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Theoretical modelling of electromagnetic freak (rogue) waves in beam-plasma interactions

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Outline

1. **Introduction**
   * Electromagnetic solitary waves in plasmas: modeling & background
   * *Rogue waves (freak waves)* – physical setting & preliminaries
   * *Nonlinear amplitude modulation*: basic phenomenology
2. **Framework for EM wavepacket modulation**
   * Multiscale perturbation technique
   * Evolution equation for the wavepacket envelope
   * Modulational (in)stability analysis
3. **Electromagnetic (EM) rogue waves** – from first principles
4. **Discussion & Summary**

Part 1: Intro #1 – Electromagnetic (EM) pulses in plasmas

- Old problem in relativistic electrodynamics in plasmas
  Akhiezer and Polovin (1956); Gerstein & Tzoar, PRL (1975); Marburger & Tooper, PRL (1975); Tsintsadze & Tskhakaya, JETP (1977), Kozlov, Litvak and Suvorov, JETP (1979)
- Recently gained impetus in the field of intense laser-plasma interaction:
  * 1D theoretical investigations: ***Kaw, Sen and Katsouleas, PRL (1992)***; Kuehl, Zhang PRE (1993); Sudan et al, Phys. Plasmas (1997); Esirkepov et al., JETP Lett. (1998); Farina and Bulanov, PRL (2001); Farina and Bulanov, Plasma Phys. Rep., (2001); Hadzievski et al PoP (2002); Poornakala et al, PoP (2002); G. Lehmann, E.W. Laedke, K.H. Spatschek, PoP (2006); Lontano, Passoni, Bulanov, PoP (2003); Borhanian et al. PLA & PoP (2009); Sanchez-Arriaga et al, PoP (2011).
  * PIC or fluid simulations: Bulanov et al., Plasma Phys. Rep. (1995); Bulanov et al., PRL (1999); Sentoku et al. Phys. Rev. Lett. (1999); Naumova et al., PRL (2001); Tushentsov et al, PRL (2001); Esirkepov et al, PRL (2002), Esirkepov et al, PRL (2004); Saxena et al, PoP (2006, 2007).
- Old problem in nonlinear physics: originally treated via reductive perturbation theory [Hasegawa, PRA (1970); Phys. Fluids (1972); Taniuti and co., JPSJ (1972)]

Relativistic EM solitons

- *Relativistic solitons* are self-trapped, localized (finite-size) electromagnetic excitations of relativistic intensity that propagate without spreading due to diffraction.
- Dominant mechanisms are:
  - *relativistic nonlinearity* (relativistic mass variation)
  - *striction nonlinearity* (density perturbation due to ponderomotive effects)
  - *dispersion effects* due to finite particle inertia.
- EM plasma solitons consist of electron (and ion) density depressions and intense electromagnetic field concentrations with a larger amplitude and a lower frequency than those of the laser pulse.
- EM energy localization!

Intro #2: Rogue waves – an emerging unifying concept

- *Rogue waves* are localized excitations (events) of extreme amplitude, exceeding twice the average strength of background turbulence level;

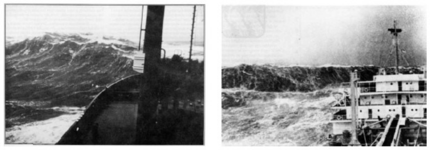
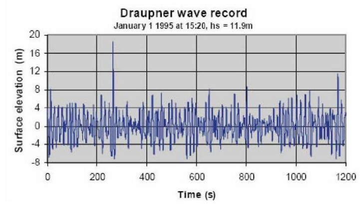


Fig. 2. Various photos of rogue waves.



Data from the Draupner platform event in Norway (Jan. 1995).

Credit: Kharif & Pelinovsky, Eur. Journal of Mechanics B/Fluids **22**, 603 (2003).

Catastrophic ship encounters with rogue waves - stats

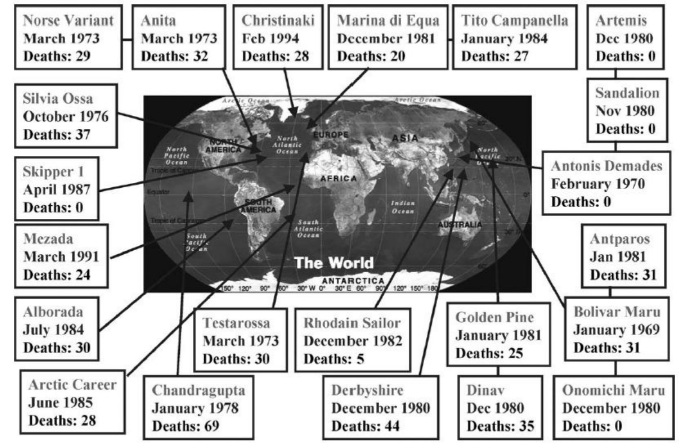


Fig. 1. Statistics of the super-carrier collision with rogue waves for 1968–1994.

Credit: Kharif & Pelinovsky, Eur. Journal of Mechanics B/Fluids **22**, 603 (2003).

Catastrophic encounters: Durban #1

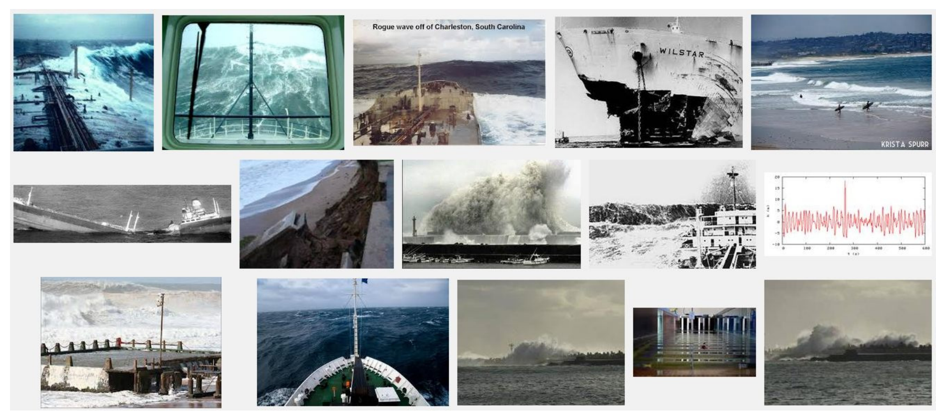
1909: The SS Waratah, sometimes referred to as *Australia's Titanic*, was a 500-foot (150 m) long steamship that operated between Europe and Australia in the early 1900s. On July 29 1909, SS Waratah disappeared on her return (2nd) voyage from Sydney to London, while en route from South Africa's Durban to Cape Town. To this day, no trace of the ship has been found.

Credit: http://en.wikipedia.org/wiki/Waratah_(ship).

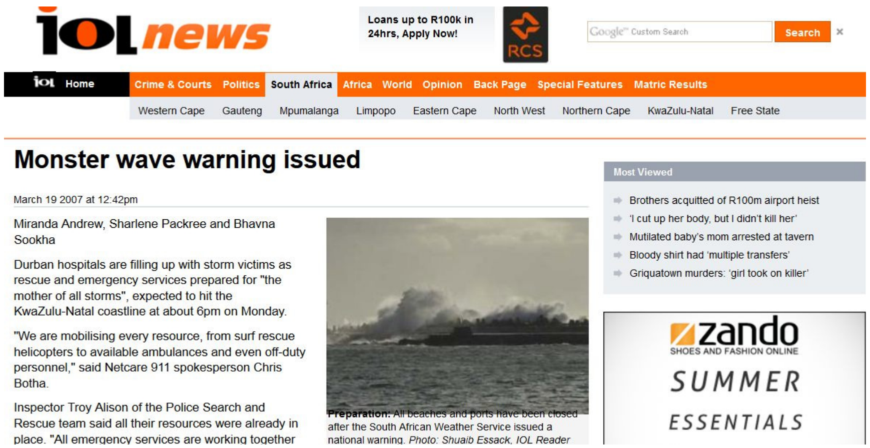
http://freaquewaves.blogspot.com/2006/07/list-of-freaque-wave-encounters.html.

"Monsters of the deep – Huge, freak waves may not be as rare as once thought". Economist, Sept. 17, 2009.

Catastrophic encounters: #2



Catastrophic encounters: Durban #3



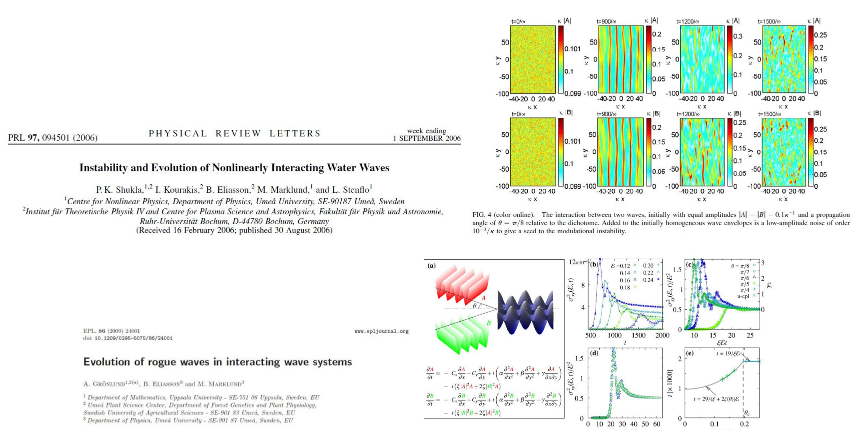
Credit: http://www.news24.com/SouthAfrica/News/Freak-wave-hits-Durban-beach-20080224.

