(\frac{2\sqrt{\pi}}{3}\right)^4 C(\nu) \left(\frac{l_{slab}}{l_{2D}}\right)^2 \frac{\delta B_{slab}^6}{B_0^2\,\delta B_{2D}^4} \\ \tilde{z}_{c,2} \equiv \frac{z_{c,2}}{l_{slab}} &= \left(\frac{1}{2\pi C(\nu)}\right) \frac{l_{2D}}{l_{slab}} \left(\frac{\delta B_{slab}}{B_0}\right)^{-2} \\ \tilde{z}_{c,3} \equiv \frac{z_{c,3}}{l_{slab}} &= \frac{2^4}{3}\left(\frac{\pi}{2}\right)^{3/4} C(\nu)^{3/2} \left(\frac{l_{2D}}{l_{slab}}\right)^3 \left(\frac{\delta B_{2D}}{B_0}\right)^3 . \end{aligned} \tag{40}$$

In particular, we shall retain the requirement $\tilde{z}_{cr,1} \gg 1$, for diffusion to be possible, which yields an explicit criterion in the form:

$$\delta \equiv \left(\frac{l_{slab}}{l_{2D}}\right)^2 \frac{\delta B_{slab}^6}{B_0^2\,\delta B_{2D}^4} \gg 1 \tag{41}$$

(where a constant of the order ≈ 0.2 was neglected). The dimensionless parameter δ here defined determines the limits of diffusion occurring in this model. If $\delta \gg 1$, then a diffusive behavior may be anticipated if condition 1 (see above) holds, otherwise a superdiffusive behavior is expected if condition 2 is satisfied instead. On the other hand, if δ is of the order of unity or smaller, then superdiffusion will be dominant, and diffusion is excluded.

In view of a representative numerical application, we shall consider a typical set of cosmic plasma parameter values (here referred to as *Set 1*): $\nu = 5/6$ [hence $C(\nu) \approx 0.1188$], $\xi = l_{2D}/l_{slab} = 0.1$; a hybrid 20% slab/80% 2D turbulence topology is assumed [i.e., setting $\delta B_{2D}^2/B_0^2 = 4\,\delta B_{slab}^2/B_0^2 = 0.8$]. For these values, we find $\alpha \approx 0.05$ and $\beta \approx 0.22$, which lead to: $\tilde{z}_{cr,1} \approx 0.29$, $\tilde{z}_{cr,2} \approx 0.67$ and $\tilde{z}_{cr,3} \approx 0.0002$. One easily sees that diffusion is excluded in this case, while superdiffusion will be dominant for $z \gg l_{slab}$. On the other hand, we have considered the opposite extreme case of a dominant slab contribution, by assuming a hybrid 90% slab/10% 2D turbulence topology is assumed [i.e., setting $\delta B_{slab}^2/B_0^2 = 9\delta B_{2D}^2/B_0^2 = 0.9$] (all remaining parameter values are taken to be the same as in *Set 1*); this choice of parameters will be referred to as *Set 2*. One finds, for this model, a critical value $\tilde{z}_{cr,1} \approx 1500$, while $\tilde{z}_{cr,2} \approx 0.15$ and $\tilde{z}_{cr,3} \approx 10^{-5}$. According to our regime identification above, this fact allows for an extended range of values (for z/l_{slab} well below $\tilde{z}_{cr,1}$) where FL wandering will be quasi-diffusive, while a superdiffusive regime will, once more, dominate for higher z values. These predictions are indeed confirmed by the numerical analysis below (refer to Figs. 2 and 3, and the analysis in Section V below).

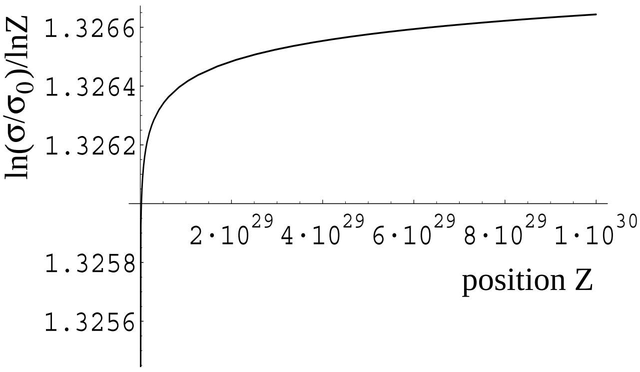


FIG. 1: The logarithm ratio $\ln(\sigma/\sigma_0)/\ln Z$, where $\sigma_0 = (\beta/2\alpha)^2$, is depicted against the reduced position variable $Z = 6\alpha^2 z/(\beta^{3/2} l_{slab})$; this plot is dictated by the anticipated relationship $\sigma = aZ^b$, which implies $b = \lim_{Z\to\infty}(\ln\sigma/\ln Z)$. The value of σ is obtained by combining relations (36) - (43). See that the predicted value $b = 4/3$ is obtained, in the asymptotic (large Z) limit.

G. Exact solution of (35) for the MSD

Eq. (35) is a cubic polynomial equation in $Y^{1/2}$, which may be solved analytically by employing standard algebraic techniques. The procedure leads to a real root $Y_1(Z)$ [34]

$$Y = \frac{1}{2}(1 + A^{-1/3} + A^{1/3}), \tag{42}$$

with

$$\begin{aligned} A &= 2(Z-1)^2 + 2\sqrt{(Z-1)^4 - (Z-1)^2} - 1 \\ &= 1 + 2Z(Z-2) + 2\sqrt{Z(Z-2)(Z-1)^2}, \end{aligned} \tag{43}$$

in addition to two complex roots $Y_{2/3}(Z)$, whose lengthy analytical form is omitted here, for brevity. We shall add, omitting details, that the asymptotic analysis of the 3 roots in terms of the argument Z leads to the anticipated result obtained above. In particular, for $Z \ll 1$, we find $Y_{2/3}(Z) \approx \mp\sqrt{2Z/3}$, which leads exactly, after a short calculation (inverting to the original scaled variables), to the diffusive behavior $\sigma \approx \sqrt{\beta}\tilde{z}$ presented in paragraph IV C above. On the other hand, in the large Z limit, i.e. for $Z \gg 1$, one finds that $Y_1(Z) \approx (Z/\sqrt{2})^{2/3}$. It is straightforward to show that we thus recover the superdiffusive behavior $\sigma \approx (3\sqrt{\alpha}\tilde{/}2)^{4/3}$ presented in paragraph IV D above [35].

The exact value of $\sigma \sim Y^2$ can be numerically evaluated from expressions (42, 43). In Figure 1, we have depicted the ratio $\ln\sigma/\ln Z$ [making use of (42) and (43)] against the (reduced) position variable Z, in fact inspired by the anticipated relationship $\sigma = aZ^b$, which implies $b = \lim_{Z\to\infty}(\ln\sigma/\ln Z$. We see that the predicted value $b = 4/3$ is obtained, in the asymptotic (large Z) limit.

Resuming our findings so far, the relation (42) [together with the definitions in (36) and (43)] provides a solution of (25) [cf. (26)]. The correct asymptotic result, namely superdiffusion ($\sigma \sim z^{4/3}$), is reproduced to leading order (see that $Y \sim A^{1/3} \approx 4Z^2$ for $Z \gg 1$); this is true for ($\alpha \neq 0$ and) $\beta = 0$ (pure-2D turbulence limit), yet also in general (for $\alpha, \beta \neq 0$). We stress that the condition for this superdiffusive result to be valid implies $z \gg \beta^{3/2}/(6\alpha^2)\, l_{slab}$ (cf. the analysis above), which can always be satisfied for sufficiently large $\tilde{z}$, provided that the ratio $\beta^{3/2}/(6\alpha^2)$ remains finite (or zero), viz. $\alpha \neq 0$ (i.e., a small 2D contribution exists).

H. Asymptotic large $\tilde{z}$ behavior for arbitrary α and β

Eq. (42) has provided a basis for an investigation of the dependence of the FL-MSD $\sigma \sim Y^2$ versus $Z \sim \tilde{z}$, both analytically and numerically. We shall here advance the analytical calculation in the asymptotic limit $Z \gg 1$. At a first step, one is tempted to consider $Y \approx A^{1/3}/2 \approx (2Z)^{2/3}/2$ (i.e. omitting terms $\sim \mathcal{O}(Z^0)$ or smaller, in the infinite Z limit). However, this choice trivially leads to the same result as obtained above in the pure-2D limit (for $\beta = 0$), as previously mentioned. To gain analytical insight, even approximately, one can keep the leading two orders in the asymptotic expansion (for large Z), i.e., $Y \approx (1 + A^{1/3})/2 \approx [1 + (2Z)^{2/3}]/2$, where terms $\mathcal{O}(Z^{-2/3})$ or smaller were now omitted. Reversing the transformation (36), we thus obtain

$$\sigma \approx \left(\frac{3\sqrt{\alpha}}{2}\right)^{4/3} \tilde{z}^{4/3} + \frac{\beta}{2}\left(\frac{3}{2\alpha}\right)^{2/3} \tilde{z}^{2/3} . \tag{44}$$

We note, for rigor, that the same relation is obtained by combining (36) - (43) into an expression for $\sigma = \sigma(\tilde{z})$ and then linearizing in β. We remark that the second term in the *rhs* is due to the slab contribution – and vanishes without it, i.e. for $\beta = 0$ – although we find out, not without surprise, that it is not linear in z. Recovering the original dimensions of all quantities, we find (in the composite model)

$$\begin{aligned}\left\langle (\Delta x(z))^2 \right\rangle_{comp} \approx & \left(\frac{9\sqrt{\pi}C(\nu)}{\sqrt{2}}\right)^{2/3} \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^{2/3}\, z^{4/3} \\ & + \left(\frac{3\pi^{5/2}C(\nu)^2}{2}\right)^{2/3} \left(\frac{\delta B_{slab}}{B_0}\right)^4 \left(\frac{\delta B_{2D}}{B_0}\right)^{-4/3} l_{slab}^2\, l_{2D}^{-2/3}\, z^{2/3} .\end{aligned} \tag{45}$$

We shall retain the latter expression as an improved approximate (reduced) expression for the MSD in the two-component turbulence model. We see that the diffusive character of the MSD in the slab model is entirely lost in the presence of 2D turbulence.

V. NUMERICAL ANALYSIS IN THE TWO-COMPONENT MODEL

We have shown above that the asymptotic value of the MSD $y(z)$, which exactly obeys Eq. (24), is approximately given by Eq. (33). This result was obtained analytically, by applying simplifying qualitative arguments in expression (25) (focusing on its large $z \gg 1$ behavior). In the following, we shall test this prediction by directly solving (24) numerically.

We have carried out an accurate numerical evaluation of (24), for $\nu = 5/6$ (thus $C(\nu) \approx 0.1188$), $\xi = l_{2D}/l_{slab} = 0.1$, and assuming a hybrid 20% slab/80% 2D turbulence topology [i.e., setting $\delta B_{2D}^2/B_0^2 = 4\,\delta B_{slab}^2/B_0^2 = 0.8$]. In order to take into account the intrinsic nonlinearity of this integral equation – manifested by the appearance of σ in the left-hand side (*lhs*) *and* inside the integrand in the *rhs* – we have employed an appropriate recursive algorithm, which is outlined in the following. First, the range of values of the position variable $\tilde{z}$ is determined, say from 0 to $\tilde{z}_{max}$, and spanned by sampling at N different points, i.e., separated by an interval $h = \tilde{z}_{max}/N$. We have chosen $\tilde{z}_{max} = 10^3$ and $N = 10^4$ (thus $h = 10^{-1}$) in the plots presented below, although the recipe was also tested for different combination of values, for comparison, and was found to yield practically identical results. The iterative solution scheme consists in substituting σ in the *rhs* at every step n (for $n = 0, 1, 2, ...$) by a specific function $\sigma_n(\tilde{z})$, and then evaluating the *lhs* numerically. The next step trial function $\sigma_{n+1}(\tilde{z})$ is thus obtained, to be subsequently substituted in the *rhs* in the next iteration, and so forth. The anticipated convergence of the algorithm, viz. $\sigma_{n+1}(\tilde{z}) \simeq \sigma_n(\tilde{z})$ (for $n \geq n_{max}$, i.e., after some order of iteration) was effectively ensured, up to a accuracy of $\sim 10^{-4}$ or better, after as few as $n_{max} = 10$, roughly, iterations (yet a few more iterations were carried out to ascertain convergence was achieved). The (quasilinear) assumption $\sigma_0(z) = 0$ was used as an initial condition, yielding at the first iteration step the expected ballistic result $\sigma_1 \sim z^2$ [i.e., the exact analytical result which may be obtained by evaluating the integrals in $rhs(24)$], and then a series of functions σ_n for $n = 2, 3, ...$, which behaved as $\sim a_n \tilde{z}^{b_n}$ for $\tilde{z}$ values above, approximately, 50. The coefficients a_n and the characteristic exponents b_n were determined by least-square fitting in the $\log(\sigma)$ vs. $\log(\tilde{z})$ plane. For the parameter values mentioned above, the iteration converged to a function $\sigma \approx 0.232\,\tilde{z}^{1.328}$ (found by linear fitting of a sample of values), in agreement with the prediction (33), which yields: $\sigma \approx 0.225\,\tilde{z}^{1.333}$, for our choice of parameter values. Note that the predicted value of the characteristic exponent, namely b=4/3, was thus confirmed up to a small relative error of 0.4%. For comparison, we have repeated the same procedure for pure slab turbulence, i.e. by keeping only the first contribution in the *rhs* of (24) [setting $(\delta B_{2D}/B_0)^2 = 0$, $(\delta B_{slab}/B_0)^2 = 1$]. The expected diffusive asymptotic behavior ($\sigma \sim \tilde{z}$) was thus obtained, as $\sigma \approx 0.737\,\tilde{z}^{1.0017}$, and was actually found to agree quite well with the analytical asymptotic result $\sigma \approx 0.746\,\tilde{z}$, which can be straightforward obtained in this case [28].

