Discrete Breathers, Multibreathers and Vortices in Hexagonal & Honeycomb 2D Dusty Plasma Crystals

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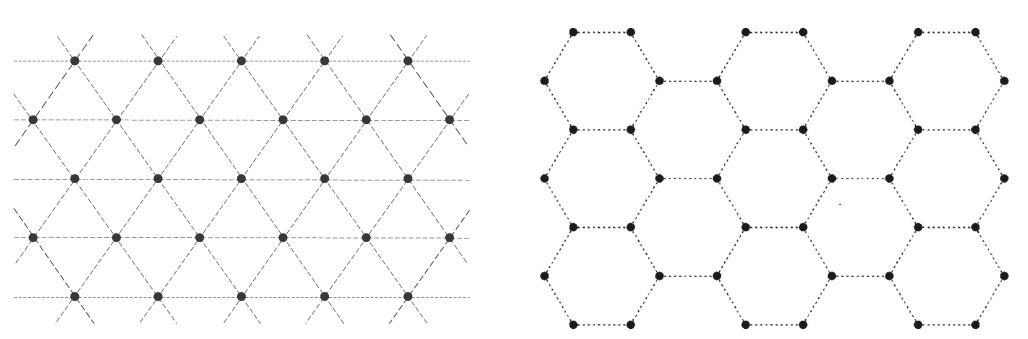
1. Introduction

Strongly-coupled dusty plasma (DP) lattices are known to occur in low-temperature gas discharge experiments, generally in the form of horizontal hexagonal two-dimensional (2D) quasi-crystalline arrangements [1]. Voronoi diagrams bearing a honeycomb structure were also recently observed in experiments [2].



(a) (b)

Fig.1: 2D structures in a DP crystal: (a) a hexagonal lattice and (b) a honeycomb lattice

Transverse (off-plane, vertical, along gravity) dust-lattice (TDL) vibrations are associated with an inverse-dispersive, backward propagating optic-like mode [3], viz. $\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2(kr_0/2)$. Discrete periodic media are today known to support single- and multi-site *Discrete Breather (DB)* excitations, a direction not yet explored in DP crystal experiments. Applying existing methodology [4, 5], we have recently investigated [6, 7, 8] the occurrence of DBs in 2D DP crystals, and explored their stability properties, in terms of the dimensionless parameters $\epsilon = \omega_0^2/\omega_g^2$, $\alpha' = \alpha r_0/\omega_g^2$ and $\beta' = \beta r_0^2/\omega_g^2$ (damping is omitted). Here ω_g and ω_0 are the TDL mode eigenfrequency and (linear) coupling frequency, r_0 is the lattice spacing and α and β are related to the anharmonicity of the plasma sheath potential – details in [9] – (primes to be dropped). Our main results are summarized below.

2. The Klein-Gordon methodology

We consider a 2D array of one dimensional nonlinear point mass oscillators, modelling dust grains, in an on-site potential $V(x)$ with $V''(x) > 0$. Each oscillator is linearly coupled to its nearest neighbors. The Hamiltonian for both configurations is of the form $H = \sum_i \frac{p_i^2}{2} + V(x_i) - \frac{\varepsilon}{2}\sum_{i,j}(x_j - x_i)^2$, where indices i and j run over all sites and their first neighbors, respectively. The minus sign in the coupling term is due to the inverse-dispersive character of TDL oscillations. The corresponding discrete equations of motion read

$$\ddot{x}_i = -V'(x_i) - \varepsilon\Big(\sum_{j\in\mathbb{N}} x_j - N x_i\Big)\,, \qquad (1)$$

where $\mathbb{N}$ is the set of neighbors of the i-site and N is the cardinality of $\mathbb{N}$ which is 6 in the case of the hexagonal lattice and 3 in the case of the honeycomb lattice.