Catastrophic encounters: Durban #4

Credit: http://www.news24.com/SouthAfrica/News/Freak-wave-hits-Durban-beach-20080224.

Theoretical attempts to explain freak wave formation

... via water surface envelope mode interaction:



[Credit: M. Onorato, A.R. Osborne and M. Serio, Phys. Rev. Lett., **96** 014503 (2006); P. K. Shukla, I. Kourakis, B. Eliasson, M. Marklund and L. Stenflo, Phys. Rev. Lett. **97**, 094501 (2006); A. Grönlund, B. Eliasson and M. Marklund, EPL, **86** 24001 (2009).]

PRL 106, 204502 (2011) PHYSICAL REVIEW LETTERS week ending 20 MAY 2011

Rogue Wave Observation in a Water Wave Tank

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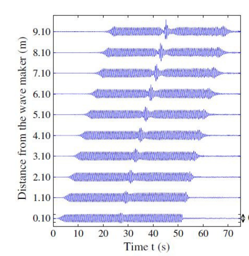
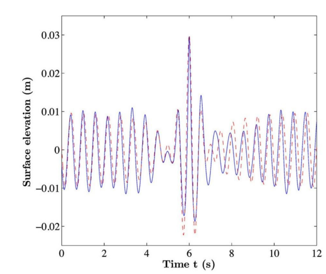


FIG. 3 (color online). Temporal evolution of the water surface height at various distances from the wave maker.



[Credit: A. Chabchoub *et al*, Phys. Rev. Letters **106**, 204502 (2011); (right plot) A. Chabchoub/Hamburg University of Technology (online).]

nature Vol 450|13 December 2007|doi:10.1038/nature06402

LETTERS

Optical rogue waves

D. R. Solli[1], C. Ropers[1,2], P. Koonath[1] & B. Jalali[1]

Recent observations show that the probability of encountering an extremely large rogue wave in the open ocean is much larger than expected from ordinary wave-amplitude statistics[1–3]. Although considerable effort has been directed towards understanding the physics behind these mysterious and potentially destructive events, the complete picture remains uncertain. Furthermore, rogue waves have not yet been observed in other physical systems. Here, we introduce the concept of optical rogue waves, a counterpart of

Although the physics behind rogue waves is still under investigation, observations indicate that they have unusually steep, solitary or tightly grouped profiles, which appear like "walls of water"[10]. These features imply that rogue waves have relatively broadband frequency content compared with normal waves, and also suggest a possible connection with solitons—solitary waves, first observed by J. S. Russell in the nineteenth century, that propagate without spreading in water because of a balance between dispersion and nonlinearity. As

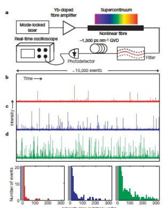


Figure 1 | Experimental observation of optical rogue waves. a, Schematic of experimental apparatus. **b–d**, Single-shot time traces containing roughly 15,000 pulses each and associated histograms (bottom of figure: left, **b**; middle, **c**; right, **d**) for average power levels 0.8 μW (red), 3.2 μW (blue) and 12.8 μW (green), respectively. The grey shaded area in each histogram demarcates the noise floor of the measurement process. In each measurement, the vast majority of events (>99.5% for the lowest power) are buried in this low intensity range, and the rogue events reach intensities of at least 30–40 times the average value. These distributions are very different from those encountered in most stochastic processes.

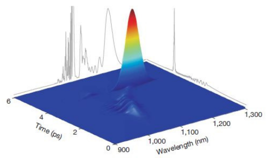


Figure 3 | Time–wavelength profile of an optical rogue wave obtained from a short-time Fourier transform. The optical wave has broad bandwidth and has extremely steep slopes in the time domain compared with the typical events. It appears as a 'wall of light' analogous to the 'wall of water' description of oceanic rogue waves. The rogue wave travels a curved path in time–wavelength space because of the Raman self-frequency shift and group velocity dispersion, separating from non-solitonic fragments and remnants of the seed pulse at shorter wavelengths. The grey traces show the full time structure and spectrum of the rogue wave. The spectrum contains sharp spectral features that are temporally broad and, thus, do not reach large peak power levels and do not appear prominently in the short-time Fourier transform.

Credit: D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Nature **450**, 1054 (2007).

LETTERS
PUBLISHED ONLINE: 22 AUGUST 2010 | DOI: 10.1038/NPHYS1740
nature physics

The Peregrine soliton in nonlinear fibre optics

B. Kibler[1], J. Fatome[1], C. Finot[1], G. Millot[1], F. Dias[2,3], G. Genty[4], N. Akhmediev[5] and J. M. Dudley[6]*

The Peregrine soliton is a localized nonlinear structure predicted to exist over 25 years ago, but not so far experimentally observed in any physical system[1]. It is of fundamental significance because it is localized in both time and space, and because it defines the limit of a wide class of solutions to the nonlinear Schrödinger equation (NLSE). Here, we use an analytic

Our experiments are designed using the breather formalism of ref. 2. With dimensionless field $\psi(\xi,\tau)$, the self-focusing NLSE is:

$$i\frac{\partial\psi}{\partial\xi}+\frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2}+|\psi|^2\psi=0 \quad (1)$$

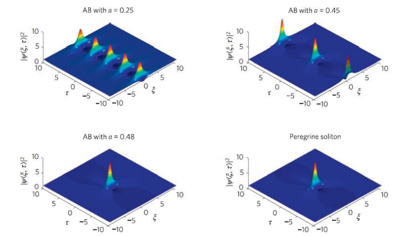


Figure 1 | Plotted Akhmediev breather solutions using equation (2) for modulation parameter $a = 0.25$, $a = 0.45$ and $a = 0.48$, as well as the ideal **Peregrine soliton of equation (3), the limiting case of the Akhmediev breather as $a \to 1/2$.** Maximum temporal compression occurs at normalized distance $\xi = 0$. The differences between the Akhmediev breather (AB) with $a = 0.48$ and the Peregrine soliton can be seen with close inspection of the decay of the peak to the wings; they are shown more clearly in Fig. 2.

[B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev & JM Dudley, Nat. Phys. **6**, 790 (2010)]

PRL **101**, 065303 (2008) PHYSICAL REVIEW LETTERS week ending 8 AUGUST 2008

Observation of an Inverse Energy Cascade in Developed Acoustic Turbulence in Superfluid Helium

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(Received 20 May 2008; published 8 August 2008)

We report observation of an inverse energy cascade in second sound acoustic turbulence in He II. Its onset occurs above a critical driving energy and it is accompanied by giant waves that constitute an acoustic analogue of the rogue waves that occasionally appear on the surface of the ocean. The theory of the phenomenon is developed and shown to be in good agreement with the experiments.

DOI: 10.1103/PhysRevLett.101.065303 PACS numbers: 67.25.dk, 47.20.Ky, 47.27.−i, 67.25.dt

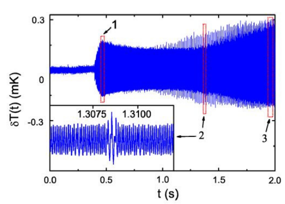

FIG. 3 (color online). Transient evolution of the 2nd sound wave amplitude δT after a steplike shift of the driving frequency to the 96th resonance at time $t = 0.397$ s. Signals in frames 1 and 3 are similar to those obtained in steady-state measurements, Figs. 1(a) and 1(c), respectively. Formation of isolated "rogue" waves is clearly evident. Inset: Example of a rogue wave, enlarged from frame 2.

[A. N. Ganshin *et al*, PRL **101**, 065303 (2008)]

Rogue waves everywhere ?

- Rogue wave formation has been investigated in various frameworks, including:
  - * oceanic *freak waves* (or *ghost waves*, or rogons, or WANDTs *"Waves that Appear from Nowhere and Disappear without a Trace"*)
    [Akhmediev et al, PLA (2009); Kharif & Pelinovsky, Eur. J. Mech. B/Fluids (2003)]
  - * *surface waves* generated in water tank experiments [Chabchoub, PRL (2011)]
  - * extreme intensity events (*"rare solitons"*) in *nonlinear optics*
    [Solli et al, Nature (2007); Kibler et al, Nat. Phys. (2010) & Nature/Sci.Rep. (2012)]
  - * *errors in data communications* [Savory et al, J. Lightwave Technol. (2006)]
  - * *anomalous acoustic turbulence* in superfluid Helium [Ganshin et al, PRL (2008)]
  - * rogue cells – forerunners of metastatic cancer [Kaiser, Science (2010)]
  - * stock market dynamics: crashes, asset pricing (*Black-Scholes* theory) ...
- Unlike solitary waves (which are propagating excitations which are localized in space), rogue waves may be localized in space *and* in time (*"ghost waves"*).