The numerical results for *Set 1* (weakly-slab composite turbulence) are depicted in Figs. 2 and 3. The (dimensionless) running "diffusion coefficient" $d_{xx}(\tilde{z}) \equiv \sigma/(2\tilde{z}) = \left\langle (\Delta x(z))^2 \right\rangle/(2\, l_{slab}\, z)$ is depicted in Fig. 2, as a function of

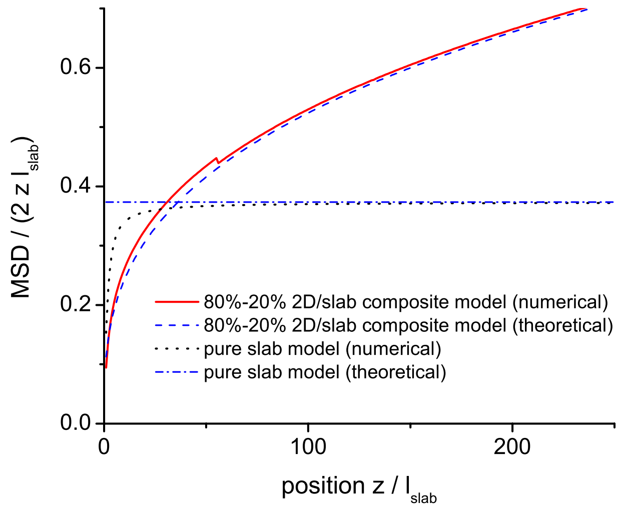


FIG. 2: (color online) The "running diffusion coefficient" $\left\langle (\Delta x(z))^2 \right\rangle / (2 l_{slab} z)$ is shown, as a function of the position z/l_{slab}. The numerical result (solid curve) is compared to the analytical asymptotic result (33) (dashed curve), for 20 % slab / 80 % 2D composite geometry. The corresponding slab-model results (numerical, dotted curve, and theoretical, dash-dotted line) are provided, for comparison. The analytical and numerical results are obviously in good agreement.

the reduced position variable $\tilde{z} = z/l_{slab}$. The composite model appears to attain a clearly superdiffusive regime (viz. $d_{xx} \neq$ cst., for large $\tilde{z}$), which is close to the anticipated (approximate) theoretical result, given by (33). The pure-slab model leads to the expected diffusive result (viz. $d_{xx} \approx$ cst., for large $\tilde{z}$), which is also provided for comparison. The superdiffusive character of FL random walk is clearly confirmed by a log-log plot of the FL-MSD $\left\langle (\Delta x(z))^2 \right\rangle$ vs. position $\tilde{z}$ (see Fig. 3). In particular, a linear dependence of the form $\ln\sigma = -1.459 + 1.328 \ln\tilde{z}$ is obtained by curve fitting of the rectilinear data on Fig. 3 (upper dashed curve), in (approximate) agreement with the theoretical result (33), as stated in the previous paragraph.

The numerical results for *Set 2* (dominantly-2D composite turbulence) are depicted in Figs. 4 to 6. In Fig. 4, it is observed that the value of the diffusion coefficient remains in the vicinity of the (diffusive) slab result for a while (cf. Fig. 4b), and then for large z approaches the superdiffusive behavior predicted analytically (under practically the same slope) for very large z (expected to be attained in the infinite limit). Three different regimes are witnessed in Fig. 5, namely: a ballistic initial regime (slope ≈ 2; cf. Fig. 6a)), a diffusive intermediate regime (slope ≈ 1; cf. Fig. 6b)), and a superdiffusive asymptotic regime (slope $\approx 4/3$; cf. Fig. 6c)). Note that the second (quasi-diffusive) regime was absent in *Set 1* (cf. Figs. 2-3). We conclude that the predictions put forward in Section IV F are indeed confirmed by the numerical analysis.

VI. ISOTROPIC TURBULENCE MODEL

A standard hypothesis in the analysis of magnetized plasmas consists in considering isotropic turbulence. This Section is devoted to an analytical investigation of FLRW in isotropic turbulent plasmas, in view of a critical comparison to the composite turbulence model adopted above.

We shall adopt a correlation tensor element of the form

$$P_{xx}(\mathbf{k}) = \frac{G(k)}{k^2} \left(1 - \frac{k_x^2}{k^2} \right) , \tag{46}$$

where the (spherically symmetric) magnetic turbulence spectrum is given by

$$G(k) = \frac{1}{2\pi} \, l_0 \, C(\nu) \, \delta B^2 \, (1 + k^2 l_0^2)^{-\nu} \, , \tag{47}$$

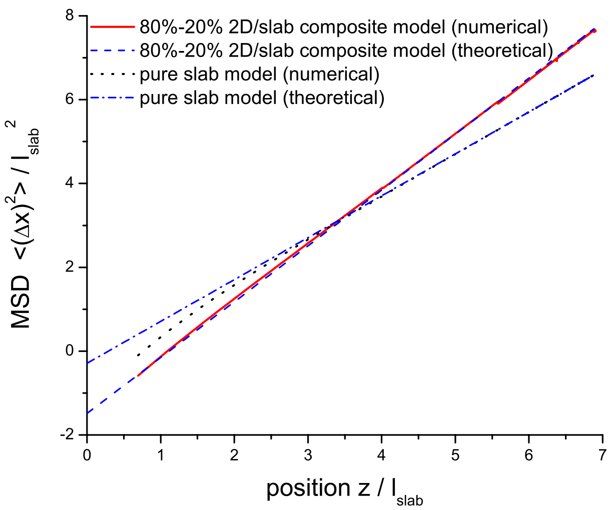


FIG. 3: (color online) The field-line MSD $\left\langle (\Delta x(z))^2 \right\rangle$ is depicted, as a function of the position z/l_{slab}. The numerical result (solid curve) is compared to the analytical result (33) (dashed curve), for 20 % slab / 80 % 2D composite geometry. The corresponding slab-model results (numerical, dotted curve, and theoretical, dash-dotted line) are provided, for comparison. The analytical and numerical results are in excellent agreement.

defining the bendover length scale l_0, the strength of the turbulent field δB, and the inertial-range spectral index 2ν. The normalization constant $C(\nu)$ was defined above.

Employing spherical Fourier space coordinates $\{k, \Psi, \phi\}$, viz. $k_x = k\cos\Psi\sin\phi$, $k_y = k\sin\Psi\sin\phi$ and $k_z = k\cos\phi$ (or, $k_\parallel = k_z = \eta k$ and $k_\perp = \sqrt{k_x^2 + k_y^2} = k\sqrt{1-\eta^2}$, where $\eta = \cos\phi$), it is straightforward to find that (11) takes the form

$$\begin{aligned} \frac{d^2}{dz^2}\left\langle (\Delta x(z))^2 \right\rangle &= l_0 C(\nu)\frac{\delta B^2}{B_0^2} \\ &\times \int_{-1}^{1} d\eta \int_0^\infty dk\, (1 + l_0^2 k^2)^{-\nu}(1+\eta^2)\, \cos(k\eta z) e^{-\frac{1}{2}\left\langle (\Delta x(z))^2 \right\rangle k^2(1-\eta^2)}\,, \end{aligned} \tag{48}$$

or, in dimensionless form,

$$\frac{d^2\sigma(\tilde{z})}{d\tilde{z}^2} = C(\nu)\frac{\delta B^2}{B_0^2}\int_0^\infty dx\,(1+x^2)^{-\nu}\int_{-1}^{1} d\eta\,(1+\eta^2)\,\cos(x\eta\tilde{z})e^{-\frac{1}{2}\sigma x^2(1-\eta^2)}\,, \tag{49}$$

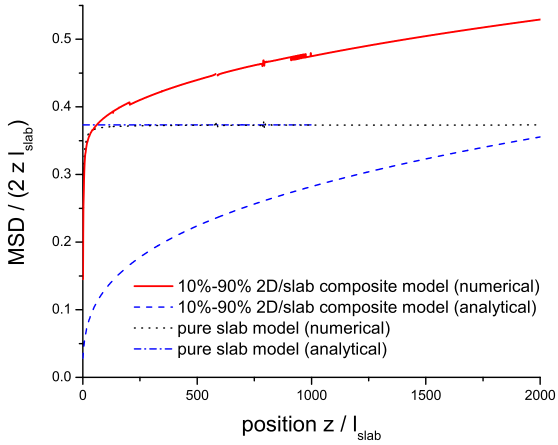
where $\sigma = \left\langle (\Delta x(z))^2 \right\rangle / l_0^2$, $\tilde{z} = z/l_0$ and $x = kl_0$. Note the appearance of the FL-MSD σ in both sides; cf. (25) above. In the following, we shall solve the ODE (49) in various limits of interest.

A. The quasilinear-theoretical (QLT) limit

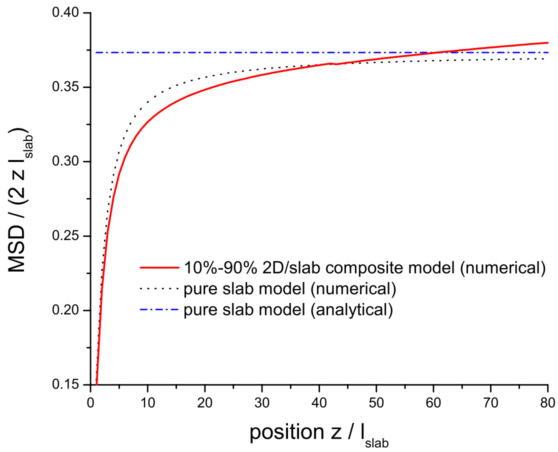
Substituting with $\sigma = 0$ in $rhs(49)$ [viz. setting $e^{-\frac{1}{2}\sigma x^2(1-\eta^2)} = 1$], the angle integral is evaluated as

$$\int_{-1}^{1} d\eta\,(1+\eta^2)\,\cos(x\eta\tilde{z}) = \frac{4}{x^3\tilde{z}^3}[x\tilde{z}\cos(x\tilde{z}) + (x^2\tilde{z}^2 - 1)\sin(x\tilde{z})]\,. \tag{50}$$

One is now interested in carrying out the $x-$ integration in (49), and, finally, solving for the MSD σ, in view of tracing its dependence on the position variable $\tilde{z}$.



(a)



(b)

FIG. 4: (color online) (a) The "running diffusion coefficient" $\left\langle (\Delta x(z))^2 \right\rangle / (2 l_{slab} z)$ is shown, as a function of the position z/l_{slab}. The numerical result (solid curve) is compared to the analytical asymptotic result (33) (dashed curve), for 90 % slab / 10 % 2D composite geometry. The corresponding slab-model results (numerical, dotted curve, and theoretical, dash-dotted line) are provided, for comparison. The analytical (asymptotic) and numerical results bear the same slope (i.e., the same exponent) in the large z region, and tend to coincide for infinite z. (b) The same plot, focusing on lower values of the position variable z.

Alternatively, the $x-$ integration in (49) may be first evaluated (still in the QLT limit):

$$\int_0^\infty dx\,(1+x^2)^{-\nu}\,\cos(x\eta\tilde{z}) = \frac{\sqrt{\pi}}{\Gamma(\nu)}\left(\frac{|\eta|\tilde{z}}{2}\right)^{\nu-1/2}\mathcal{K}_{\nu-1/2}(|\eta|\tilde{z})\,, \tag{51}$$

where $\mathcal{K}_n(x)$ denotes a modified Bessel function of the 1st kind [26].

In the following, we shall evaluate the solution of (49) analytically and numerically.

We shall here set $\nu = 1$ in (49), for simplicity (the shape of real observed spectra is indeed qualitatively reproduced this way). Eq. (49) thus becomes:

$$\frac{d^2\sigma(\tilde{z})}{d\tilde{z}^2} = C(1)\frac{\delta B^2}{B_0^2}\left[\pi\left(\frac{1}{\tilde{z}}+\frac{2}{\tilde{z}^3}\right) - 2\pi e^{-\tilde{z}}\left(\frac{1}{\tilde{z}}+\frac{1}{\tilde{z}^2}+\frac{1}{\tilde{z}^3}\right)\right], \tag{52}$$

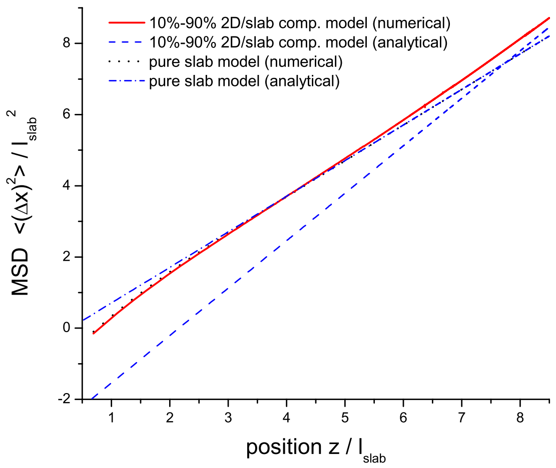


FIG. 5: (color online) The field-line MSD $\langle(\Delta x(z))^2\rangle$ is depicted, as a function of the position z/l_{slab}, in a logarithmic plot. The numerical result (dashed curve) is compared to the analytical result (solid curve), for 90 % slab / 10 % 2D composite geometry. The corresponding slab-model results (numerical, dotted curve, and theoretical, dash-dotted line) are provided, for comparison. The analytical (asymptotic) and numerical results tend to coincide in the large z region.

which may be integrated twice, assuming $\sigma(0) = \sigma'(0) = 0$, to give

$$\sigma(\tilde{z}) = \frac{1}{2}\frac{\delta B^2}{B_0^2}\left[\tilde{z}\ln\tilde{z} + \frac{1}{\tilde{z}}(1-e^{-\tilde{z}}) - e^{-\tilde{z}} - \tilde{z}\mathrm{Ei}(-\tilde{z}) + \left(\gamma - \frac{1}{2}\right)\tilde{z}\right]. \tag{53}$$

We have here employed the Euler constant $\gamma \approx 0.5772$ and the exponential integral function $\mathrm{Ei}(z) = -\int_{-z}^{\infty} dt\, e^{-t}/t$ (a *very* rapidly decreasing, in absolute value, function of $\tilde{z}$). We have used $C(1) = 1/(2\pi)$.