Hexagonal lattice. Consider a hexagonal lattice with a quartic on-site potential $V(x) = x^2/2 + ax^3/3 + bx^4/4$ and $a = 0.01, b = -0.04$ [10]. We consider single particle vibrations in V (anticontinuum limit), and then switch on the coupling via a continuation of orbits for $\epsilon \neq 0$. The stability of a single site breather is determined by its Floquet multipliers i.e., the breather remains stable as long as they remain in the unit circle of the complex plane. The numerical investigation shows that the single site breather configuration remains stable for all $\varepsilon < 0.05$ (which includes $\varepsilon = 0.034$, as in [10]). For details see in [6].

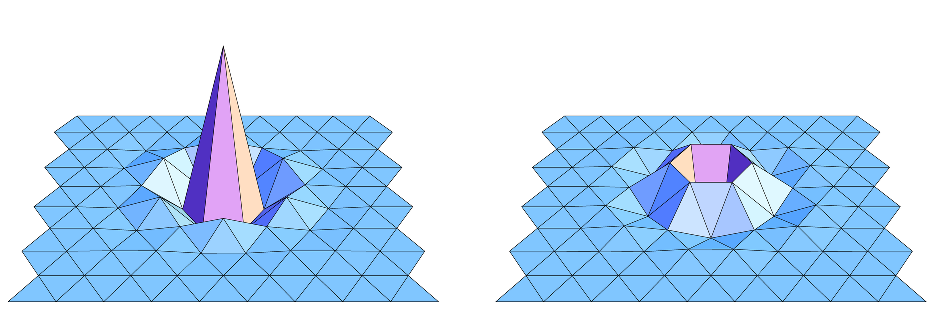
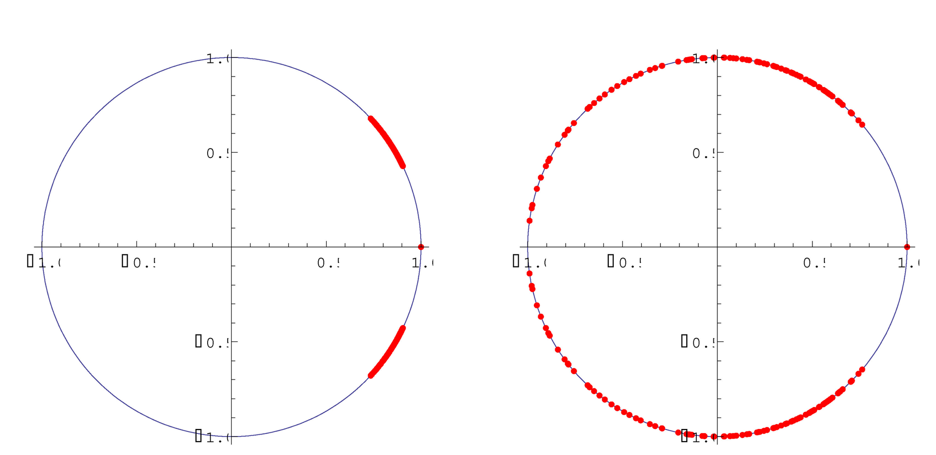


Fig.2: A single site discrete breather



$\varepsilon = 0.01$ $\varepsilon = 0.05$

Fig.3: The Floquet multipliers of a single site discrete breather. The breather remains stable for all $\varepsilon < 0.05$

Consider now three moving adjacent central oscillators. The conditions for the phase difference ϕ_i among successive oscillators for three-site breathers to exist read $\phi_i = 0$, $\phi_i = \pi$ or $\phi_i = 2\pi/3$, which correspond to an *in-phase*, an *out of phase* and a *vortex three-site breather* respectively. In the case of [10], for the latter two configurations, either one or two pairs of multipliers leave the unit circle for arbitrary small ε. Thus, the only stable configuration is the in-phase one and it remains stable until the multipliers of the central sites collide with the linear spectrum and leave the unit circle [6], which in our case occurs for $\varepsilon = 0.017$. Stable *in-phase* 3-breathers are excluded in [10] (where $\varepsilon = 0.034$).