Analytical models for rogue waves

- Breather-type solutions of the *nonlinear Schrödinger (NLS) equation* were proposed by Dysthe & Trulsen (*) as possible analytical models for rogue waves.

Physica Scripta. Vol. T82, 48–52, 1999

Note on Breather Type Solutions of the NLS as Models for Freak-Waves

Kristian B. Dysthe[1] and Karsten Trulsen[2†]

[1]Department of Mathematics, University of Bergen, Johs.Brunsgt.12, 5008 Bergen, Norway
[2]Instituto Pluridisciplinar, Universidad Complutense de Madrid, Paseo Juan XXIII 1, E-28040 Madrid, Spain,

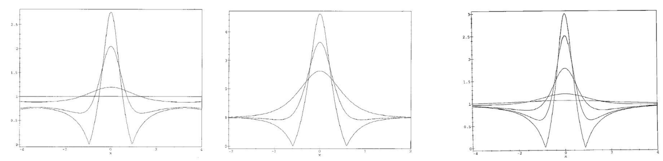

Fig. 1. The Akhmediev solution (6) for $\varphi = 0.5$. The envelope $|q|$ is shown as a function of x and t in (a), and the time evolution of one period $2\pi/p$ of the spatial profile is shown in (b).

Fig. 2. The Ma-breather (13) for $\varphi = 1.2$. The envelope $|q|$ is shown as a function of x and t in (a), and the spacial profile of the envelope through a period $2\pi/\Omega$ is shown in (b).

Fig. 3. The Peregrine solution (8). The envelope $|q|$ is shown as a function of x and t in (a), and the time evolution of the spacial profile of the envelope is shown in (b).

[(*) K.B. Dysthe and K. Trulsen, Phys. Scripta T**82**, 48 (1999)]

Rogue waves in plasmas? (1)

- The rogue wave paradigm was recently employed in plasmas as a possible mechanism for magnetic hole generation (*).

- The generation of Alfvén type freak waves described by the *Derivative Nonlinear Schrödinger (DNLS)* equation was proposed (*).

Eur. Phys. J. Special Topics **185**, 57–66 (2010)

DOI: 10.1140/epjst/e2010-01238-7

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Freak waves in laboratory and space plasmas

Freak waves in plasmas

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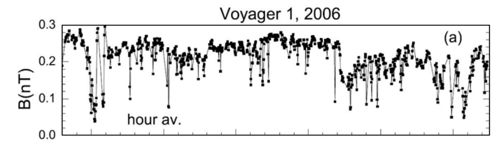


Fig. 1. Voyager 1 observations of hour averages of the magnetic field strength B in the heliosheath. The magnetic field magnitude shows many spike-like dips that are too narrow to be resolved in the hour average. Figure taken from Ref. [10].

[(*) MS Ruderman, Eur. Phys. J. Special Topics **185**, 57 (2010)]

Rogue waves in plasmas? (2)

Rogue waves have been considered recently in various plasma contexts:

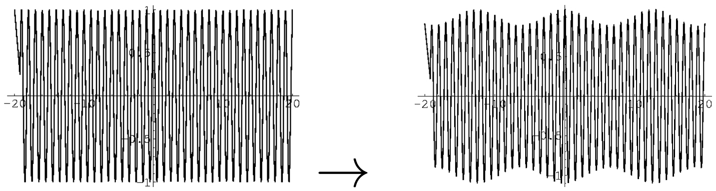
- *Alfvén-type rogue waves* [Shukla et al, Physics Letters A (2012)]
- *Langmuir rogue waves in electron-positron plasmas* [Moslem, PoP 2011]
- *Electrostatic waves in e-p-i plasmas* [El-Awady & Moslem, Phys. Plasmas 2011; El-Labany *et al*, Astrophys. Space Sci. 2012]
- *Dusty plasmas* [Abdelsalam, *et al*, Phys. Plasmas 2011; Moslem *et al*, PRE 2011]
- *Surface plasma waves* [Moslem, Shukla and Eliasson, Europhys. Lett. 2011].
- Most of these studies have relied on a phenomenological analogy between rogue waves and large amplitude solutions of nonlinear model PDEs, e.g. KdV/mKdV or NLS equations (families).

Intro #3: Nonlinear Amplitude Modulation (prerequisites)

- Harmonic variation of the amplitude of a plasma wavepacket
- Amplitude not constant, may vary weakly in space and time
- Ubiquitous nonlinear mechanism, associated with
  - * *secondary harmonic generation*
  - * *modulational instability*
  - * *envelope solitons*: localized forms with periodic internal structure
- ***Energy localization***: lumps of energy are formed and propagate in the plasma; dynamics to be harnessed for applications

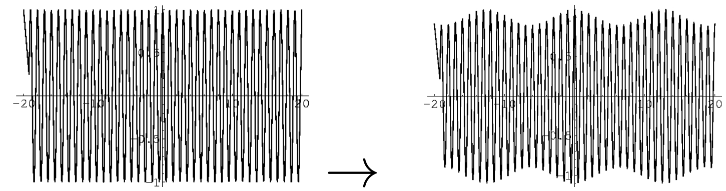
Intro: Prerequisites *(continued)*

The *amplitude* of a harmonic wave may vary in space and time:

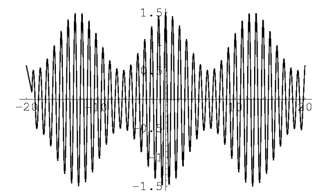


Intro: Prerequisites *(continued)*

The *amplitude* of a harmonic wave may vary in space and time:



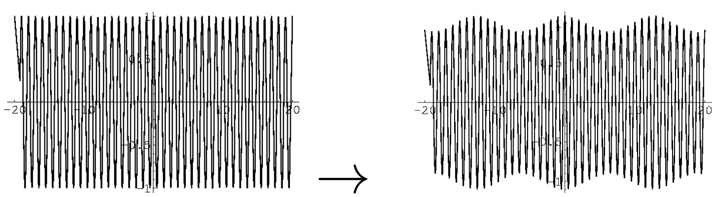
This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* (modulational instability) or ...



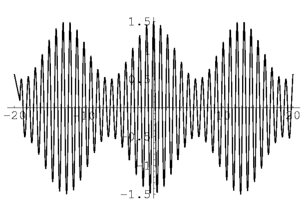
→ ?

Intro: Prerequisites *(continued)*

The *amplitude* of a harmonic wave may vary in space and time:



This *modulation* (due to nonlinearity) may be *strong* enough to lead to wave *collapse* (modulational instability) or to the formation of *envelope solitons*:



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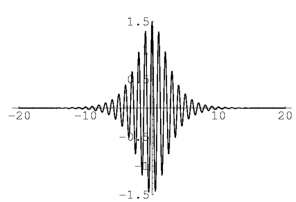
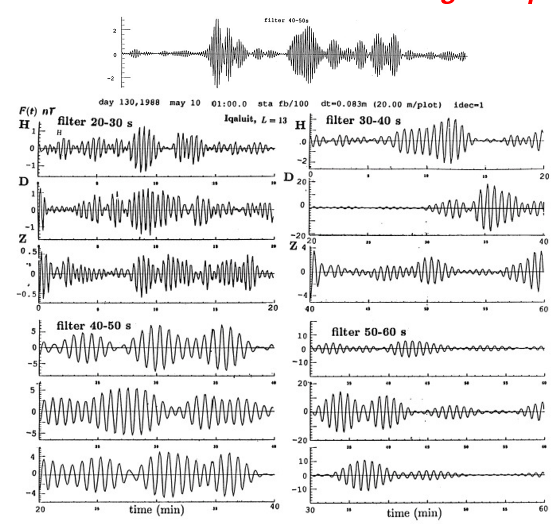


Modulated structures occur widely, e.g. in EM field measurements in the magnetosphere, ...



(from: [Ya. Alpert, Phys. Reports **339**, 323 (2001)])

Part 2: Framework for EM wave modulation

Electron fluid-dynamical evolution equations + Maxwell laws:

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial^2 \phi}{\partial t \partial x}\hat{x} + \frac{n}{\gamma}\mathbf{P}, \qquad (1)$$

$$\frac{\partial^2 \phi}{\partial x^2} = n - 1, \qquad \frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0, \qquad (2)$$

$$\frac{\partial(\mathbf{P}-\mathbf{A})}{\partial t} = \frac{\partial(\phi - \gamma)}{\partial x}\hat{x} + \mathbf{V}\times\nabla\times(\mathbf{P}-\mathbf{A}) - \mathbf{V}\times\mathbf{B_0}, \qquad (3)$$

- $\mathbf{A}$ and ϕ are the vector and scalar potentials, respectively; $\mathbf{B_0} = \Omega\,\hat{x}$ is the ambient magnetic field.
- $\mathbf{V} = \mathbf{P}/\gamma$ is the fluid velocity ($\mathbf{P}$ is the electron momentum);
- γ is the relativistic factor $\gamma = \sqrt{1+P^2}$.
- We have considered $\nabla\cdot = \frac{\partial\cdot}{\partial x}\hat{x}$;
- ALL quantities are dimensionless; we have normalized:
  - ⋆ the scalar and vector potentials by mc^2/e, the electric field $\mathbf{E}$ by $mc\omega_{pe}/e$,
  - ⋆ the magnetic field $\mathbf{B}$ by $m\omega_{pe}/e$,
  - ⋆ the momentum by mc, the density by $n_{e,0}$, the electron velocity by the speed of light c.
- Space and time are scaled by the skin length c/ω_{p0} and the inverse plasma frequency ω_{p0}^{-1}.