For practical purposes, we retain, for large $\tilde{z} \gg 1$

$$y(\tilde{z}) \approx \frac{1}{2}\frac{\delta B^2}{B_0^2}(\tilde{z}\ln\tilde{z} + c\tilde{z}) = \frac{1}{2}\frac{\delta B^2}{B_0^2}\tilde{z}\ln\left(\frac{\tilde{z}}{\tilde{z}_1}\right), \tag{54}$$

where the constant c here equals $c = \gamma - 1/2 \approx 0.0772 = -\ln\tilde{z}_1$ (thus $\tilde{z}_1 \approx 0.9257$).

Expression (54) is our strong result to be retained at this stage; it determines the behavior of the FL-MSD for large (yet finite) values of the position variable z. Quite interestingly, in the *infinite* z limit, a classical diffusive exponent is thus obtained. Precisely, anticipating $\sigma(\tilde{z}) = a\tilde{z}^b$ (for $\tilde{z} \to \infty$), one may define the exponent $b = \lim_{\tilde{z}\to\infty}(\ln\sigma(\tilde{z})/\ln\tilde{z})$ and then determine its value by making use of De l'Hôpital's rule, as: $\lim_{x\to\infty}[\ln(c_1 x\ln x + c_2 x)/\ln x] = 1$ (for arbitrary finite real constants c_1 and c_2). One thus finds $b = 1$, as can also be shown numerically. On the other hand, in the small z limit, a simple expansion in (53) shows that the MSD behaves as $\sigma \sim 2\tilde{z}^2/3$; a ballistic behavior is therefore obtained, for $\tilde{z} \ll 1$, as

$$\sigma(\tilde{z}) \approx \frac{1}{3}\frac{\delta B^2}{B_0^2}\tilde{z}^2.$$

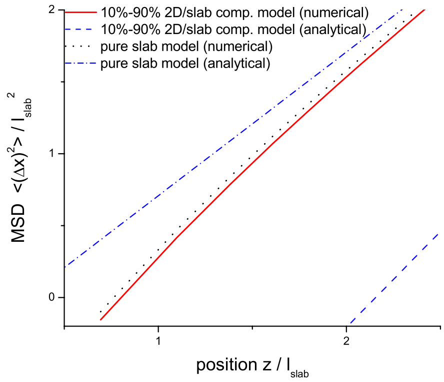
Note that the derivative of the MSD σ is related to the running diffusion coefficient; cf. (12). We thus find:

$$\sigma'(\tilde{z}) = \pi C(1)\frac{\delta B^2}{B_0^2}\left[\ln\tilde{z} + \frac{e^{-\tilde{z}}}{\tilde{z}} - \frac{1-e^{-\tilde{z}}}{\tilde{z}^2} - \mathrm{Ei}(-\tilde{z}) + \gamma + \frac{1}{2}\right], \tag{55}$$

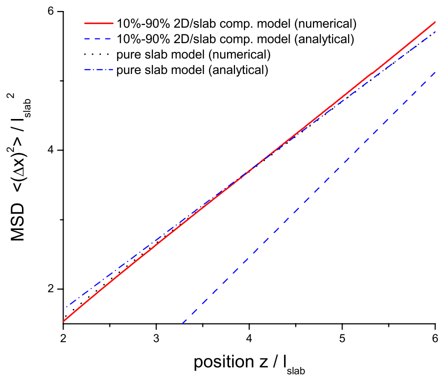
i.e., essentially (recovering dimensions), for large z [cf. (12)],

$$d_{xx}(\tilde{z}) \approx \frac{l_0}{4}\frac{\delta B^2}{B_0^2}\ln\left(\frac{z}{l_0}\right), \tag{56}$$

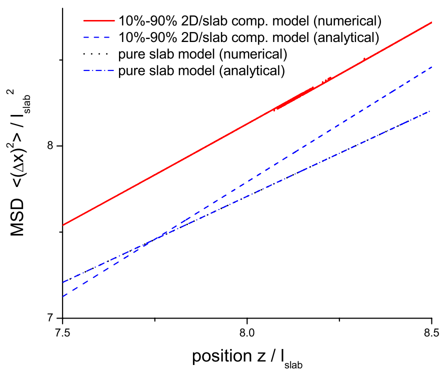
for large $\tilde{z} \gg 1$.



(a)



(b)



(c)

FIG. 6: (color online) The same plot as in Fig. 5 is provided, focusing on: (a) the low z ballistic regime (see that the composite and slab model results coincide in this region); (b) the intermediate z quasi-diffusive regime (the composite model result practically coincides with the diffusive asymptotic slab limit); (c) the large z superdiffusive regime (the composite model and the superdiffusive analytical prediction converge asymptotically).

B. Quasilinear field line diffusion in the general isotropic case: an exact treatment

In this Section, we will show that, for field line diffusion coefficient in isotropic turbulence, quasilinear theory indeed provides a good description. We start with the general Eq. (9), where the Fourier magnetic correlation R_{xx} is to be substituted by

$$R_{xx} = \int \mathrm{d}^3\mathbf{k}\, P_{xx} \cos\bigl(k_\parallel z\bigr) \ . \tag{57}$$

Adopting expressions (46) and (47), and using spherical coordinates [see the definitions preceding Eq. (48) above], we obtain

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle_{\mathrm{FL}} =& \frac{\delta B^2}{B_0^2}\, C(\nu) l_0 \int_0^\infty \mathrm{d}k\ \bigl(1+l_0^2k^2\bigr)^{-\nu} \int_{-1}^1 \mathrm{d}\eta\ \bigl(1+\eta^2\bigr) \\ & \times \int_0^z \mathrm{d}z'\ (z-z') \cos(k\eta z')\ .\end{aligned} \tag{58}$$

See that this relation is tantamount (upon a double differentiation) to (49) above. However, instead of evaluating the latter, we shall now first calculate the integral over z to obtain

$$\left\langle (\Delta x)^2 \right\rangle_{\mathrm{FL}} = \frac{\delta B^2}{B_0^2}\, C(\nu) l_0 \int_0^\infty \mathrm{d}k\ \bigl(1+l_0^2k^2\bigr)^{-\nu} \int_{-1}^1 \mathrm{d}\eta\ \bigl(1+\eta^2\bigr)\, \frac{1-\cos(k\eta z)}{(k\eta)^2} \tag{59}$$

$$= \frac{\delta B^2}{B_0^2}\, C(\nu) l_0 \int_0^\infty \mathrm{d}k\ \bigl(1+l_0^2k^2\bigr)^{-\nu}\, \frac{kz\cos kz - \sin kz + (kz)^2\, \mathrm{Si}(kz)}{k^3 z}, \tag{60}$$

where

$$\mathrm{Si}(x) = \int_0^x \mathrm{d}t\ \frac{\sin t}{t} \tag{61}$$

denotes the sine integral function.

Recalling the scaled variable definitions $x = kl_0$ and $\tilde z = z/l_0$, one obtains

$$\sigma(\tilde z) \equiv \left\langle \left(\frac{\Delta x}{l_0}\right)^2 \right\rangle_{\mathrm{FL}} = \frac{\delta B^2}{B_0^2}\, C(\nu) \int_0^\infty \mathrm{d}x\ \bigl(1+x^2\bigr)^{-\nu}\, \mathcal{J}(x,\tilde z), \tag{62}$$

where

$$\mathcal{J}(x,\tilde z) = \frac{x\tilde z\cos x\tilde z - \sin x\tilde z + (x\tilde z)^2\, \mathrm{Si}(x\tilde z)}{x^3\tilde z}. \tag{63}$$

A Taylor expansion of the function $\mathcal{J}(x,\tilde z)$ yields

$$\mathcal{J}(x,\tilde z) \simeq \begin{cases} \dfrac{2\tilde z^2}{3}, & x \ll \dfrac{3\pi}{4\tilde z} \\ \dfrac{\pi\tilde z}{2x}, & x \gg \dfrac{3\pi}{4\tilde z} \end{cases} \tag{64}$$

By integrating separately over the three intervals $[0, 3\pi/4\tilde z]$, $[3\pi/(4\tilde z), 1]$ and $[1,\infty]$, we obtain

$$\sigma(\tilde z) \simeq \frac{\delta B^2}{B_0^2}\, C(\nu) \frac{\pi}{2} \left[\tilde z \ln\left(\frac{4\tilde z}{3\pi}\right) + \left(1 + \frac{1}{2\nu}\right)\, \tilde z\right], \tag{65}$$

where we have omitted terms vanishing for large $\tilde z$. See that the logarithmic behavior obtained above is here also qualitatively recovered, as

$$\sigma(\tilde z) \simeq \frac{\pi}{2} \frac{\delta B^2}{B_0^2}\, C(\nu) \left(\tilde z \ln \tilde z + c\tilde z\right), \tag{66}$$

where the real constant c here equals $c = 1 + 1/2\nu + \ln[4/(3\pi)] \approx 0.143 + 1/2\nu$.

We thus obtain for the second derivative

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tilde z^2} \simeq \frac{\delta B^2}{B_0^2}\, C(\nu)\, \frac{3\pi^2}{8\tilde z} \ \sim \frac{1}{\tilde z}\,, \tag{67}$$

in qualitative agreement with our findings in the previous paragraphs.

The validity of the results in this paragraph will be tested in Section VII below.

C. Approximate treatment in the general case

Let us consider the ODE (49) once again. In search for analytical tractability, we may approximate the spectrum function $(1+x^2)^{-\nu}$ therein by replacing as

$$\frac{C(\nu)}{2\pi}(1+x^2)^{-\nu} \to \frac{1}{4\pi^{3/2}} e^{-x^2}\,, \tag{68}$$

where normalization to unity is preserved.

The x-integration in (49) may now be carried out, yielding

$$\int_0^\infty dx\, e^{-x^2}\; \cos(x\eta\tilde{z})\; e^{-\frac{1}{2}\sigma x^2(1-\eta^2)} = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2+(1-\eta^2)\sigma}}\; e^{-\frac{\eta^2\tilde{z}^2}{2[2+(1-\eta^2)\sigma]}}\,. \tag{69}$$

Eq. (49) now becomes

$$\frac{d^2\sigma(\tilde{z})}{d\tilde{z}^2} \approx \frac{1}{\sqrt{2}}\frac{\delta B^2}{B_0^2}\int_0^1 d\eta\,(1+\eta^2)\,\frac{1}{\sqrt{2+(1-\eta^2)\sigma}}\; e^{-\frac{\eta^2\tilde{z}^2}{2[2+(1-\eta^2)\sigma]}}\,. \tag{70}$$

Neglecting the contribution from η values far from zero (thanks to the exponential decrease of the integrand), the latter expression can be approximated as

$$\begin{aligned}\frac{d^2\sigma(\tilde{z})}{d\tilde{z}^2} &\approx \frac{1}{\sqrt{2}}\frac{\delta B^2}{B_0^2}\int_0^1 d\eta\,\frac{1}{\sqrt{2+\sigma}}\; e^{-\frac{\eta^2\tilde{z}^2}{2(2+\sigma)}}\\ &\approx \frac{1}{\sqrt{2}}\frac{\delta B^2}{B_0^2}\frac{1}{\sqrt{\sigma}}\int_0^1 d\eta\, e^{-\frac{\eta^2\tilde{z}^2}{2(2+\sigma)}}\\ &\approx \frac{1}{\sqrt{2}}\frac{\delta B^2}{B_0^2}\frac{1}{\sqrt{\sigma}}\sqrt{\frac{\pi}{2}}\frac{\sqrt{\sigma}}{\tilde{z}}\,\mathrm{erf}\left(\frac{\tilde{z}^2}{2\sigma}\right)\\ &\approx \frac{\sqrt{\pi}}{2}\frac{\delta B^2}{B_0^2}\frac{1}{\tilde{z}}\,,\end{aligned} \tag{71}$$

where $\tilde{z}, \sigma \gg 1$ and the large-argument behavior of the error function $\mathrm{erf}(x)$ were taken into account in the second and last step(s) [**?**]. Integrating (71) twice, we thus obtain rigorously

$$\sigma(\tilde{z}) = \frac{\sqrt{\pi}}{2}\frac{\delta B^2}{B_0^2}\,(\tilde{z}\ln\tilde{z} + c_1\tilde{z} + c_2) \tag{72}$$

(c_1 and c_2 are real integration constants). We note that the qualitative behavior of the MSD $\sigma\ \tilde{z}\ln\tilde{z}$ (for large $\tilde{z}$) found above is thus reproduced.