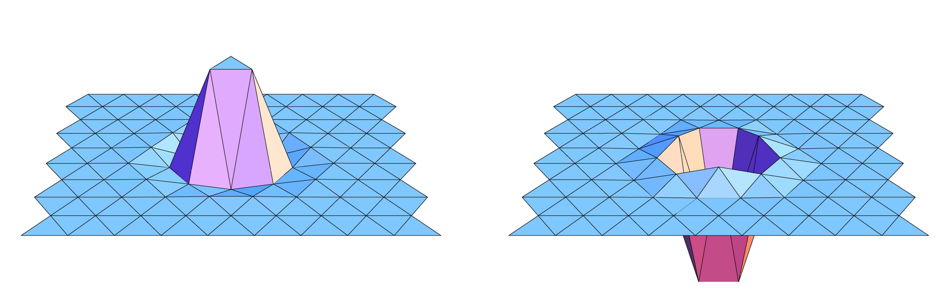
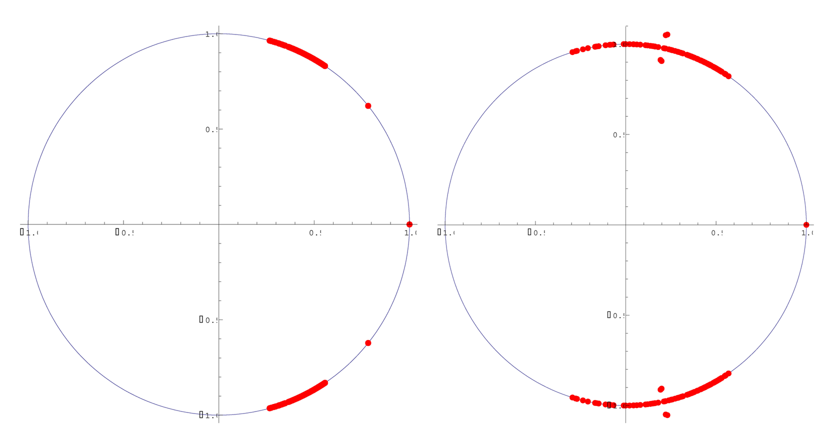


Fig.4: An in-phase three site discrete breather



$\varepsilon = 0.01$ $\varepsilon = 0.025$

Fig.3: The Floquet multipliers of the in-phase three site discrete breather.

Honeycomb lattice. Consider now six adjacent central oscillators forming a unit cell in a honeycomb lattice; see Fig.1b. The conditions for six-site breathers to exist are [8]: $\phi_i = 0$, $\phi_i = \pi$, $\phi_i = \frac{\pi}{3}$ or $\phi_i = 2\pi/3$. The first two cases correspond to an *in-phase* and an *out of phase* six site breather. The latter two correspond to the "charge-one" and the "charge-two" vortex six-site breathers respectively. In this case the linearly stable configurations for ε small enough are the in-phase breather and the charge-one vortex.

3. The Discrete Nonlinear Schrödinger (DLNS) description.

To illustrate the generality of our findings, we also briefly consider the case of the discrete nonlinear Schrödinger (DNLS) model. The latter is a general envelope wave model for discrete nonlinear wave equations of the Klein-Gordon type [11]. The DNLS is also a model of particular relevance, in its own right, in the nonlinear optics of fabricated AlGaAs waveguide arrays [12], as well as in dynamics of Bose-Einstein condensate droplets in the presence of optical lattice potentials [13].

In the DNLS setting, we will present a unified treatment of six site and three site excitations. The relevenat model reads:

$$i\frac{du_{m,n}}{dt} = \epsilon\left(\sum_{<m',n'>\in N} u_{m',n'} - |N|u_{m,n}\right) - |u_{m,n}|^2 u_{m,n}, \qquad (2)$$

where the summation is over the set N of nearest neighbors (denoted by $< m', n' >$) of the site (m, n), and $u_{m,n}$ represents the relevant complex field; notice that for the inter-site coupling ϵ, the opposite than the standard sign has been used, as explained also above in the KG case.