[G. Lehmann et al, Phys. Plasmas **13**, 092302 (2006); J. Borhanian et al, Phys. Lett. A **373**, 3667 (2009).]

We set: $\mathbf{A} = A_y\,\hat{y} + A_z\,\hat{z}\,, \qquad \mathbf{P} = P_y\,\hat{y} + P_z\,\hat{z} + \gamma u\,\hat{x}$

For CP EM pulses: $\mathbf{A}_1 = A_1(x,t)(\hat{y} + i\alpha\hat{z})\,, \mathbf{P}_1 = p_1(x,t)(\hat{y}+i\alpha\hat{z}) + \gamma u_1(x,t)$

- $p_{y,z}(x,t)$ and $\gamma\,u(x,t)$ denote the transverse and longitudinal component(s) of the electron momentum;
- $\alpha = +1$ $(\alpha = -1)$ for left- (right-) hand circularly polarized electromagnetic (CPEM) waves.

The fluid-Maxwell system of equations then becomes

$$\frac{\partial^2 A_{y,z}}{\partial x^2} - \frac{\partial^2 A_{y,z}}{\partial t^2} = \frac{n}{\gamma}p_{y,z}\,, \qquad \frac{\partial^2 \phi}{\partial t \partial x} + nu = 0 \qquad (4)$$

$$\frac{\partial^2 \phi}{\partial x^2} = n - 1\,, \qquad \frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \qquad (5)$$

$$\frac{\partial}{\partial t}(\gamma u) = \frac{\partial}{\partial x}(\phi - \gamma) + \frac{1}{\gamma}\left[p_y\frac{\partial}{\partial x}(p_y - A_y) + p_z\frac{\partial}{\partial x}(p_z - A_z)\right], \qquad (6)$$

$$\frac{\partial}{\partial t}(p_{y,z} - A_{y,z}) + u\frac{\partial}{\partial x}(p_{y,z} - A_{y,z}) = \mp\Omega\frac{p_{z,y}}{\gamma} \qquad \left(\gamma = \sqrt{\frac{1+p^2}{1-u^2}}\right) \qquad (7)$$

[cf. Kaw *et al*, PRL (1992); Esirkepov *et al*, JETP Lett. (1998); Poornakala *et al*, PoP (2002); Farina *et al*, PRL (2001) ; agreement to first order].

Multiscale perturbative technique for envelope dynamics

- Following the multiple scales (reductive perturbation) technique of Taniuti and coworkers (JMP & JPSJ 1969), we consider the stretched variables

$$X_n = \epsilon^n x\,; \quad T_n = \epsilon^n t\,; \qquad n = 0, 1, 2, ...$$

- We define the state vector $\mathbf{S} = (n, u, p, \phi, A)$, and
- proceed by expanding near the equilibrium state $\mathbf{S}^{(0)} = (1, 0, 0, 0, 0)$ as

$$\mathbf{S} = \mathbf{S}^{(0)} + \sum_{n=-\infty}^{n} \epsilon^n \mathbf{S}^{(n)}$$

where

$$\mathbf{S}^{(n)} = \sum_{l=-n}^{n} \mathbf{S}_l^{(n)}\, e^{il(kx-\omega t)}$$

denotes the amplitude of the $n-$th order contribution, as a series of the $l-$th harmonic amplitude(s) $\mathbf{S}^{(n)}_{(l)} = \mathbf{S}^{(n)}_{(l)}(X_j, T_j)$ (*slow*, for $j \geq 1$).

Perturbative scheme – results

- The leading order system ($\sim\epsilon$) gives $\phi_1^{(1)} = n_1^{(1)} = u_1^{(1)} = 0$, along with:

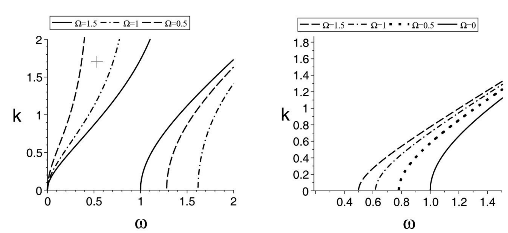
$$(\omega^2 - k^2)A_1^{(1)} = p_1^{(1)}\,, \qquad \omega(p_1^{(1)} - A_1^{(1)}) = \alpha\Omega p_1^{(1)}$$

as expected [Hasegawa, Phys. Fluids 1972; Lehmann & Spatschek, Phys. Plasmas 2006].

- *Dispersion relation*:

$$\omega^2 - k^2 = \frac{\omega}{\omega - \alpha\Omega}\,.$$

Here, α is $+1/-1$ for L-/R- CPEM waves (cf. book by Swanson 2003):



NLS equation for the vector potential (amplitude) $A_1^{(1)}$

- In order $\sim \epsilon^3$, an explicit compatibility condition is imposed for *annihilation of secular terms* (which would otherwise lead to a divergent solution).
- This analytical requirement can be expressed in the form

$$i\left(\frac{\partial A_1^{(1)}}{\partial T_2} + v_g \frac{\partial A_1^{(1)}}{\partial X_2}\right) + P\frac{\partial^2 A_1^{(1)}}{\partial X_1^2} + Q\,|A_1^{(1)}|^2 A_1^{(1)} = 0$$

- The dispersion coefficient P is given by

$$P = \frac{1}{2}\frac{d^2\omega}{dk^2} = \frac{v_g}{2k} + \frac{v_g^2}{\omega - \alpha\Omega} - \frac{(3\omega - \alpha\Omega)v_g^3}{2k(\omega - \alpha\Omega)}\,. \qquad (8)$$

- The nonlinearity coefficient Q is

$$Q = \frac{v_g}{k}(\omega^2 - k^2)^4 \qquad (9)$$

Equilibrium solution & NL frequency shift

- The NLSE admits the *harmonic wave solution* for the electric potential amplitude :

$$\psi \;=\; \hat{\psi}\, e^{iQ|\hat{\psi}|^2\tau} \;+\; \text{c.c.}$$

- The total potential disturbance then reads:

$$\phi \simeq \epsilon\, \hat{\psi} e^{iQ|\hat{\psi}|^2\tau} \exp i(kx - \omega t) + \cdots$$

which takes the form

$$\phi \simeq \epsilon\; \hat{\psi}\; \exp i[kx - (\omega - \epsilon^2 Q|\hat{\psi}|^2)\, t] + \cdots$$

- the net result is a *nonlinear frequency shift*

$$\omega \qquad \to \qquad \omega - \epsilon^2 Q|\hat{\psi}|^2$$

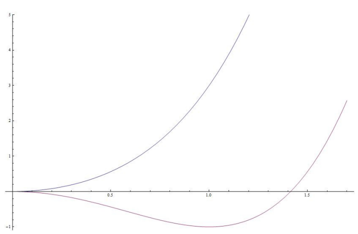
which has been verified experimentally!

Modulational (in)stability analysis

- *Perturb* the amplitude by setting: $\hat{\psi} \;=\; \hat{\psi}_0 \;+\; \epsilon\, \hat{\psi}_{1,0} \cos(\tilde{k}\zeta - \tilde{\omega}\tau)$
- We obtain the *(perturbation)* dispersion relation:

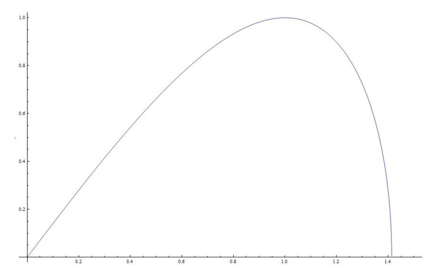
$$\tilde{\omega}^2 \;=\; P^2\, \tilde{k}^2 \left(\tilde{k}^2 \;-\; 2\,\frac{Q}{P}\,|\hat{\psi}_{1,0}|^2\right).$$

- If $PQ < 0$: the amplitude ψ is *stable* to external perturbations:



Modulational (in)stability analysis (continued...)

- If $PQ > 0$: the amplitude ψ is *unstable* for $\tilde{k} < \sqrt{2\,\frac{Q}{P}}\,|\psi_{1,0}|$.



- Maximum (instability) growth rate: $\sigma = Q|\psi_{1,0}|^2$, occurs at $\tilde{k}_m < \sqrt{\frac{Q}{P}}\,|\psi_{1,0}|$
- Instability occurs in the "window": $0 < \tilde{k} < \sqrt{2\,\frac{Q}{P}}\,|\psi_{1,0}|$.
- The wave may either "blow up", or localize its energy towards the formation of (envelope) solitons.