VII. NUMERICAL ANALYSIS IN THE ISOTROPIC MODEL

We have tested the results in §VI B numerically. First, in order to establish the validity of the expansion (64), which led to (65), we have computed the integral in Eq. (62), both exactly [i.e., by directly integrating Eq. (62) numerically] and approximately [i.e., evaluating (65), which relied on (64)]. The result of the numerical evaluation is shown in Figure 7. We see that the two curves practically coincide for large $\tilde{z} = z/l_0$. We conclude that the expansion (64) indeed provides a good analytical approximation of the integrand.

We have investigated the appropriateness of our result (65) as the final result for the MSD in the isotropic turbulence model, and have naturally anticipated that it might provide a solution of the original ODE (11). In order to test the validity of (65) [also related to (67)] for this purpose, we have inserted the MSD σ from Eq. (65) into the original nonlinear ODE (11), and have then evaluated the *rhs* and *lhs* independently. As illustrated in Fig. 8, Eq. (65) provides a very good approximation, for high $\tilde{z}$. The qualitative result $\sigma''(\tilde{z})\ 1/\tilde{z}$ is thus confirmed, validating the preceding analysis.

We conclude that, for field line random walk in isotropic turbulence, quasilinear theory provides a satisfactory description, at least in the large scale regime (i.e., for $z \gg l_0$). The result provided by expression (66) is to be retained, as the correct asymptotic behavior of the FL-MSD for large z.

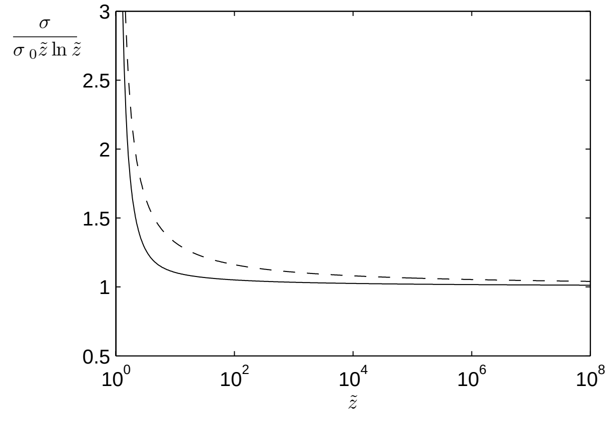


FIG. 7: The quasilinear mean square displacement σ, divided by $\sigma_0 \tilde{z} \ln \tilde{z}$ [with $\sigma_0 = \frac{\pi}{2}\, C(\nu) \frac{\delta B^2}{B_0^2}$; cf. Eq. (66)], is depicted as a function of the position variable $\tilde{z} = z/l_0$. A comparison is provided of the exact numerically evaluated values from Eq. (62) (solid line) to the analytical approximation from Eq. (65) (dashed line). For large $\tilde{z}$, the two curves agree asymptotically.

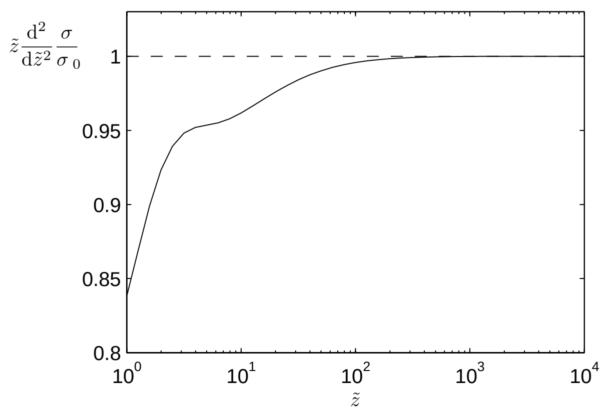


FIG. 8: A test of the quasilinear theory for the normalized field line mean square displacement is attempted by computing the 2nd derivative of the FL-MSD σ. We depict the right-hand side of Eq. (11) (solid line), in comparison to the left-hand side of Eq. (11) (dashed line), both calculated with σ from Eq. (65). The numerical values in both curves have been multiplied by $\tilde{z}$. Values in the vertical axis have been scaled by $\sigma_0 = \frac{3\pi^2}{8}\, C(\nu)\, \frac{\delta B^2}{B_0^2}$; cf. Eq. (67). For large $\tilde{z} \gg 1$ (i.e., $z \gg l_0$), the two curves coincide, asymptotically.

VIII. CONCLUSIONS

We have investigated the random displacement of magnetic field lines in the presence of magnetic turbulence in plasmas, from first principles. A two-component (slab/two-dimensional composite) turbulence model was employed, and an analytical evaluation of the asymptotic behavior of the field-line mean square displacement was carried out. It was found that quasilinear theory was not appropriate in this model, although it can be recovered by "switching" off the 2D component in the model (purely slab turbulence limit). We have shown that the magnetic field lines behave superdiffusively for large values of the position variable z and, in fact, the FL-MSD varies as $\sigma_{FL} \sim z^{4/3}$. Furthermore, a diffusive regime may occur, for intermediate (finite) values of z, while a ballistic behavior is witnessed for low z (here coinciding with the prediction of the slab model). These predictions were indeed confirmed numerically.

In view of a critical comparison with the hybrid (composite slab/2D) model, an isotropic turbulence model was also considered. Quasi-linear theory was proven to provide an adequate description for the latter (isotropic turbulence) model. The asymptotic FL-MSD behavior was shown to be weakly superdiffusive, in fact behaving as $\sigma \sim z \ln z$ for large values of the position variable z. Qualitatively speaking, the logarithmic behavior of the MSD stems from the fact that, in the energy range, the spectrum is essentially constant.

We point out that this nonlinear theory for field-line random walk was shown in ref. [20] to be agreement with results obtained from test-particle numerical simulations, when combined with a generalized compound diffusion model [19]. Furthermore, it can be shown to agree with cosmic observations (refer to the discussion in Ref. [20]). However, it must be noted that these results seem to contradict the diffusive behavior suggested in Ref. [7], which was nevertheless found only under questionable assumptions; see the discussion in Ref. [28].

Our results are relevant in space and astrophysical environments, where the stochastic magnetic field line topology plays an important role in transport phenomena related to cosmic ray propagation and may, in fact, determine the characteristics of the observed cosmic particle and energy flow observed by relevant experiments.

Acknowledgments

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[31] See 3.383.5 in Ref. [27], setting $p = y/2\xi^2$, $a = 1$ and $q = 1/2$ therein; also, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
[32] Anticipating a leading contribution in the asymptotic form $\sigma \sim \tilde{z}^b$, we have here imposed that $\sigma = 0$ for $\tilde{z} = 0$.
[33] See that (35) is identically satisfied for $\alpha = 0$, where $Y = Z = 0$, while it is not valid for $\beta = 0$, unlike (34); one thus has to revert back to (34), should one wish to consider the pure-2D ($\beta = 0$) turbulence limit as a special case.
[34] Note, for rigor, that this expression is valid for $Z > 2$, in order for the positivity of the quantity under the square root to be ensured.
[35] It is added, for rigor, that we find $Y_{2/3} \approx (Y_1 - 3/2)e^{i(1\mp 1/3)\pi}$ for $Z \gg 1$; therefore, upon evaluating $\sigma \sim |Y_{2/3}|^2 \approx Y_1^2$, we obtain exactly the same superdiffusive asymptotic result.

APPENDIX A: THE INTEGRALS IN (25)

1. The first integral in (25)

We shall evaluate the first integral in (25), say I_1. Making use of tables of integrals [27], one finds

$$I_1(\tilde{z}) = \int_0^\infty dx \, \frac{1}{(1+x^2)^\nu} \frac{\cos(x\tilde{z})}{x^2} = \sqrt{\pi}\left(\frac{\tilde{z}}{2}\right)^{\nu-1/2} \frac{K_{\nu-1/2}(\tilde{z})}{\Gamma(\nu)} \,, \tag{A1}$$

where $K_n(x)$ denotes the modified Bessel function of order n. For small values of the argument $\tilde{z}$ (i.e. $\tilde{z} \ll 1$), this behaves as

$$K_{\nu-1/2}(\tilde{z}) \approx \left(\frac{2}{\tilde{z}}\right)^{\nu-1/2} \frac{1}{2}\Gamma\left(\nu - \frac{1}{2}\right)$$

(see 9.6.9, p. 375 in Ref. [26]), so that the above integral, say $I_1(\tilde{z})$, behaves as

$$I_1(\tilde{z}) \approx \frac{\sqrt{\pi}}{2} \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} \equiv \frac{1}{4C(\nu)} \tag{A2}$$

(cf. the definition of $C(\nu)$ in the text). For high values of the argument $\tilde{z}$ (i.e. $\tilde{z} \gg 1$), the modified Bessel function $K_n(\tilde{z})$ behaves as

$$K_{\nu-1/2}(\tilde{z}) \approx \sqrt{\frac{\pi}{2\tilde{z}}}\, e^{-\tilde{z}} + \mathcal{O}(\tilde{z}^{-3/2})$$

(see 9.7.2, p. 378 in Ref. [26]). It is straightforward to see that this implies a negligible value of the slab contribution (first integral) in (25) [as compared to the 2D contribution, i.e. the second integral], for large $\tilde{z}$ values.

2. The second integral in Eqs. (25)

The second integral in (25)

$$I_2 = \int_0^\infty dx \; (1+x^2)^{-\nu} \, e^{-yx^2/(2\xi^2)}$$

can be evaluated analytically, by making use of the Kummer (confluent) hypergeometric function of the first kind $M(a,b,z) = {}_1F_1(a,b,z)$ (where ${}_1F_1$ denotes the generalized – or Barnes extended – hypergeometric function; see e.g. in [30] and Refs. therein). The exact result reads:

$$I_2(y) = \frac{\pi}{2}\left[-\frac{\sqrt{\pi}}{\Gamma(\nu)} \frac{{}_1F_1(\frac{1}{2}, \frac{3}{2}-\nu, \frac{y}{2\xi^2})}{\Gamma(\frac{3}{2}-\nu)} + \frac{{}_1F_1(\nu, \nu+\frac{1}{2}, \frac{y}{2\xi^2})}{\Gamma(\nu+\frac{1}{2})} \left(\frac{y}{2\xi^2}\right)^{\nu-\frac{1}{2}} \right] \sec(\nu\pi) \,. \tag{A3}$$

For a given value of ν, this expression provides the right-hand side (rhs) of the ODE (23), which is then to be integrated for a solution for $y(z)$ $(= \sigma^2)$ to be obtained. In the following, we shall consider two asymptotic limits, namely the small y limit, and the large y (or vanishing l_{slab}) limit.

Asymptotic form of $\hat{I}_2(y)$ *for small* $y \ll 1$. Using the asymptotic expansion:

$${}_1F_1(a,b,x) \approx 1 + \frac{ax}{b} + \frac{a(1+a)}{2b(1+b)}x^2 + \mathcal{O}(x^3)$$

[26] (see 13.1.2, p. 504 therein), in combination with Γ function recursive relations, viz. $n\Gamma(n) = \Gamma(n+1)$, we obtain

$$\hat{I}_2(y) \approx -\frac{\pi^{3/2}}{2} \frac{\sec(\nu\pi)}{\Gamma(\nu)\Gamma(\frac{3}{2}-\nu)} + \frac{\pi^{3/2}}{2} \frac{\sec(\nu\pi)}{8\Gamma(\nu)\Gamma(\frac{5}{2}-\nu)} \frac{y}{\xi^2} + \frac{\pi}{2} \frac{\sec(\nu\pi)}{\Gamma(\nu+\frac{1}{2})} \left(\frac{y}{2\xi^2}\right)^{\nu-1/2} \,, \tag{A4}$$

where the neglected terms are of the order $\mathcal{O}(\min\{y^2, y^{\nu+1/2}\})$ or higher. This approximate expression may therefore be inserted in (23), and the resulting ODE may then be integrated twice, by using appropriate boundary conditions (here, we assume that $y(0) = y'(0) = 0$). Keeping only the first (constant) term in the rhs (A4), one thus obtains

$$\sigma_{2D}^2(z) \approx \frac{1}{2}\left(\frac{\delta B_{2D}}{B_0}\right)^2 z^2 \,. \tag{A5}$$

Asymptotic form of the integral $\hat{I}_2$ for large $y \gg 1$. Upon the variable transformation $x'^2 := y'$, the integral $\hat{I}_2(y)$ in (23) takes the form

$$\begin{aligned} I_2(y) &= \frac{1}{2}\int_0^\infty dy' \, \frac{(1+y')^{-\nu}}{\sqrt{y'}} \, e^{-y'y/2\xi^2} \\ &= \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}, \frac{3}{2}-\nu, \frac{y}{2\xi^2}\right) \\ &\approx \frac{1}{2}\sqrt{\pi}\left(\frac{y}{2\xi^2}\right)^{-1/2} + \mathcal{O}[(y)^{-3/2}] \end{aligned} \tag{A6}$$

(for large $y \gg 1$) [31], where we have considered the asymptotic behavior of the characteristic function $\psi(a, b, z)$ [27].

We conclude that

$$I_2(y) \approx \sqrt{\frac{\pi}{2}}\,\frac{\xi}{\sqrt{y}} + \mathcal{O}(y^{-3/2}) \,. \tag{A7}$$

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A new theory for perpendicular transport of cosmic rays

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ABSTRACT

We present an improved nonlinear theory for the perpendicular transport of charged particles. This approach is based on an improved nonlinear treatment of field-line random walk in combination with a generalized compound diffusion model. The generalized compound diffusion model employed is more systematic and reliable, in comparison with previous theories. Furthermore, the theory shows remarkably good agreement with test-particle simulations and solar wind observations.