In the, so-called, anti-continuum (AC) limit of $\epsilon \to 0$ explicit solutions over contours of nodes indexed by j can be represented without loss of generality as $u_j = \exp(i\theta_j)\exp(it)$, where $\theta_j \in [0, 2\pi)$. Then, following the considerations of [5], for such solutions with M excited adjacent sites to persist for $\epsilon \neq 0$, $g_j \equiv \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0$, should be satisfied for all $j = 1, \ldots, M$. The stability can also be determined from the eigenvalues γ_j of the $|M| \times |M|$ *Jacobian* $\mathcal{J}_{jk} = \partial g_j/\partial\theta_k$. In particular, it can be proved that the eigenvalues λ_j of the full problem satisfy $\lambda_j = \pm\sqrt{-2\gamma_j\epsilon}$. In the case of phase increments of $|\theta_{j+1} - \theta_j| = \Delta\theta$, it is in fact possible to compute the relevant Jacobian eigenvalues explicitly and obtain for the full problem (near-zero) eigenvalues the general, analytical expression

$$\lambda_j = \pm\sqrt{-8\epsilon\cos(\Delta\theta)\sin^2\left(\frac{\pi j}{|M|}\right)}, \qquad (3)$$

This expression can be used *both* for hexagonal and for honeycomb lattices. Furthermore, it can be used both for $M = 3$ site and for $M = 6$ site configurations. In the defocusing cases of interest herein, it predicts that the in-phase configuration will be the only stable 3-site configuration, while among 6-site configurations the in-phase and the vortex of charge 1 are going to be the stable ones (while the out-of-phase and charge 2 vortex will be unstable). Typical examples of the 6-site waveforms are shown with their eigenvalue ($\lambda = \lambda_r + i\lambda_i$) analysis in Figs. 6 and 7.

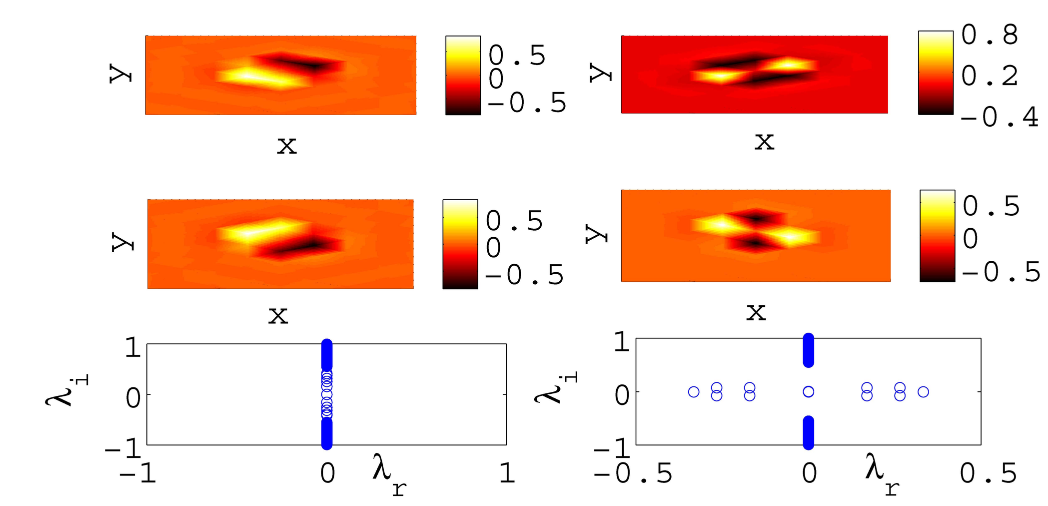


Fig.6: Vortices of charge 1 (left) and 2 (right).The top row shows the real part and middle the imaginary part. The waveforms and the spectrum of linearization around them (bottom row) are shown in both case for $\epsilon = 0.05$.