Localized envelope excitations (solitons)

- The NLSE accepts various solutions in the form: $\psi = \rho\, e^{i\Theta}$;

  the *total* electric potential is then: $\phi \approx \epsilon\, \rho\, \cos(\mathbf{kr} - \omega t + \Theta)$

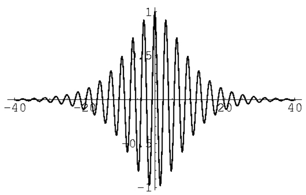
  where the *amplitude* ρ and *phase correction* Θ depend on ζ, τ.

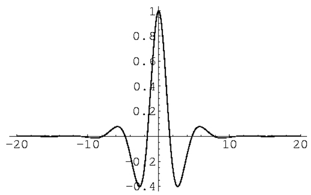
- Bright–type envelope soliton (pulse):

$$\rho = \rho_0\, \mathrm{sech}\left(\frac{\zeta - v\,\tau}{L}\right), \qquad \Theta = \frac{1}{2P}\left[v\,\zeta - (\Omega + \frac{1}{2}v^2)\tau\right].$$

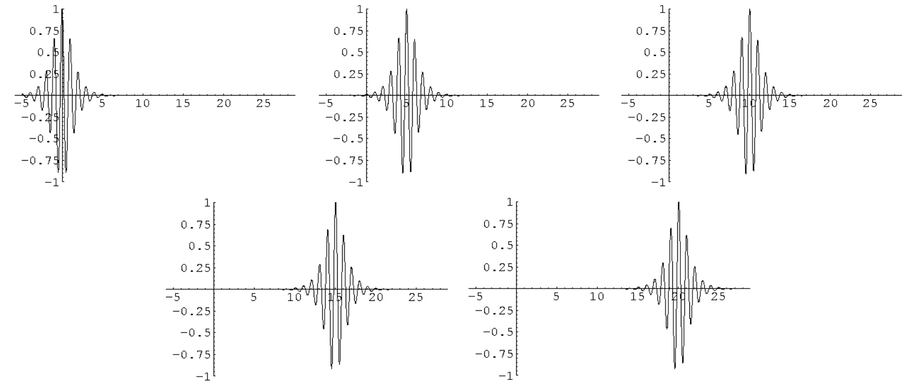
$L = \sqrt{\frac{2P}{Q}\frac{1}{\rho_0}}$

This is a propagating (and *oscillating*) localized pulse:

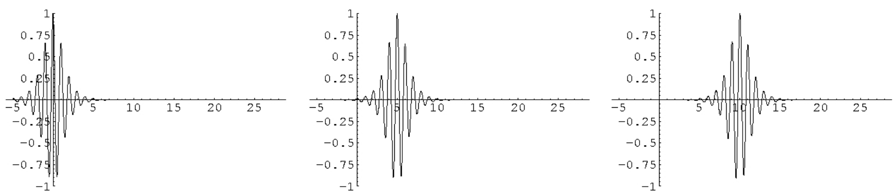




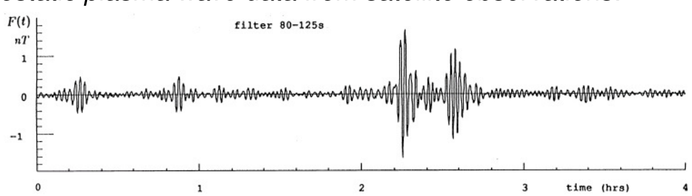
Propagation of a bright envelope soliton (pulse)



Propagation of a bright envelope soliton (pulse)

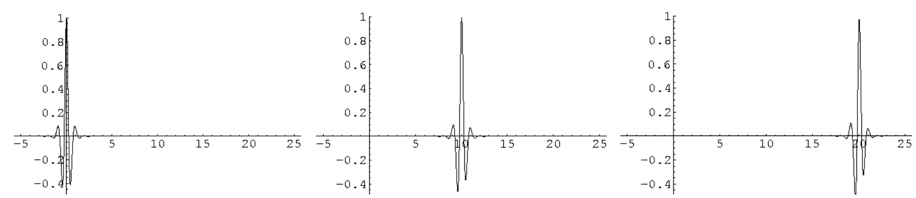


Cf. electrostatic plasma wave data from satellite observations:

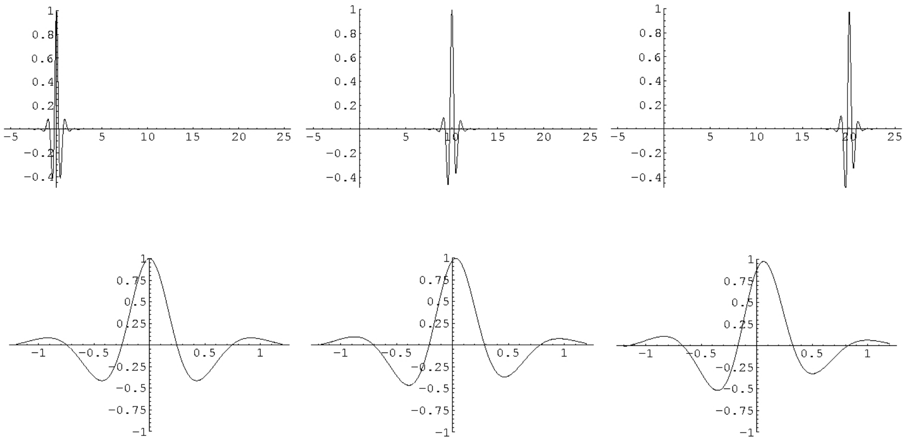


(from: [Ya. Alpert, Phys. Reports **339**, 323 (2001)])

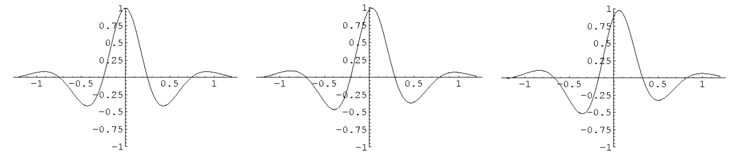
Propagation of a bright envelope soliton (continued...)



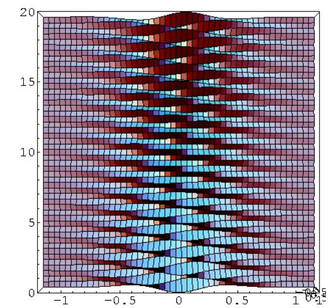
Propagation of a bright envelope soliton (continued...)



Propagation of a bright envelope soliton (continued...)



Rem.: *Time-dependent phase* $\rightarrow$ *breathing* effect (at rest frame):



Localized envelope excitations

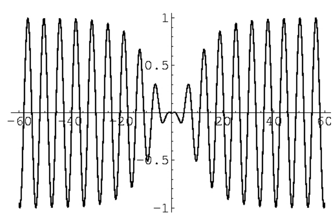
- Dark–type envelope solution (*hole soliton*):

$$\rho = \pm\rho_1 \left[1 - \mathrm{sech}^2\left(\frac{\zeta - v\tau}{L'}\right)\right]^{1/2} = \pm\rho_1 \tanh\left(\frac{\zeta - v\,\tau}{L'}\right),$$

$$\Theta = \frac{1}{2P}\left[v\,\zeta - \left(\frac{1}{2}v^2 - 2PQ\rho_1^2\right)\tau\right]$$

$$L' = \sqrt{2\left|\frac{P}{Q}\right|}\frac{1}{\rho_1}$$

This is a *propagating localized* hole (zero density void):



Localized envelope excitations

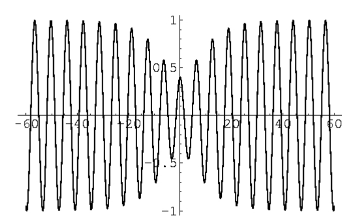
- Grey–type envelope solution (*void soliton*):

$$\rho = \pm\rho_2 \left[1 - a^2\,\mathrm{sech}^2\left(\frac{\zeta - v\,\tau}{L''}\right)\right]^{1/2}$$

$$\Theta = \ldots$$

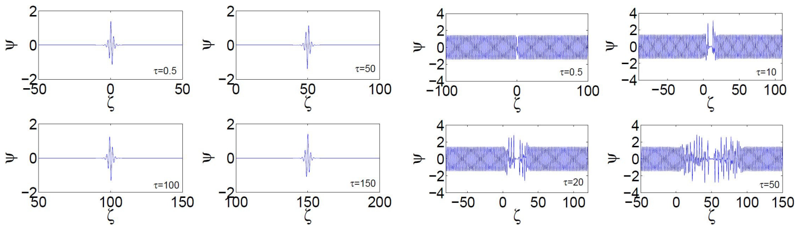
$$L'' = \sqrt{2\left|\frac{P}{Q}\right|}\frac{1}{a\rho_2}$$