Key words. diffusion – scattering – turbulence – gamma rays: theory

1. Introduction

The long-standing problem of particle transport perpendicular to a magnetic background field in turbulent magnetized plasmas is revisited in this article. Although this problem has been discussed in several papers (e.g. Jokipii 1966; Bieber & Matthaeus 1997; Kóta & Jokipii 2000; Matthaeus et al. 2003; Shalchi et al. 2004; Webb et al. 2006; Shalchi 2006), a final solution has not been provided so far. The perpendicular transport of charged particles is a central problem in astrophysics, since the knowledge of the diffusion tensor for particle transport parallel and perpendicular to the prescribed external magnetic field is essential for describing solar energetic particles (Dröge 2000), the modulation of Galactic cosmic rays (Burger & Hattingh 1998), diffusive shock acceleration, and the lifetime of cosmic rays in the Galaxy (Jokipii & Parker 1969). Furthermore, there are measurements of perpendicular mean free paths in the heliosphere (Chenette et al. 1977; Palmer 1982; Burger et al. 2000) that still await a theoretical explanation.

An early treatment of particle transport has relied on a quasilinear description of cosmic ray propagation (Jokipii 1966). In the quasilinear theory (QLT) it is assumed that particles follow the magnetic field-lines while they move unperturbed in the direction parallel to the background field. Thus, the corresponding result is often referred to as the field-line random walk limit (FLRW-limit). According to this result, which has originally been derived for the slab turbulence model, the perpendicular mean-square deviation (MSD) of the particle increases linearly with time, viz. $\left\langle (\Delta x)^2 \right\rangle = 2\kappa_{xx} t$. This linear time dependence is usually referred to as a classical Markovian diffusion process. Thirty-four years later, Kóta & Jokipii (2000) formulated a compound diffusion model that assumes that the particle moves along the magnetic field-lines while it is scattered diffusively in the parallel direction. Relying on the Taylor-Green-Kubo-formulation (e.g. Kubo 1957), in combination with the assumption of diffusive field-line wandering, Kóta & Jokipii (2000) have found a subdiffusive behavior of particle transport of the form $\left\langle (\Delta x)^2 \right\rangle \sim \sqrt{t}$. In the same years, particle propagation in magnetized plasmas was explored by making use of test-particle simulations (e.g. Giacalone & Jokipii 1999; Mace et al. 2000; Qin et al. 2002a,b), where it was clearly confirmed that $\left\langle (\Delta x)^2 \right\rangle \sim \sqrt{t}$, so long as a slab model is considered for the turbulence geometry (see Qin et al. 2002a). By using improved test-particle codes (see Qin et al. 2002b for instance), it has been demonstrated that for a non-slab model, diffusion is recovered (though only partially, as demonstrated in Sect. 3 of this paper). This recovery of diffusion cannot been explained by the method of Kóta & Jokipii (2000).

Various other theories have been proposed for perpendicular transport, mainly based on nonlinear extensions of QLT, such as the nonlinear closure approximation of Owens (1974) or model-based approaches such as the one proposed by Bieber & Matthaeus (BAM 1997). However, these theories provide a diffusive behavior of perpendicular transport for the slab model, in disagreement with simulations. One more promising theory, namely the nonlinear guiding-center theory (NLGC-theory), has been derived by Matthaeus et al. (2003). Although this theory shows agreement with some test-particle simulations in slab/2D geometries, the theory cannot reproduce subdiffusion for the slab model. An extended nonlinear guiding-center (ENLGC) theory was therefore formulated by Shalchi (2006), which agrees with simulations for slab and non-slab models. However, this theory is very close to the original NLGC-theory and uses nearly the same crude approximation: exponential form of the velocity correlation function, magnetic fields and particle velocities are uncorrelated.

In this paper we propose a more reliable theoretical approach that uses less ad-hoc assumptions and *ansätze* than the NLGC-theory. The layout of this article goes as follows. In Sect. 2, we argue that field-line wandering behaves superdiffusively for non-slab models. In Sect. 3, we employ a generalized compound diffusion model to deduce an analytic form for the perpendicular MSD of particles. By comparing with test-particle simulations (Sect. 4) and solar wind observations (Sect. 5) we show that our theory provides the correct result.

2. Nonlinear description of field-line wandering

The key input into our new formulation is the MSD of the magnetic field-lines $\langle (\Delta x(z))^2 \rangle_{\mathrm{FL}}$. In several previous papers (e.g.

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Jokipii 1966; Matthaeus et al. 1995; Ragot 2006; Ruffolo et al. 2006), field-line wandering (or random walk) has been described by applying linear and nonlinear formulations. In a recent paper (Shalchi & Kourakis 2007), an improved formulation for nonlinear field-line random walk in magnetostatic turbulence has been developed, thus criticizing the validity of those previous approaches. The approach of Shalchi & Kourakis (2007) is a direct generalization of the theory proposed by Matthaeus et al. (1995) and further applied by Ruffolo et al. (2006). For a diffusive behavior of field-lines this theory can be obtained by the theory of Shalchi & Kourakis (2007) as a special limit. However, our theory can also be applied in non-diffusive transport cases. A further advantage of this theory is its analytical tractability, which enriches and complements the existing numerical toolbox on field-line wandering (see e.g. Ragot 2006).

In view of modeling field-line random walk, the turbulence model has to be specified in terms of the magnetic correlation tensor $P_{ij}(\boldsymbol{k}) = \langle \delta B_i(\boldsymbol{k}) \delta B_j(\boldsymbol{k}) \rangle$. According to Bieber et al. (1994) the so-called slab/2D composite model is a realistic model for solar wind turbulence. This two-component model ignores the usually smaller parallel field turbulent variance (δB_z) and only includes excitations with wavevectors either purely parallel or purely perpendicular to the mean magnetic field $\boldsymbol{B}_0$, leading to the following form of the correlation tensors: $P_{xx}(\boldsymbol{k}) = P_{xx}^{\text{slab}}(\boldsymbol{k}) + P_{xx}^{\text{2D}}(\boldsymbol{k})$ with $P_{xx}^{\text{slab}}(\boldsymbol{k}) = g^{\text{slab}}(k_\parallel)\delta(k_\perp)/k_\perp$ and $P_{xx}^{\text{2D}}(\boldsymbol{k}) = g^{\text{2D}}(k_\perp)\delta(k_\parallel)k_y^2/k_\perp^3$. For the two wave spectra $g^{\text{slab}}(k_\parallel)$ and $g^{\text{2D}}(k_\perp)$, we employ a standard form that has also been proposed by Bieber et al. (1994)

$$\begin{aligned} g^{\text{slab}}(k_\parallel) &= \frac{C(\nu)}{2\pi} l_{\text{slab}} \delta B_{\text{slab}}^2 \left(1 + k_\parallel^2 l_{\text{slab}}^2\right)^{-\nu} \\ g^{\text{2D}}(k_\perp) &= \frac{2C(\nu)}{\pi} l_{\text{2D}} \delta B_{\text{2D}}^2 \left(1 + k_\perp^2 l_{\text{2D}}^2\right)^{-\nu}. \end{aligned} \tag{1}$$

Here we have defined the normalization constant $C(\nu) = \Gamma(\nu)/(2\sqrt{\pi}\Gamma(\nu - 1/2))$, where Γ denotes the Euler Gamma function. We have defined the slab- and 2D bendover length scales l_{slab} and l_{2D}, the strength of the turbulent fields δB_{slab} and δB_{2D}, and the inertial-range spectral index 2ν.

It can easily be demonstrated (see e.g. Jokipii 1966; Matthaeus et al. 1995) that, for pure slab geometry, the field-lines behaves diffusively

$$\left\langle (\Delta x(z))^2 \right\rangle_{|z|\to\infty} \approx 2\kappa_{\text{FL}}\, |z|, \tag{2}$$

for large $|z|$. In several previous papers (e.g. Matthaeus et al. 1995) it has been explicitly assumed that the form of Eq. (2) also holds in two-component turbulence. More precisely, field-line wandering always behaves diffusively, at least as long as there is a finite slab contribution. However, by applying an improved formulation of field-line random walk (Shalchi & Kourakis 2007), it has been shown that

$$\begin{aligned} \left\langle (\Delta x(z))^2 \right\rangle_{|z|\to\infty} &= \left(9\sqrt{\frac{\pi}{2}} C(\nu) \right)^{2/3} \\ &\quad \times \left(\frac{\delta B_{\text{2D}}}{B_0} \right)^{4/3} l_{\text{2D}}^2 \left(\frac{|z|}{l_{\text{2D}}} \right)^{4/3}. \end{aligned} \tag{3}$$

The only assumptions that have been applied to derive this result are Corrsin's independence hypothesis (Corrsin 1959) and the assumption of a Gaussian distribution of field-lines (see Matthaeus et al. 1995). For two-component turbulence, Eq. (3) is the correct asymptotic limit that can be obtained in the limit $z \to \infty$. A quasilinear description of the field-line random walk is not possible for these length scales (see Shalchi & Kourakis 2007). For completeness, it should be noted that the different regimes where a quasilinear description is valid or not are discussed in Isichenko (1991a,b).

3. Generalized compound diffusion of charged particles

In the previous section, we discussed results regarding the field-line wandering in the slab/2D composite model as a function of z. However, charged particles experience parallel scattering while moving through the turbulence. Thus, the parameter z becomes a statistical variable in particle transport studies. If we assume that the particles (or, more precisely, their guiding-centers) follow the magnetic field-lines (guiding center approximation), we have

$$\left\langle (\Delta x(t))^2 \right\rangle_P = \int_{-\infty}^{+\infty} \mathrm{d}z \, \left\langle (\Delta x(z))^2 \right\rangle_{\text{FL}} f_P(z,t). \tag{4}$$

Here the index P denotes the perpendicular MSD of the charged particle, and $f_P(z,t)$ is the particle distribution in the direction parallel to the background field.

Equation (4) can also be obtained from the Chapman-Kolmogorov equation (see, e.g., Webb et al. 2006), which has the form

$$f_\perp(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}z \, f_{\text{FL}}(x,z) f_P(z,t) \tag{5}$$

with the particle distribution in the perpendicular direction $f_\perp(x,t)$ and the field-line distribution function $f_{\text{FL}}(x,z)$. By calculating the second moment of $f_\perp(x,t)$, Eq. (4) can be deduced from Eq. (5).

A further standard assumption in the cosmic ray transport theory is the assumption of a Gaussian particle distribution, see e.g. Matthaeus et al. (2003):

$$f_P(z,t) = \left(2\pi \left\langle (\Delta z(t))^2 \right\rangle_P \right)^{-1/2} \mathrm{e}^{-\frac{z^2}{2\left\langle (\Delta z(t))^2 \right\rangle_P}}. \tag{6}$$

It should be noted, however, that the assumption of a Gaussian distribution is a certain hypothesis that might be inaccurate for certain parameter regimes. A non-Gaussian distribution might be more appropriate mainly for a strong non-diffusivity of parallel particle propagation. However, as demonstrated in Sect. 4, the real particle motion is very close to the diffusive behavior, thus Eq. (6) should be a good approximation. Forthcoming work will be devoted to exploring the influence of non-Gaussian statistics.

By using Eq. (3) for the field-line MSD in combination with Eq. (6) for the particle distribution, we can evaluate Eq. (4) to find

$$\left\langle (\Delta x)^2 \right\rangle_P = \alpha(\nu) \left(\frac{\delta B_{\text{2D}}}{B_0} \right)^{4/3} \left[l_{\text{2D}} \left\langle (\Delta z(t))^2 \right\rangle_P \right]^{2/3}, \tag{7}$$

with

$$\alpha(\nu) = \frac{\Gamma(7/6)}{\sqrt{\pi}} \left(18\sqrt{\frac{\pi}{2}} C(\nu) \right)^{2/3}. \tag{8}$$

This is a tractable analytical result that can easily be applied to the parameter regimes considered in test particle simulations or for turbulence parameters appropriate for the solar wind. In observed spectra, it was clearly found that $\nu = 5/6$ and thus $\alpha(\nu = 5/6) \approx 0.5$. A (time-dependent) diffusion coefficient as obtained from test-particle simulations can be defined as $\kappa_{xx}(t) = \langle(\Delta x)^2\rangle/(2t)$. In general, one may adopt the assumption $\langle(\Delta z(t))^2\rangle_P \sim t^{b_\parallel+1}$, implying a parallel diffusion coefficient $\kappa_{zz} \sim t^{b_\parallel}$. By assuming $\kappa_{xx} \sim t^{b_\perp}$, it is straightforward to find from Eq. (7) the relation

$$b_\perp = \frac{2b_\parallel - 1}{3}. \tag{9}$$

Therefore, knowledge of $b_\parallel$ (e.g., from simulation data) leads to an evaluation of $b_\perp$, within this model. For instance, if parallel transport behaves diffusively ($b_\parallel = 0$), we find $b_\perp = -1/3$ (subdiffusion). A diffusive behavior of perpendicular transport ($b_\perp = 0$) can only be obtained for $b_\parallel = 1/2$ (superdiffusion).

For pure slab geometry, however, we deduce by combining Eqs. (2) and (6) with Eq. (4):

$$\left\langle(\Delta x)^2\right\rangle_P\Big|_{t\to\infty} = 2\sqrt{\frac{2}{\pi}}\kappa_{\mathrm{FL}}\sqrt{\left\langle(\Delta z(t))^2\right\rangle_P} \tag{10}$$

and therefore

$$b_\perp^{\mathrm{slab}} = \frac{b_\parallel^{\mathrm{slab}} - 1}{2}. \tag{11}$$

For parallel diffusion ($b_\parallel^{\mathrm{slab}} = 0$), we cleary obtain the well-known result $b_\perp^{\mathrm{slab}} = -1/2$ (see e.g. Kóta & Jokipii 2000; Shalchi 2005). While the slab result is well-known, the relations (7) and (9) are entirely new. The formulation proposed by this paper allows a systematic and reliable discription of perpendicular transport requiring knowledge of parallel transport. We refer to this new approach as the *Generalized Compound Diffusion (GCD)*-model. In the next section, the GCD-model will be tested via test-particle simulations.