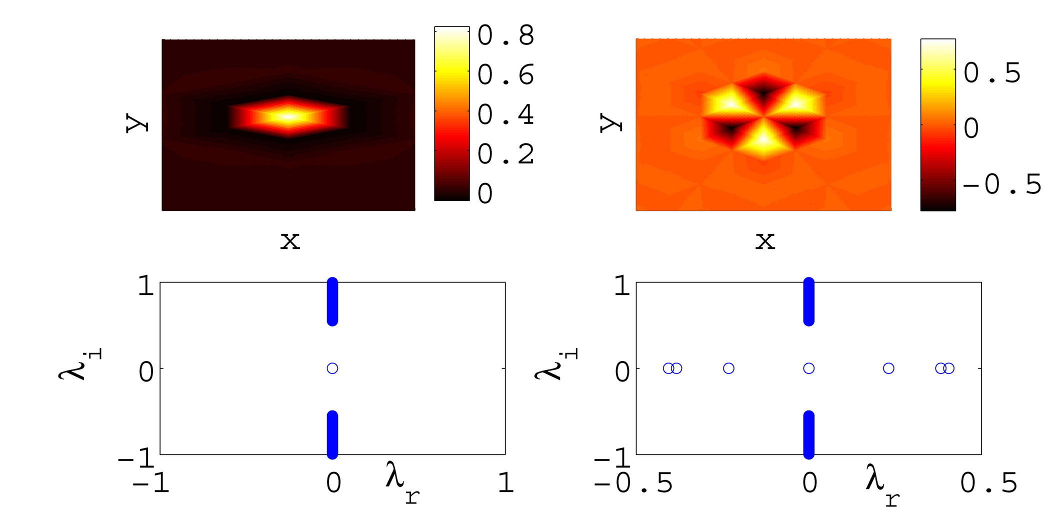


Fig.7: In-phase (left) and out-of-phase hexapoles (left). The waveforms and the spectrum of linearization around them (bottom row) are shown in each case for $\epsilon = 0.05$.

4. Conclusions.

In this work, we have presented a series of novel results regarding the potential of formation of nonlinear breathing excitations in quasi-crystalline, non-square lattice arrangements in dusty plasmas. Both Klein-Gordon and discrete nonlinear Schrödinger models were used to illustrate the stability of in-phase structures for 3-site contours and in-phase, as well as single-charge vortex excitations in 6-site contours. It would be of particular interest to consider such configurations in low-temperature, gas-discharge experiments.

References

[1] G .E. Morfill *et al*, *Plasma Phys. Cont. Fusion* **44**, B263 (2002).

[2] O. Ishihara, in *5th Int. Conf. Phys. Dusty Plasmas*, Ponta Delgada, Azores (2008).

[3] S.V. Vladimirov, P.V. Shevchenko and N.F. Cramer, *Phys. Rev. E*, **56**, R74 (1997).

[4] V. Koukouloyannis and R.S. MacKay, *J. Phys. A: Math. Gen.* **38**, 1021 (2005).

[5] D.E Pelinovsky, P.G. Kevrekidis, and D.J. Frantzeskakis, *Phys. D* **212**, 1 ; *idem*, 20 (2005).

[6] V. Koukouloyannis and I. Kourakis, *Existence and stability of breathers in two dimensional dusty plasma crystals*, in preparation.

[7] K.J.H. Law, P.G. Kevrekidis, V. Koukouloyannis, I. Kourakis, D.J. Frantzeskakis, and A.R. Bishop, *Discrete soliton and vortex states in triangular DNLS lattices: existence, stability and dynamics*, in preparation.

[8] V. Koukouloyannis, P.G. Kevrekidis and K.J.H. Law, I. Kourakis, and D.J. Frantzeskakis, *temporary title 2*, in preparation.

[9] I Kourakis and P. K. Shukla, *Int. J. Bif. Chaos* **16**, 1711 (2006).

[10] A. Melzer, *private communication*: typical experimental values for a, b and ε.

[11] Yu.S. Kivshar and M. Peyrard, Phys. Rev. A **46**, 3198 (1992).

[12] D. N. Christodoulides, F. Lederer, and Y. Silberberg, Nature **424**, 817-823 (2003); A. A. Sukhorukov, Yu. S. Kivshar, H. S. Eisenberg, and Y. Silberberg, IEEE J. Quant. Elect. **39**, 31-50 (2003).

[13] V. A. Brazhnyi and V. V. Konotop, Mod. Phys. Lett. B **18**, 627 (2004); O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179-215 (2006).