This is a propagating *(non zero-density)* void:

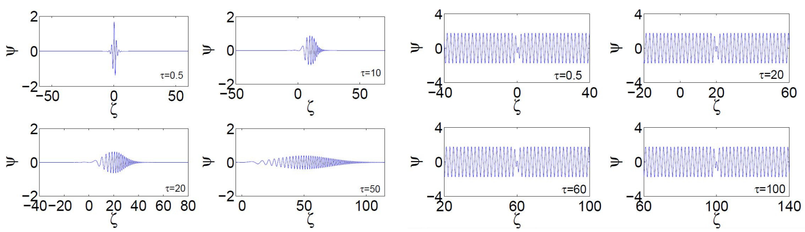


Envelope solitons in action (1): *anomalous* vs. *normal* dispersion

Case $PQ > 0$ (**"Anomalous dispersion"**): stable bright (left plot)/ unstable dark (right plot) envelopes:



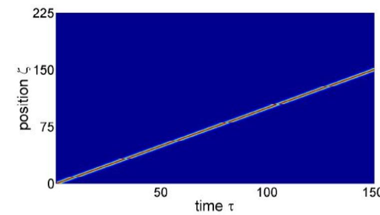
Case $PQ < 0$ (**"Normal dispersion"**): unstable bright (left plot) / stable dark (right plot) envelopes:

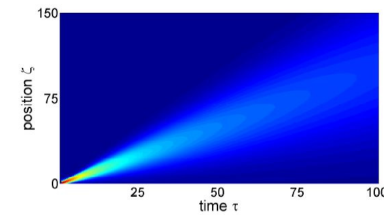


[Numerical results by Sharmin Sultana, Queen's University Belfast (2011)]

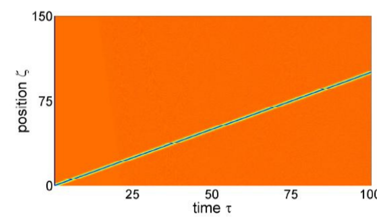
Envelope solitons in action (2): *anomalous* vs. *normal* dispersion

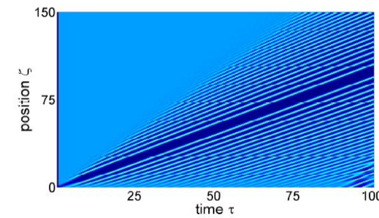
Bright envelope solitons on the space-time plane: stable vs unstable:





Dark-type envelope solitons on the space-time plane: stable vs unstable:





[Numerical results by Sharmin Sultana, Queen's University Belfast (2011).]

Modulational (in)stability: parametric dependence on Ω

The magnetic field may either *enhance MI* (LCP, $\Omega < \omega_{p,e}$; top left plot) or -generally- *suppress MI* (reduced growth rate for higher Ω).

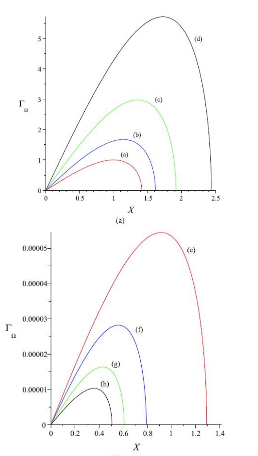


Fig. 2. Variation of the growth rate Γ_Ω (LCP wave) with X for different values of Ω with $\omega = 1.6$. (a) $\Omega = 0$, (b) $\Omega = 0.2$, (c) $\Omega = 0.4$, (d) $\Omega = 0.6$, (e) $\Omega = 25.0$, (f) $\Omega = 30.0$, (g) $\Omega = 35.0$, and (h) $\Omega = 40.0$. (Color online.)

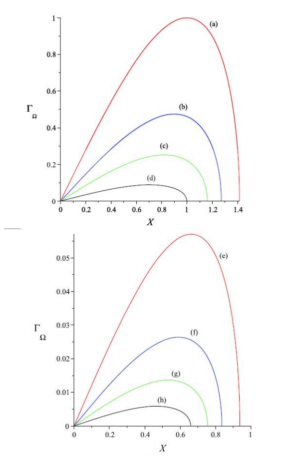
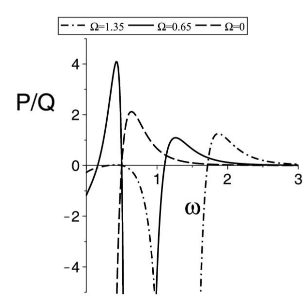
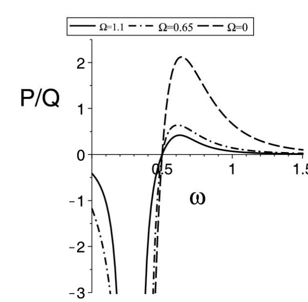


Fig. 3. Variation of the growth rate Γ_Ω (RCP wave) with X for different values of Ω with $\omega = 5.0$. (a) $\Omega = 0$, (b) $\Omega = 1.0$, (c) $\Omega = 2.0$, (d) $\Omega = 4.0$, (e) $\Omega = 5.0$, (f) $\Omega = 7.0$, (g) $\Omega = 9.0$, (h) $\Omega = 12.0$. (Color online.)

Variation of P/Q with ω (for variable Ω)

- $PQ < 0$ (wavepackets stable) for low ω; MI above threshold ω_{cr}
- L- CPEM: Lower instability threshold for weakly magnetized plasma ($\omega_c < \omega_p$); higher for strongly magnetized plasma ($\omega_c > \omega_p$)
- R- CPEM: Instability threshold practically stable, but strong dependence of P/Q ratio ($\rightarrow$ soliton width) on Ω.





[G. Veldes, J. Borhanian, V. Saxena, D. Franzeskakis and I. Kourakis, in preparation (2013)]

Part 3: Analytical models for rogue waves

Various solutions of the NLS equation have been proposed as model candidates for rogue waves.

We distinguish:

- The *Peregrine soliton*
  [D. H. Peregrine, J. Austral. Math. Soc. B **25**, 16 (1983); K. B. Dysthe, and K. Trulsen, Physica Scripta **T82**, 48 (1999); V. I. Shrira, and V. V. Geogjaev, J. Eng. Math. **67**, 11 (2010); B. Kibler, J. Fatome, et al., Nature Physics **6**, 790 (2010)]

- The *Kuznetsov-Ma breather*
  [Ya C. Ma, Stud. Appl. Math. **60**, 43 (1979)];

- The *Akhmediev breather*
  [N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Theor. Math. Phys. 72, 809 (1987)];

In the following, we have considered the above paradigms, with an aim to investigate their dependence on relevant plasma parameters (Ω in particular).

Peregrine Soliton as a model for rogue waves

- As a first approach to rogue waves, we consider the Peregrine soliton:

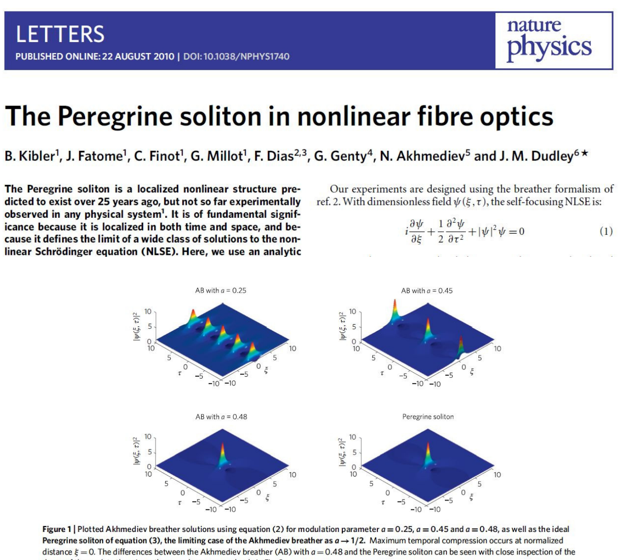
$$\psi(\xi,\tau) = \left[1 - \frac{4(1+i2Q\tau)}{1+2Q\xi^2/P+4Q^2\tau^2}\right] \exp\left(iQ\tau\right)$$

[D. H. Peregrine, J. Austral. Math. Soc. B **25**, 16 (1983); K. B. Dysthe & K. Trulsen, Physica Scripta **T82**, 48 (1999); V. I. Shrira & V. V. Geogjaev, J. Eng. Math. **67**, 11 (2010); B. Kibler, *et al.*, Nat. Phys. **6**, 790 (2010)]

- The Peregrine paradigm as a prototypical model for rogue waves has recently been employed successfully in NL optics [Kibler *et al*, Nat. Phys. (2010)];

- Recalling the functional dependence of P and Q on plasma parameters, this model allows one to investigate the parametric dependence on the magnetic field Ω and wavenumber k (reduced variables).

- *Ab initio* analytical predictions, numerical confirmation.