4. Test particle simulations

In this section numerical tests are employed to assess the accuracy of the GCD-model. By choosing the same turbulence model as used for the application of the GCD-model (magnetostatic turbulence, slab/2D composite geometry, the wave spectra of Eq. (1)) test-particle simulations can be performed easily. The diffusion coefficient is computed numerically using procedures described previously (Giacalone & Jokipii 1999; Qin et al. 2002a,b).

We have considered test-particle dynamics for the following set of parameters: $l_{2\mathrm{D}} = 0.1\, l_{\mathrm{slab}}$, $\nu = 5/6$, and 20%/80% slab/2D composite geometry. In Fig. 1, we depict the ratio of perpendicular and parallel diffusion coefficients κ_{xx}/κ_{zz} as a function of the dimensionless time $\tau = vt/l_{\mathrm{slab}}$ for the dimensionless cosmic ray rigidity value $R = R_L/l_{\mathrm{slab}} = 0.001$. We have chosen a low value of R to ensure that the guiding-center approximation is valid. The new results are compared to those obtained from the NLGC- and ENLGC-theories and also to test-particle simulations. For the NLGC-results we have assumed a parameter value of $a^2 = 1$, which corresponds to the assumption that guiding-centers follow magnetic field-lines. Obviously the GCD-model provides a result much closer to the simulations than the NLGC-theory and the ENLGC-theory. However, it should be emphasized that the main advantage of the GCD-model is its more systematic nature.

By assuming the simple form $\tilde{\kappa}_{xx}(t) = a\tau^b$, we can deduce the time dependence from numerical data by using $b = (\ln\tilde{\kappa}_{xx}(\tau) - \ln a)/\ln\tau \approx (\ln\tilde{\kappa}_{xx}(\tau))/\ln\tau$ in the high time limit ($\tilde{\kappa}_{xx}$ denotes the dimensionless diffusion coefficients obtained by the simulations). The exponents for the parallel $b_\parallel$ and perpendicular $b_\perp$ diffusion coefficients are depicted in Fig. 2 for different values of the parameter R. As shown, the test particle code provides a weakly superdiffusive behavior of parallel transport, in addition to a weakly subdiffusive behavior of perpendicular transport. In all cases considered, the GCD-model agrees well with the simulations.

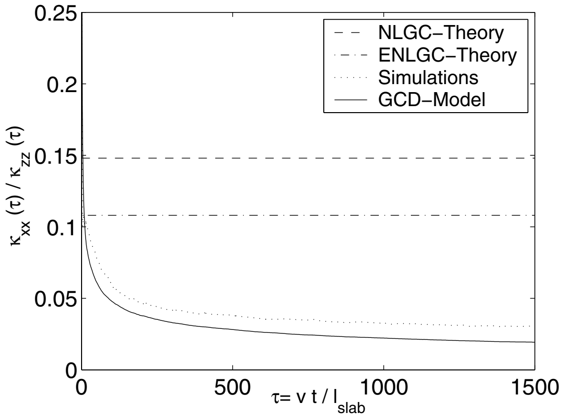

Fig. 1. The ratio of perpendicular and parallel diffusion coefficients ($\kappa_{xx}(t)/\kappa_{zz}(t)$) for $R = R_L/l_{\mathrm{slab}} = 0.001$. The results from test-particle simulations (dotted line) are compared to various theoretical results: NLGC-theory (dashed line), ENLGC-theory (dash-dotted line), and our GCD-model (solid line).

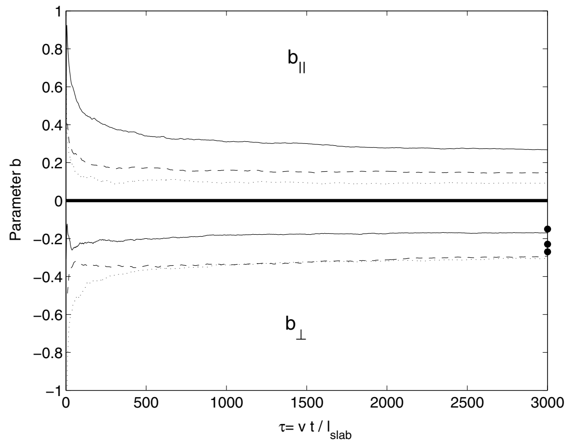

Fig. 2. The parameters $b_\parallel$ and $b_\perp$ as a function of time for different values of the dimensionless rigidity: $R = 10^{-3}$ (dotted line), $R = 10^{-2}$ (dashed line), and $R = 10^{-1}$ (solid line). The dots denote the values predicted by the GCD-model. Clearly we find a weakly superdiffusive behavior of parallel transport ($b_\parallel > 0$) and a weakly subdiffusive behavior of perpendicular transport ($b_\perp < 0$).

5. Comparison with solar wind observations

It is difficult to directly compare our new (non-diffusive) result with solar wind observations. In this section, we attempt a rough comparison by averaging our non-diffusive result over the characteristic scattering time

$$t_{\mathrm{c}} = \frac{\lambda_\parallel}{v}, \tag{12}$$

where we have defined the parallel mean free path $\lambda_\parallel$ and the velocity v of the charged particle. First, we replace the parallel mean-square deviation in Eq. (7) by a diffusive behavior ($\langle(\Delta z(t))^2\rangle_P \approx 2t\kappa_{\|}$) to get

$$\left\langle(\Delta x)^2\right\rangle_P = \alpha(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3}(2l_{2D}t\kappa_{\|})^{2/3} \tag{13}$$

and thus one obtains for the (time-dependent) perpendicular diffusion coefficient

$$\kappa_{\perp}(t) = 2^{-1/3}\alpha(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3}(l_{2D}\kappa_{\|})^{2/3}\,t^{-1/3}. \tag{14}$$

To proceed, we average over the scattering time (see Eq. (12)) to get

$$\begin{aligned}\overline{\kappa}_{\perp} &= \frac{1}{t_c}\int_0^{t_c}\kappa_{\perp}(t)\\ &= \frac{3}{2^{4/3}}\alpha(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3}(l_{2D}\kappa_{\|})^{2/3}\left(\frac{v}{\lambda_{\|}}\right)^{1/3}.\end{aligned} \tag{15}$$

By using $\lambda_{\|} = 3\kappa_{\|}/v$ and $\lambda_{\perp} = 3\kappa_{\perp}/v$, we find an analytical expression for the perpendicular mean free path

$$\overline{\lambda}_{\perp} = \left(\frac{3}{2}\right)^{4/3}\alpha(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^{2/3}\lambda_{\|}^{1/3}. \tag{16}$$

For $\nu = 5/6$ and $\delta B_{2D}^2/B_0^2 = 0.8$, as proposed by Bieber et al. (1994), we obtain

$$\overline{\lambda}_{\perp} = 0.75\, l_{2D}^{2/3}\,\lambda_{\|}^{1/3}. \tag{17}$$

Palmer (1982) suggested that the parallel mean free path in the solar wind is $0.08\ \mathrm{AU} \leq \lambda_{\|,\mathrm{Palmer}} \leq 0.3\ \mathrm{AU}$ and the perpendicular spatial diffusion coefficient is $\kappa_{\perp}c/v \approx 10^{21}\ \mathrm{cm}^2\,\mathrm{s}^{-1}$ and thus $\lambda_{\perp,\mathrm{Palmer}} \approx 0.007$ AU. By taking the average value for the parallel mean free path $\lambda_{\|,\mathrm{Palmer}} \approx 0.2$ and by applying Eq. (17) we find $\lambda_{\perp,\mathrm{GCD}} \approx 0.009$ AU (for $l_{2D} = 0.1 l_{\mathrm{slab}} \approx 0.003$ AU, as suggested by e.g. Matthaeus et al. 2003), which is therefore very close to the measurements. Obviously there is good agreement between solar wind observations and our new theoretical approach.

The values for the perpendicular mean free paths obtained from Jovian electrons (Chenette et al. 1977) and Ulysses measurements of Galactic protons (Burger et al. 2000) are similar. Thus we conclude that the generalized compound diffusion model can reproduce solar wind observations for the perpendicular mean free paths.

To reproduce these observations we applied Eq. (16), which can be obtained from the more general result of Eq. (7) by averaging over the scattering time. According to our new theory and test-particle simulations, parallel as well as perpendicular transport behaves non-diffusively. However, the non-diffusivity is very weak, thus Eqs. (16) and (17) should be good approximations.

6. Summary and conclusion

By combining a compound diffusion model – cf. Eq. (4) – with a nonlinear treatment of field-line wandering (namely Eq. (3)), a new theoretical treatment for the perpendicular transport of cosmic rays is presented in this article. In Table 1, the assumptions of this new theory are compared to the NLGC-theory, as representative of existing transport theories. Obviously the new approach relies on less approximations and model assumptions. Therefore the GCD-model is less restricted and thus more reliable. In Table 2, we compare different theories and their results for the parameter $b_{\perp}$, which denotes the time dependence of the diffusion coefficient via $\kappa_{xx}(t) \sim t^{b_{\perp}}$. Furthermore, the theory is easily applicable due to its simple analytical form (see Eqs. (7) and (8)).

Table 1. Comparison between the assumptions used in our GCD-model and the assumptions used in the NLGC-theory.

Assumption	NLGC	GCD
Guiding-center approximation	YES	YES
Gaussian statistics	YES	YES
Corrsin's hypothesis	YES	YES
Velocities and fields are uncorrelated	YES	NO
Exponential velocity correlation	YES	NO
Diffusion approximation	YES	NO

Table 2. Comparison between results for the parameter $b_{\perp}$ from various theories: QLT, BAM-model, NLGC-theory, WNLT, ENLGC-theory, and the GCD-model. Negative values of $b_{\perp}$ correspond to subdiffusion, positive values to superdiffusion, and $b_{\perp} = 0$ corresponds to diffusion.

Theory	slab turbulence	slab/2D composite
Simulations	−0.5	≈−0.2
QLT	0	1.0
BAM	0	0
NLGC	0	0
WNLT	0	0
ENLGC	−1/2	0
GCD-model	−1/2	≈−0.2

Through comparison with direct numerical simulations of test particles, we have demonstrated that the GCD-model behaves very well and provides a noticeably improved description of perpendicular transport compared to several other theories considered in the tables for reference. Furthermore, by averaging over the scattering time, we have derived a simple formula (Eq. (16)) for the perpendicular mean free path. This formula can easily be applied for solar wind parameters and can be compared with observations. As demonstrated, there is very good agreement between the GCD-model and the observations discussed by Chenette et al. (1977), Palmer (1982), and Burger et al. (2000), similar to the results of the NLGC-theory and its extended version. However, the NLGC-theory, as well as the ENLGC-theory, predict very large perpendicular mean free paths for certain limits (see Bieber et al. 2004; Shalchi 2006). These limits do not exist as shown in the current article, hence Eqs. (7) and (8) represent perpendicular tranport for all parameter regimes within the two-component model. Thus, besides the weak superdiffusivity discovery in the article, the GCD-model clearly disagrees with the NLGC-theory that has been applied in several transport theory studies. It is the subject of our current work to study the application of the present theory in space physics and astrophysics in the hope that an improved formulation of perpendicular transport might be useful in solving a number of observational puzzles. Mainly the non-diffusivity of particle transport for turbulence models that have been considered in the past as realistic models for solar wind turbulence could be important for reproducing heliospheric observations.

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Generalized compound transport of charged particles in turbulent magnetized plasmas

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Abstract

The transport of charged particles in partially turbulent magnetic systems is investigated from first principles. A generalized compound transport model is proposed, providing an explicit relation between the mean-square deviation of the particle parallel and perpendicular to a magnetic mean field, and the mean-square deviation which characterizes the stochastic field-line topology. The model is applied in various cases of study, and the relation to previous models is discussed.

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1. Introduction

The analytical description of magnetic field-line wandering, or *random walk*, in partially turbulent systems is a long standing problem in astrophysics, space physics and turbulence physics. In a number of previous works (e.g., Krommes *et al* (1983), Matthaeus *et al* (1995), Ruffolo *et al* (2004), Ragot (2006), Shalchi and Kourakis (2007)), detailed linear and nonlinear calculations have been performed in order to understand the field-line random walk, in association with the random behavior of dynamical charged particle trajectories in turbulent plasma systems.

The transport of charged particles perpendicular to a large scale magnetic field (e.g., the magnetic field of the Sun, if particle propagation in the solar system is investigated) is often modeled in relevance with field-line wandering. In certain studies, it was assumed that field-line wandering behaves diffusively, yet without giving a justification for this assumption (e.g. Kóta and Jokipii (2000), Webb *et al* (2006)). Other theoretical approaches, such as the nonlinear guiding-center theory (Matthaeus *et al* 2003), the extended NLGC theory (Shalchi 2006) or the weakly nonlinear theory (Shalchi *et al* 2004) have extended that methodology, yet neither using any assumptions about field-line wandering nor assuming that the field-line behavior has a direct influence onto particle propagation.