LETTERS
PUBLISHED ONLINE: 22 AUGUST 2010 | DOI: 10.1038/NPHYS1740
nature physics

The Peregrine soliton in nonlinear fibre optics

B. Kibler[1], J. Fatome[1], C. Finot[1], G. Millot[1], F. Dias[2,3], G. Genty[4], N. Akhmediev[5] and J. M. Dudley[6]★

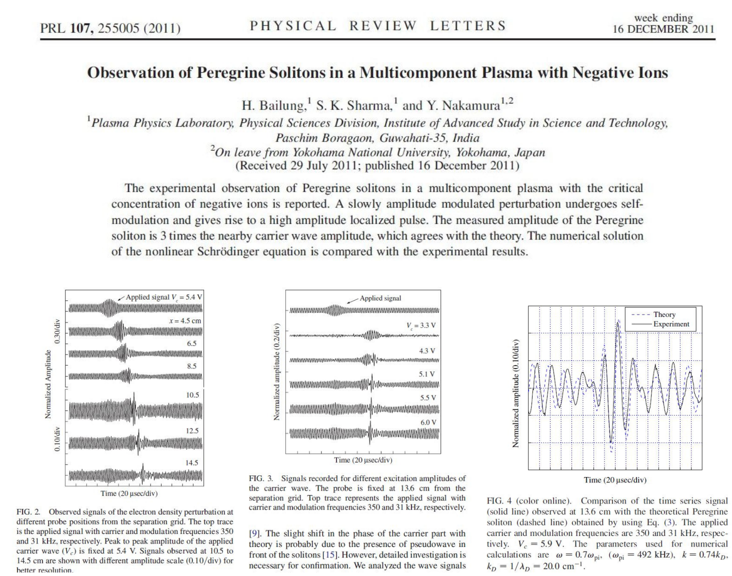
The Peregrine soliton is a localized nonlinear structure predicted to exist over 25 years ago, but not so far experimentally observed in any physical system[1]. It is of fundamental significance because it is localized in both time and space, and because it defines the limit of a wide class of solutions to the nonlinear Schrödinger equation (NLSE). Here, we use an analytic

Our experiments are designed using the breather formalism of ref. 2. With dimensionless field $\psi(\xi,\tau)$, the self-focusing NLSE is:

$$i\frac{\partial\psi}{\partial\xi} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2} + |\psi|^2\psi = 0 \quad (1)$$

Figure 1 | Plotted Akhmediev breather solutions using equation (2) for modulation parameter $a = 0.25$, $a = 0.45$ and $a = 0.48$, as well as the ideal Peregrine soliton of equation (3), the limiting case of the Akhmediev breather as $a \to 1/2$. Maximum temporal compression occurs at normalized distance $\xi = 0$. The differences between the Akhmediev breather (AB) with $a = 0.48$ and the Peregrine soliton can be seen with close inspection of the decay of the peak to the wings; they are shown more clearly in Fig. 2.

[B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev & JM Dudley, Nat. Phys. **6**, 790 (2010)]


PRL **107,** 255005 (2011) PHYSICAL REVIEW LETTERS week ending 16 DECEMBER 2011

Observation of Peregrine Solitons in a Multicomponent Plasma with Negative Ions

H. Bailung,[1] S. K. Sharma,[1] and Y. Nakamura[1,2]
[1]*Plasma Physics Laboratory, Physical Sciences Division, Institute of Advanced Study in Science and Technology, Paschim Boragaon, Guwahati-35, India*
[2]*On leave from Yokohama National University, Yokohama, Japan*
(Received 29 July 2011; published 16 December 2011)

The experimental observation of Peregrine solitons in a multicomponent plasma with the critical concentration of negative ions is reported. A slowly amplitude modulated perturbation undergoes self-modulation and gives rise to a high amplitude localized pulse. The measured amplitude of the Peregrine soliton is 3 times the nearby carrier wave amplitude, which agrees with the theory. The numerical solution of the nonlinear Schrödinger equation is compared with the experimental results.

FIG. 2. Observed signals of the electron density perturbation at different probe positions from the separation grid. The top trace is the applied signal with carrier and modulation frequencies 350 and 31 kHz, respectively. Peak to peak amplitude of the applied carrier wave (V_c) is fixed at 5.4 V. Signals observed at 10.5 to 14.5 cm are shown with different amplitude scale (0.10/div) for better resolution.

FIG. 3. Signals recorded for different excitation amplitudes of the carrier wave. The probe is fixed at 13.6 cm from the separation grid. Top trace represents the applied signal with carrier and modulation frequencies 350 and 31 kHz, respectively.

[9]. The slight shift in the phase of the carrier part with theory is probably due to the presence of pseudowave in front of the solitons [15]. However, detailed investigation is necessary for confirmation. We analyzed the wave signals

FIG. 4 (color online). Comparison of the time series signal (solid line) observed at 13.6 cm with the theoretical Peregrine soliton (dashed line) obtained by using Eq. (3). The applied carrier and modulation frequencies are 350 and 31 kHz, respectively. $V_c = 5.9$ V. The parameters used for numerical calculations are $\omega = 0.7\omega_{pi}$, ($\omega_{pi} = 492$ kHz), $k = 0.74k_D$, $k_D = 1/\lambda_D = 20.0\ cm^{-1}$.

[H. Bailung, S.K. Sharma and Y. Nakamura, PRL 107, 255005 (2011)]

Kuznetsov-Ma breather as a model for rogue waves

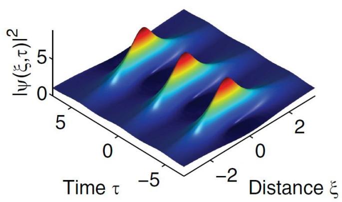
- *Kuznetsov - Ma breather*:

$$\psi(\xi,\tau) = \left[\frac{\cos(\frac{1}{2}s'Q\tau - 2i\phi) - \cosh\phi\cosh(s\sqrt{\frac{Q}{2P}}\xi)}{\cos(\frac{1}{2}s'Q\tau) - \cosh\phi\cosh(s\sqrt{\frac{Q}{2P}}\xi)}\right]\exp(iQ\tau)$$

where $\phi \in \Re$, $s = 2\sinh\phi$, $s' = 2\sinh(2\phi)$

[Credit: Ya C. Ma, *Stud. Appl. Math.* **60**, 43 (1979).]

- The KM breather was observed in optical fibers [Kibler *et al*, Nature/Sci. Rep. (2012)];



Akhmediev breather as a model for rogue waves

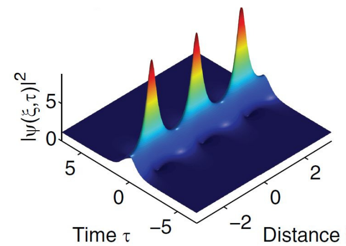
$$\psi(\xi,\tau) = \left[1 + \frac{2(1-2a)\cosh(bQ\tau) + ib\sinh(bQ\tau)}{\sqrt{2a}\cos(\omega\sqrt{\frac{Q}{2P}}\xi) - \cosh(bQ\tau)}\right]\exp(iQ\tau)$$

where

$$\alpha \in (0, 1/2], \qquad \omega = 2\sqrt{1-2\alpha}, \qquad b = \sqrt{8a(1-2a)} \;.$$

[Credit: N. Akhmediev, V. M. Eleonskii and N. E. Kulagin, Theor. Math. Phys. **72**, 809 (1987).]

- The A-breather is periodic in space, but localized in time:



[Figure from: Kibler *et al*, Nat. Phys. (2010) & Nature/Sci.Rep. (2012).]

Akhmediev breather as a model for rogue waves

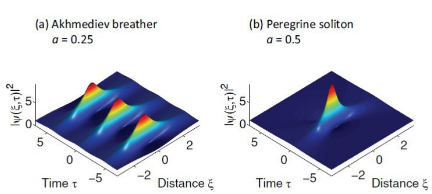
$$\psi(\xi,\tau) = \left[1 + \frac{2(1-2a)\cosh(bQ\tau) + ib\sinh(bQ\tau)}{\sqrt{2a}\cos(\omega\sqrt{\frac{Q}{2P}}\xi) - \cosh(bQ\tau)}\right]\exp(iQ\tau)$$

where

$$\alpha \in (0, 1/2], \qquad \omega = 2\sqrt{1-2\alpha}, \qquad b = \sqrt{8a(1-2a)} \;.$$

- The *Peregrine* soliton is recovered in some (aperiodic) limit:

$$\psi_P = \lim_{a\to\frac{1}{2}} \psi_A = e^{iq\tau}\left[1 - \frac{4(1+2iq\tau)}{1 + \frac{2q}{p}\xi^2 + 4q^2\tau^2}\right]$$



[Credit: Kibler *et al*, Nat. Phys. (2010) & Nature/Sci.Rep. (2012).]

Parametric analysis of CPEM rogue waves in plasmas: three (3) dispersion modes

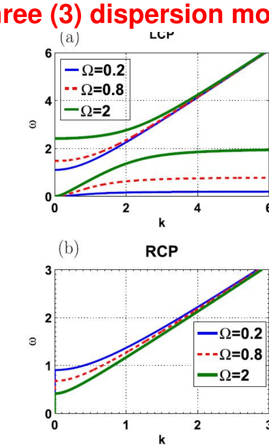


Figure 1. The dispersion relation showing the normalized frequency ω as a function of the normalized wavenumber k for LCP—(top panel) and RCP—(bottom panel) EM waves. The thin solid (blue), dashed (red) and bold (green) lines show the dispersion relation for different values of Ω, i.e., $\Omega = 0.1$, $\Omega = 0.8$, and $\Omega = 2$, respectively.