An open issue in the cosmic ray transport theory is the subdiffusive behavior of perpendicular transport in slab models and the recovery of diffusion for non-slab geometry, as was observed in test-particle simulations (e.g. Qin *et al* 2002a, 2002b). Although the extended nonlinear guiding-center theory (Shalchi 2006) can explain subdiffusion in the slab model as well as the recovery of diffusion in slab/2D composite geometry quantitatively, no physical explanation is yet available, to our knowledge, for these different regimes to be understood. In this paper, we address this problem by relating field-line transport coefficients with particle transport parameters. Via a generalized compound diffusion model, relying on a flexible parametrization of the field-line diffusion and of (in relation with) the actual particle trajectory diffusion in the directions parallel and perpendicular to the magnetic field, we aim to show that different particle random walk regimes obtained in the past come out to be precisely the consequence of underlying assumptions concerning the random behavior of magnetic field lines. The outcome of this study will be important in the theoretical interpretation of charged cosmic ray transport, as provided by space observations.

2. Random walk of magnetic field lines

We shall consider a collisionless magnetized plasma system which is embedded in a uniform mean field ($\vec{B}_0 = B_0\vec{e}_z$) in addition to a turbulent magnetic field component $\delta\vec{B}$. The field-line equation in this system reads $\mathrm{d}x/\mathrm{d}z = \delta B_x/B_0$. Here we assumed a vanishing parallel component of the turbulent field $\delta B_z = 0$. We also assume that the mean field $\vec{B}_0$ is aligned parallel to the z-axis of our (cartesian) system of coordinates.

A characteristic quantity to describe field-line random-walk (FLRW) is the mean-square displacements (MSD's) $\langle(\Delta x)^2\rangle$ and $\langle(\Delta y)^2\rangle$ for large values of z. In the following we only consider the variable x, since all calculations can easily be repeated for y. For axisymmetric turbulence, which is assumed to be a good approximation for real turbulent systems, we have $\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle$. In this case the results derived in this paper for $\langle(\Delta x)^2\rangle$ can also be used for $\langle(\Delta y)^2\rangle$.

By doing this, one anticipates an asymptotic behavior in the following form:

$$\langle(\Delta x)^2\rangle|_{z\to\pm\infty} = \alpha|z|^{\beta}. \tag{1}$$

Field-line wandering is thus characterized by identifying different parameter regimes for β:

$$\begin{aligned} &0 < \beta < 1: && \text{subdiffusion} \\ &\beta = 1: && \text{normal (Markovian) diffusion} \\ &1 < \beta < 2: && \text{superdiffusion} \\ &\beta = 2: && \text{ballistic behavior.} \end{aligned} \tag{2}$$

In the past several approaches have been proposed to describe FLRW analytically.

In the so-called slab turbulence approach (i.e. assuming that the turbulent fields only depend on the parallel position variable, namely $\delta B_i(\vec{x}) = \delta B_i(z)$, for $i = x, y$) the field-line MSD can be calculated exactly. By assuming a constant wave-spectrum at large turbulence scales (energy range) we find a diffusive behavior of the field lines:

$$\langle(\Delta x)^2\rangle = 2\kappa_{\mathrm{FL}}|z| \tag{3}$$

with the field-line diffusion coefficient κ_{FL}.

The description of FLRW in non-slab turbulence models is more problematic. As an example, we consider the so-called two-component turbulence model, where we have a hybrid combination of the slab and 2D fluctuations (in the latter model, one assumes that $\delta B_i(\vec{x}) = \delta B_i(x, y)$, for $i = x, y$). In the slab/2D composite model we have, precisely,

$$\delta B_i(\vec{x}) = \delta B_i^{(\mathrm{slab})}(z) + \delta B_i^{(2D)}(x, y) \tag{4}$$

(for $i = x, y$). In this case the field-line equation takes the nonlinear form

$$\mathrm{d}x(z) = \frac{\delta B_x^{(\mathrm{slab})}(z)}{B_0}\,\mathrm{d}z + \frac{\delta B_x^{(2D)}(x, y)}{B_0}\,\mathrm{d}z, \tag{5}$$

and an analogous relation holds for the y-component.

In our knowledge, at least three different theories have been developed to describe FLRW analytically:

(i) Quasilinear theory (QLT, Jokipii (1966)) consists in replacing the field-line equation on the right-hand side of equation (5) by the unperturbed lines (i.e. the rectilinear magnetic field topology in the absence of turbulence), say at $x = y = 0$. For pure slab geometry $\left(\delta B_x^{(2D)}(x, y) = 0\right)$, the QLT for FLRW is exact. However, for pure 2D turbulence we have within the QLT

$$\mathrm{d}x(z) = \frac{\delta B_x^{(2D)}(0, 0)}{B_0}\,\mathrm{d}z = \frac{\delta B_x^{(2D)}}{B_0}\,\mathrm{d}z \tag{6}$$

and thus we find for the MSD,

$$\langle(\Delta x)^2\rangle = \frac{\delta B_x^2}{B_0^2} z^2. \tag{7}$$

This result is also obtained for slab/2D composite geometry within the QLT, since the second term in equation (5) is dominant.

(ii) Matthaeus *et al* (1995) proposed a non-perturbative approach based on three *ad hoc* assumptions (namely, the so-called Corrsin hypothesis, Gaussian statistics for the field lines and diffusive FLRW behavior). The following form of the field-line diffusion coefficient

$$\kappa_{\mathrm{FL}} = \frac{\kappa_{\mathrm{FL,slab}} + \sqrt{\kappa_{\mathrm{FL,slab}}^2 + 4\kappa_{\mathrm{FL,2D}}^2}}{2} \tag{8}$$

is thus deduced. Here, $\kappa_{\mathrm{FL,slab}}$ is the pure slab field-line diffusion coefficient and $\kappa_{\mathrm{FL,2D}}$ is the pure 2D field-line diffusion coefficient.

(iii) An improved nonlinear theory for field-line wandering has recently been proposed by Shalchi and Kourakis (2007). In comparison to the Matthaeus *et al* (1995) approach, the authors still rely on the validity of the Corrsin hypothesis and on field-line Gaussian statistics. However, instead of applying a diffusion model as used by Matthaeus *et al* (1995), an ordinary differential equation (ODE) is derived for the field-line MSD. By solving this ODE analytically in the limit $|z| \to \infty$, it is deduced, for slab/2D composite turbulence geometry,

$$\langle(\Delta x)^2\rangle = \left[9C(\nu)\sqrt{\frac{\pi}{2}} l_{2D}\right]^{2/3} |z|^{4/3}, \tag{9}$$

which is clearly a superdiffusive result. Here $C(\nu)$ is a normalization function which depends on the inertial range spectral index 2ν, l_{2D} is the 2D bendover scale of the turbulence (this parameter indicates the turnover from the energy range to the inertial range of the spectrum), and $\delta B_{2D}^2 / B_0^2$ denotes the relative strength of the turbulent magnetic fields. The theory recently presented by Shalchi and Kourakis (2007) is essentially a generalization of the Matthaeus *et al* (1995) approach, yet relies on minimum physical assumptions, thus the description of turbulence by equation (9) may be considered to be more reliable than the diffusive result (8).

In principle we know three different results of FLRW:

- $\langle(\Delta x)^2\rangle \sim z^2$: this result can be found in the initial free-streaming regime and within the QLT for two-component (composite) turbulence.
- $\langle(\Delta x)^2\rangle \sim |z|$: the diffusive result can be derived exactly for slab geometry and a constant spectrum at large turbulence scales. Furthermore, Matthaeus *et al* (1995) have claimed that FLRW also behaves diffusively for slab/2D composite geometry.
- $\langle(\Delta x)^2\rangle \sim |z|^{4/3}$: according to Shalchi and Kourakis (2007), the field lines behave superdiffusively in the two-component model.

In the following, we shall combine these different results of FLRW with the guiding-center approximation and various transport models, in view of a critical comparison among different models.

3. The guiding-center approximation

In several previous papers it has been assumed that charged particles follow magnetic field lines (Jokipii 1966, Kóta and Jokipii 2000, Matthaeus *et al* 2003):

$$\tilde{v}_x = v_z \frac{\delta B_x(\vec{x})}{B_0}, \tag{10}$$

where v_z is the parallel velocity of the charged particle and $\tilde{v}_x$ the perpendicular velocity of its guiding center. This equation can easily be deduced from the field-line equation which reads $\mathrm{d}x = \mathrm{d}z\, \delta B_x / B_0$. A formula which is equivalent to equation (10) is

$$\sigma_\perp(t) = \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z) f_\parallel(z,t). \tag{11}$$

Here we have defined the mean-square deviation (MSD) of the particle displacement in the perpendicular direction

$$\sigma_\perp(t) = \langle(\Delta x(t))^2\rangle, \tag{12}$$

the MSD of the field lines

$$\sigma_{\mathrm{FL}}(z) = \langle(\Delta x(z))^2\rangle_{\mathrm{FL}} \tag{13}$$

and the parallel particle distribution function $f_\parallel(z,t)$. It should be noted that an equivalent formula is given by Krommes *et al* (1983). In the following we discuss two models for $f_\parallel(z,t)$.

4. A Gaussian transport model for parallel scattering

We may assume a Gaussian particle distribution function

$$f_{\parallel,G}(z,t) = \frac{1}{\sqrt{2\pi\sigma_\parallel(t)}}\, \mathrm{e}^{-z^2/(2\sigma_\parallel(t))}, \tag{14}$$

where we have employed the particle MSD in the parallel direction

$$\sigma_\parallel = \langle(\Delta z(t))^2\rangle. \tag{15}$$

Equation (11) thus becomes

$$\sigma_\perp(t) = \frac{1}{\sqrt{2\pi\sigma_\parallel(t)}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z)\, \mathrm{e}^{-z^2/(2\sigma_\parallel(t))}. \tag{16}$$

4.1. General results

To proceed with, we may assume the form

$$\sigma_{\mathrm{FL}}(z) = \alpha_{\mathrm{FL}}|z|^{\beta_{\mathrm{FL}}} \tag{17}$$

to obtain

$$\sigma_{\perp}(t) = \frac{\alpha_{\mathrm{FL}}}{\sqrt{2\pi\sigma_{\parallel}(t)}} \int_{-\infty}^{+\infty} \mathrm{d}z|z|^{\beta_{\mathrm{FL}}} \,\mathrm{e}^{-z^2/(2\sigma_{\parallel}(t))}. \tag{18}$$

This integral can easily be solved (Gradshteyn and Ryzhik 2000), so one gets

$$\sigma_{\perp}(t) = \frac{\alpha_{\mathrm{FL}}}{\sqrt{\pi}}\Gamma\left(\frac{\beta_{\mathrm{FL}}+1}{2}\right)(2\sigma_{\parallel}(t))^{\beta_{\mathrm{FL}}/2}, \tag{19}$$

where we used the gamma function $\Gamma(x)$. Furthermore, we assume the forms

$$\sigma_{\parallel}(t) = \alpha_{\parallel}t^{\beta_{\parallel}}, \qquad \sigma_{\perp}(t) = \alpha_{\perp}t^{\beta_{\perp}}. \tag{20}$$

By comparing with equation (19) we obtain

$$\alpha_{\perp} = \frac{\alpha_{\mathrm{FL}}}{\sqrt{\pi}}\Gamma\left(\frac{\beta_{\mathrm{FL}}+1}{2}\right)(2\alpha_{\parallel})^{\beta_{\mathrm{FL}}/2} \tag{21}$$

and, more important by,

$$\beta_{\perp} = \frac{\beta_{\parallel}\beta_{\mathrm{FL}}}{2} \tag{22}$$

for consistency. We see that the independent parameters appearing in (21) and (22) can be used to fine-tune and distinguish different versions of the statistical theory of turbulence, and possibly explain the different results obtained via different assumptions. Our ambition in the following is to pin-point this possibility, by employing specific paradigms. For this purpose, we shall distinguish three different cases, for the field-line statistics, in the following paragraphs.

4.2. Diffusive behavior of FLRW and parallel transport

For pure slab geometry and assuming a constant spectrum in the energy range, it can be shown that field-line wandering behaves diffusively, and thus $\beta_{\mathrm{FL}} = 1$. If we additionally assume that parallel transport also behaves diffusively, namely $\beta_{\parallel} = 1$, we find

$$\alpha_{\perp} = \alpha_{\mathrm{FL}}\sqrt{\frac{2\alpha_{\parallel}}{\pi}} \tag{23}$$

and

$$\beta_{\perp} = \tfrac{1}{2}. \tag{24}$$

For the diffusive field-line behavior we may introduce the field-line diffusion coefficient κ_{FL} via

$$\sigma_{\mathrm{FL}}(z) = 2\kappa_{\mathrm{FL}}|z| \tag{25}$$

and thus $\alpha_{\mathrm{FL}} = 2\kappa_{\mathrm{FL}}$, and the parallel diffusion coefficient $\kappa_{\parallel}$ of the particle position displacement via

$$\sigma_{\parallel}(t) = 2\kappa_{\parallel}t; \tag{26}$$

hence $\alpha_{\parallel} = 2\kappa_{\parallel}$. Therefore we find

$$\alpha_{\perp} = 4\kappa_{\mathrm{FL}}\sqrt{\frac{\kappa_{\parallel}}{\pi}}. \tag{27}$$

Obviously one gets

$$\sigma_\perp(t) = 4\kappa_{\mathrm{FL}}\sqrt{\frac{\kappa_\parallel t}{\pi}} \tag{28}$$

which is clearly a non (classical) diffusive result.

Concluding this paragraph, we have seen that a direct consequence of having assumed diffusive field-line wandering and diffusive particle propagation in the parallel direction is a subdiffusive result for perpendicular particle transport, in the form $\sigma_\perp(t) \sim \sqrt{t}$. This coincides with the result(s) obtained by Krommes *et al* (1983), also within the compound diffusion model of Kóta and Jokipii (2000), and via the extended nonlinear guiding-center theory (Shalchi, 2006).

4.3. Free streaming of field lines

For small length scales the field-line mean-square deviation reads

$$\sigma_{\mathrm{FL}}(z) = \frac{\delta B_x^2}{B_0^2} z^2 \tag{29}$$

and thus

$$\alpha_{\mathrm{FL}} = \frac{\delta B_x^2}{B_0^2} \qquad \text{and} \qquad \beta_{\mathrm{FL}} = 2. \tag{30}$$

Substituting into equations (21) and (22), we find in this case

$$\alpha_\perp = \frac{\delta B_x^2}{B_0^2}\alpha_\parallel, \qquad \beta_\perp = \beta_\parallel. \tag{31}$$

Parallel and perpendicular charged particle transport therefore present the same time behavior if the field lines can be described by equation (29).

4.4. The 4/3-result of Shalchi and Kourakis (2007)

According to the results of an improved theory for field-line wandering, recently proposed by Shalchi and Kourakis (2007), we find, for slab/2D turbulence:

$$\beta_{\mathrm{FL}} = 4/3. \tag{32}$$

As a consequence, we find from (22)

$$\frac{\beta_\perp}{\beta_\parallel} = \frac{2}{3}. \tag{33}$$

It has been argued in several previous papers (see Qin *et al* 2002a, 2002b), by using test-particle simulations, that parallel and perpendicular transport behave diffusively in the case of the two-component (composite) model. The question of how this recovery of diffusion can be explained remains unanswered. However, by assuming a symmetric deviation of the diffusive regime of parallel and perpendicular transport

$$\beta_\perp = 1 - \epsilon, \qquad \beta_\parallel = 1 + \epsilon \tag{34}$$

(here we assumed a weak subdiffusive behavior of perpendicular transport and a weak superdiffusive behavior of parallel transport), it can easily be shown that

$$\epsilon = \frac{2 - \beta_{\mathrm{FL}}}{2 + \beta_{\mathrm{FL}}} \tag{35}$$

from (22), and thus for $\beta_{\mathrm{FL}} = 4/3$ this becomes

$$\epsilon = 0.2. \tag{36}$$

This very weak deviation from the diffusive regime cannot be excluded by test-particle simulations.

5. A ballistic transport model for parallel scattering

An alternative to the Gaussian distribution hypothesis adopted in the previous section is a ballistic model of the form:

$$f_{\parallel,B}(z,t) = \delta\left(z - z_0(t)\right) \tag{37}$$

where we have denoted the unperturbed orbit $z_0(t) = v\mu t$, defining the particle trajectory's pitch-angle cosine μ. In the unperturbed system ($\delta B_x = \delta B_y = 0$) the pitch angle and therefore the parameter μ are conserved. Equation (11) with equation (37) for $f_\parallel(z,t)$ becomes

$$\sigma_\perp(t) = \sigma_{\mathrm{FL}}(z = z_0(t)) \equiv \sigma_{\mathrm{FL}}(z = v\mu t). \tag{38}$$

5.1. General results

Again we assume the form of equation (17) for σ_{FL} to find

$$\sigma_\perp(t) = \alpha_{\mathrm{FL}}(vt|\mu|)^{\beta_{\mathrm{FL}}}. \tag{39}$$

By assuming the form of equation (20) for $\sigma_\perp$ we can deduce

$$\begin{aligned} \alpha_\perp &= \alpha_{\mathrm{FL}}(v|\mu|)^{\beta_{\mathrm{FL}}}, \\ \beta_\perp &= \beta_{\mathrm{FL}}. \end{aligned} \tag{40}$$

Therefore, within the ballistic model, the time exponent of the perpendicular MSD and the length exponent of the field-line MSD are the same.

In the following, we shall consider, two of the three examples exposed above, for the sake of comparison.

5.2. Diffusive behavior of field-line wandering

Applying equations (25) and (37) we get

$$\sigma_\perp = 2\kappa_{\mathrm{FL}} vt|\mu|. \tag{41}$$

Because μ itself is a statistic quantity with $-1 \leqslant \mu \leqslant +1$, the formula can be averaged by integrating with respect to μ, setting

$$\sigma_\perp = 2\kappa_{\mathrm{FL}} vt \left(\frac{1}{2}\int_{-1}^{+1} \mathrm{d}\mu|\mu|\right); \tag{42}$$

hence

$$\sigma_\perp = \kappa_{\mathrm{FL}} vt. \tag{43}$$

Therefore, within the ballistic model and for a diffusive behavior of FLRW we find the well-known quasilinear result for perpendicular transport often referred to as FLRW limit (Jokipii 1966). For the perpendicular diffusion coefficient $\kappa_\perp$ we thus find

$$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}}, \tag{44}$$

so the perpendicular particle transport coefficient $\kappa_\perp$ is efficiently associated with the field-line diffusion coefficient κ_{FL}.

5.3. Free streaming of field lines

Here, we combine equation (29) with equation (38) to find

$$\sigma_\perp = (v\mu t)^2 \frac{\delta B_x}{B_0} \tag{45}$$

and thus

$$\sigma_\perp = \frac{1}{3}\frac{\delta B_x^2}{B_0^2} v^2 t^2, \tag{46}$$

where the pitch-angle cosine variable μ was again averaged out. Equation (46) corresponds to a ballistic motion of charged particles. Furthermore, this formula is in agreement with the quasilinear result for perpendicular transport obtained by Shalchi and Schlickeiser (2004) for two-component turbulence.

6. General results in the compound transport formulation

In the last two sections, we have discussed two specific models for the parallel particle distribution function, namely the Gaussian model and the ballistic model. In this section, we shall discuss some general properties of the compound transport model, that is, without specifying the form of the parallel distribution function.

6.1. The initial free-streaming regime

A free-streaming-like behavior of field lines is found in some cases (e.g., for small length scales, or if QLT for FLRW is applied for slab/2D composite geometry). In this case we can combine equations (29) and (11) to get

$$\begin{aligned}\sigma_\perp(t) &= \frac{\delta B_x^2}{B_0^2}\int_{-\infty}^{+\infty} \mathrm{d}z\, z^2 f_\parallel(z,t) \\ &= \frac{\delta B_x^2}{B_0^2}\sigma_\parallel(t)\end{aligned} \tag{47}$$

regardless of the form of $f_\parallel(z,t)$. Therefore, for free-streaming of field lines we find

$$\frac{\sigma_\perp(t)}{\sigma_\parallel(t)} = \frac{\delta B_x^2}{B_0^2}. \tag{48}$$

In this case the temporal behavior of parallel and perpendicular transport are the same. In the solar wind at a 1 AU heliocentric distance, we have $\delta B_x^2 \approx B_0^2$. Thus, the mean free path perpendicular to the mean field becomes comparable to the parallel mean free path. For length scales where we have free-streaming of field lines we thus find strong perpendicular scattering of charged cosmic rays.

6.2. Diffusive behavior of FLRW and parallel transport

In section 4.2 we combined the Gaussian model with a diffusive behavior of parallel transport and FLRW to demonstrate that we obtain subdiffusion in the form $\sigma_\perp \sim \sqrt{t}$. In the following, we shall show that the assumption of Gaussian statistics is not necessary to get the subdiffusive perpendicular transport of charged particles. Upon differentiating the basic compound transport relation (11), one gets

$$\frac{\partial}{\partial t}\sigma_\perp(t) = \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z)\frac{\partial f_\parallel(z,t)}{\partial t}. \tag{49}$$

Diffusion in the parallel direction may be assumed, for tractability. Thus, the function $f_\parallel(z,t)$ satisfies the diffusion equation

$$\frac{\partial f_\parallel(z,t)}{\partial t} = \kappa_\parallel \frac{\partial^2 f_\parallel(z,t)}{\partial z^2}. \tag{50}$$

Furthermore, running diffusion coefficients can be introduced by

$$\begin{aligned} d_\perp(t) &\equiv \frac{1}{2}\frac{\partial}{\partial t}\sigma_\perp(t), \\ d_{\mathrm{FL}}(t) &\equiv \frac{1}{2}\frac{\partial}{\partial z}\sigma_{\mathrm{FL}}(z). \end{aligned} \tag{51}$$

Thus equation (49) becomes

$$\begin{aligned} d_\perp(t) &= \kappa_\parallel \int_0^\infty \mathrm{d}z\, \sigma_{\mathrm{FL}}(z) \frac{\partial^2 f_\parallel(z,t)}{\partial z^2} \\ &= -2\kappa_\parallel \int_0^\infty \mathrm{d}z\, d_{\mathrm{FL}}(z) \frac{\partial f_\parallel(z,t)}{\partial z} \\ &= 2\kappa_\parallel \int_0^\infty \mathrm{d}z \frac{\partial d_{\mathrm{FL}}(z)}{\partial z} f_\parallel(z,t), \end{aligned} \tag{52}$$

where we applied $d_{\mathrm{FL}}(z=0)=0$. Now we assume diffusion of field lines, so we have

$$d_{\mathrm{FL}}(z) = \kappa_{\mathrm{FL}} - \epsilon(z) \tag{53}$$

with

$$\epsilon(z=0) = \kappa_{\mathrm{FL}}, \qquad \epsilon(z\to\infty)\to 0. \tag{54}$$

Because of $\partial d_{\mathrm{FL}}(z)/\partial z = -\epsilon'(z)$ (the prime denotes differentiation) we have

$$d_\perp(t) = -2\kappa_\parallel \int_0^\infty \mathrm{d}z\, \epsilon'(z) f_\parallel(z,t). \tag{55}$$

However, if the field lines behave diffusively, the function $\epsilon(z)$ (and therefore also $\epsilon'(z)$) must decay rapidly with increasing z. Thus one gets

$$\begin{aligned} d_\perp(t) &= -2\kappa_\parallel f_\parallel(z=0,t) \int_0^\infty \mathrm{d}z\, \epsilon'(z) \\ &= 2\kappa_\parallel[\epsilon(z=0) - \epsilon(z=\infty)] f_\parallel(z=0,t). \end{aligned} \tag{56}$$

Combining with equation (54), we deduce

$$d_\perp(t) = 2\kappa_\parallel \kappa_{\mathrm{FL}} f_\parallel(z=0,t). \tag{57}$$

As an example, we may consider again the Gaussian transport model. Evaluating equation (14) at $z=0$, one can easily recover equation (28) as a special limit of equation (57). However, equation (57) is more general, and can also be applied on non-Gaussian statistics. In real physical systems, one expects that the probability to find the particle at $z=0$ decreases with time ($f_\parallel(z=0, t\to\infty)\to 0$), so consequently

$$d_\perp(t) \to 0, \tag{58}$$

which is interpreted as subdiffusion. Thus, for parallel diffusion of charged particles in combination with a diffusive behavior of FLRW, we find subdiffusion in the perpendicular direction.

7. Summary and conclusion

In this paper we have discussed the generalized compound diffusion mechanism, which relates field-line statistics (wandering) to charged particle random walk in real (position) space. By applying the guiding-center approximation (equation (11)) and a Gaussian transport model for parallel scattering (equation (14)) we deduced the general relation

$$\beta_{\perp} = \frac{\beta_{\parallel}\beta_{\rm FL}}{2}. \tag{59}$$

Here $\beta_{\perp}$ and $\beta_{\parallel}$ are the time exponents of the perpendicular and parallel MSD's of the particles whereas $\beta_{\rm FL}$ is the length exponent of the field-line MSD.

For diffusive FLRW ($\beta_{\rm FL} = 1$) and diffusive parallel motion of charged particles ($\beta_{\parallel} = 1$), we always find subdiffusion in the form $\sigma_{\perp}(t) \sim \sqrt{t}$. This relation was derived in several previous papers (e.g. Krommes *et al* (1983), Kóta and Jokipii (2000), Shalchi (2006)). However, as demonstrated in this paper, we always get this subdiffusive behavior if FLRW and parallel transport behave diffusively. Perpendicular diffusion and parallel diffusion can only be obtained for $\beta_{\rm FL} = 2$—see equation (59)—which corresponds to free-streaming of field lines.

By replacing the Gaussian transport model by a ballistic model, and by assuming diffusive behavior of FLRW one can easily recover the well-known quasilinear result often referred to as FLRW limit (Jokipii 1966). By combining the quasilinear FLRW result for two-component turbulence—see equation (7)—with the ballistic model we find superdiffusion of charged particles in the perpendicular direction. Thus, the generalized compound transport model discussed in this paper is able to obtain the well-known QLT results in appropriate special limits.

By applying the relation $\sigma_{\rm FL}(z) \sim |z|^{4/3}$, i.e. the superdiffusive result obtained by Shalchi and Kourakis (2007), and by assuming a weakly superdiffusive behavior of parallel transport (e.g. $\beta_{\parallel} = 1.2$), we find that perpendicular diffusion is nearly recovered (e.g. $\beta_{\perp} = 0.8$). This recovery of perpendicular diffusion is an effect which can be found in test particle simulations (e.g. Giacalone and Jokipii (1999), Qin *et al* (2002b)). For diffusive behavior of field lines and parallel diffusion of charged particles these simulations cannot be reproduced theoretically. However, the combination of the Shalchi and Kourakis (2007) theory for field-line wandering with the generalized compound transport model is able to describe this effect.

Our results are important in the theoretical interpretation of cosmic ray transport, following (and interpreting) measurements provided by space observations. It will be the subject of future work to compare these new results with test-particle simulations and solar wind observations.

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