[G. Veldes, J. Borhanian, V. Saxena, D.J. Frantzeskakis and I. Kourakis, to appear in *J. Optics* (IoP) (2013).]

Parametric analysis (1- LCP/low frequency)

- Rogon management/tuning by the magnetic field (via $\Omega = \omega_c/\omega_p$);
- The magnetic field suppresses the spatial extension of breathers, *and*
- ... reduces the time duration in all 3 cases.

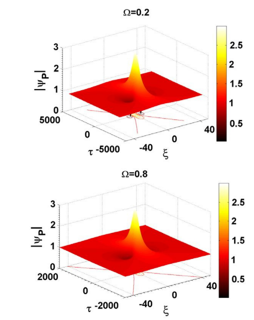


Figure 5. Peregrine soliton for LCP waves, in the low frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$.

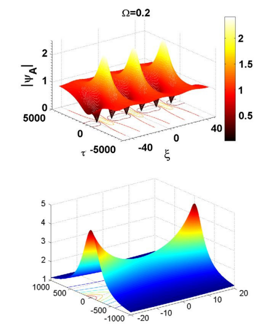


Figure 6. Akhmediev breather for LCP waves, in the low frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter α takes the value $\alpha = 0.25$.

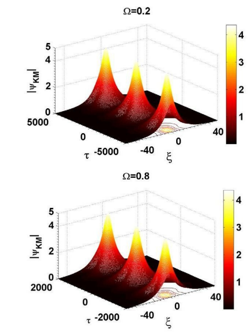


Figure 7. Kuznetsov–Ma breather for LCP waves, in the low frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter s take the value $s = 0.7$.

[G. Veldes, J. Borhanian, V. Saxena, D.J. Frantzeskakis and I. Kourakis, to appear in *J. Optics* (IoP) (2013).]

Parametric analysis (2- LCP/high frequency)

- Rogon management/tuning by the magnetic field (via $\Omega = \omega_c/\omega_p$);
- The magnetic field suppresses the spatial extension of breathers, *and*
- ... reduces the time duration in all 3 cases.

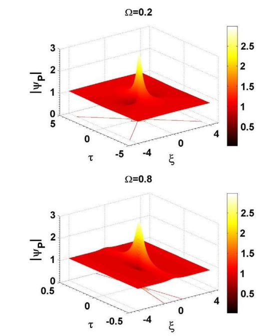


Figure 8. Peregrine soliton for LCP waves, in the high frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$.

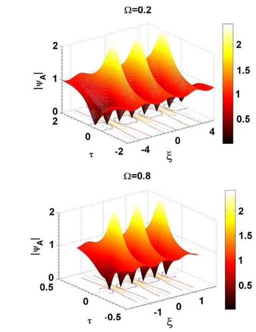


Figure 9. Akhmediev breather for LCP waves, in the high frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter α takes the value $\alpha = 0.25$.

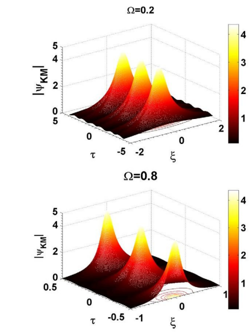


Figure 10. Kuznetsov–Ma breather for LCP waves, in the high frequency band, for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter s take the value $s = 0.7$.

[G. Veldes, J. Borhanian, V. Saxena, D.J. Frantzeskakis and I. Kourakis, to appear in *J. Optics* (IoP) (2013).]

Parametric analysis (3-RCP)

- RCP rogons are less localized for stronger magnetic fields!
- The magnetic field *stretches* the breathers in space, *and* also
- ... *extends* their time duration (in all three models).

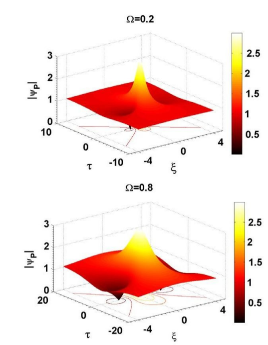


Figure 11. Peregrine soliton for RCP waves for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$.

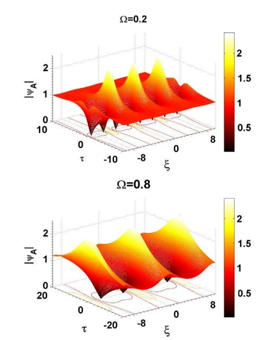


Figure 12. Akhmediev breather for RCP waves for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter α take the value $\alpha = 0.25$.

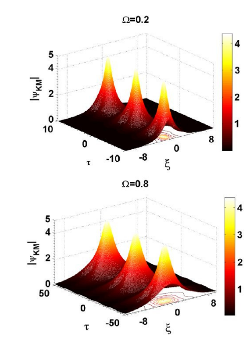


Figure 13. Kuznetsov–Ma breather for RCP waves for different values of Ω, i.e. $\Omega = 0.2$ (top panel) and $\Omega = 0.8$ (bottom panel). The wavenumber k takes the value $k = 0.5$ and the parameter s take the value $s = 0.7$.

[G. Veldes, J. Borhanian, V. Saxena, D.J. Frantzeskakis and I. Kourakis, to appear in *J. Optics* (IoP) (2013).]

Standing Esirkepov-type EM soliton interaction (1)

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Interaction of spatially overlapping standing electromagnetic solitons in plasmas

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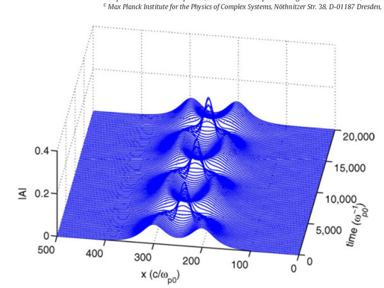


Fig. 2. The temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega = 0.999$ and initial distance $d = 100c/\omega_{p0}$.

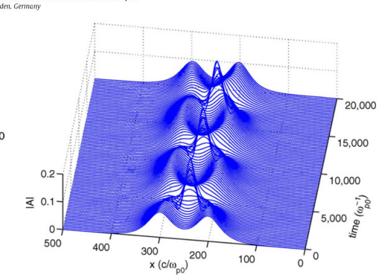
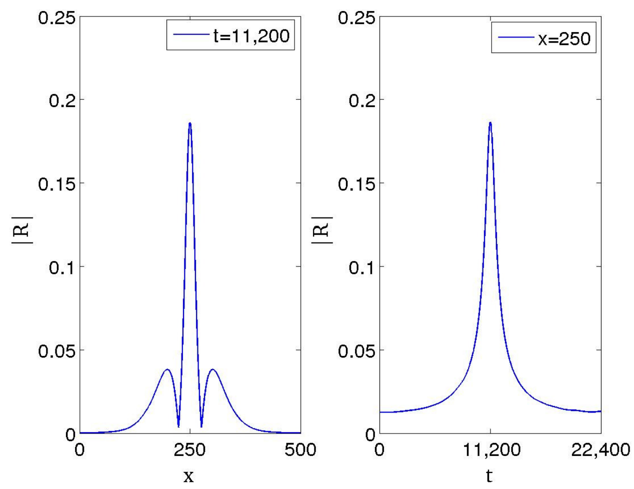


Fig. 5. Top panel: the temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega_1 = \omega_2 = 0.999$ but with a finite phase difference $\Delta\theta = 0.1\pi$. Bottom panel: the same as in the top panel, but with $\Delta\theta = 10^{-2}\pi$.

[V. Saxena, I. Kourakis, G. Sanchez-Arriaga, E. Siminos, *Phys. Lett. A*, **377**, 473 (2013).]

Standing Esirkepov-type EM soliton interaction (2)



[V. Saxena, I. Kourakis, G. Sanchez-Arriaga, E. Siminos, *Phys. Lett. A*, **377**, 473 (2013).]

Conclusions & Summary

- Multiscale methodology for EM relativistic solitons revisited
- Powerful analytical technique, provides predictions for
  – Modulational Instability thresholds and growth rate
  – Envelope modes, harmonic generation, rogue waves
- Efficient analytical toolbox for *Rogue Waves* in laser-plasma interactions
- *Rogue waves* are random events, may be tedious to detect experimentally;
- Results to be compared with large-amplitude theory (e.g., Kaw-Sen-Katsouleas or Farina-Bulanov formalism)
- Static predictions so far; need for dynamical (numerical) investigation.
- *Work in progress:* fluid simulations, PIC simulations, to be tested against theory.

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- ***Mark Dieckmann*** , Linkoping University, Sweden
- ***Jafar Borhanian***, Uni. Ardabil, Iran

– *My students:* Gina Williams, Michael McKerr, Niall Tyndall, ... (QUB)

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