FOKKER-PLANCK EQUATION FOR A TEST-PARTICLE IN WEAKLY COUPLED MAGNETIZED PLASMA

I. Kourakis

Université Libre de Bruxelles, Physique Statistique et Plasmas
Association Euratom - Etat Belge
C.P. 231, Boulevard du Triomphe, B-1050 Brussels, Belgium
email: ikouraki@ulb.ac.be

Abstract

The derivation of a kinetic equation for a charged test-particle (t.p.) weakly interacting with an electrostatic plasma in thermal equilibrium, subject to a uniform external magnetic field, is considered. From the generalized master equation a Fokker-Planck-type equation follows as a "markovian" approximation. Such an equation does not preserve the positivity of the distribution function. Applying techniques developed in the theory of open systems, a correct Fokker-Planck equation is derived. Explicit expressions for the diffusion and drift coefficients, depending on the magnetic field, are obtained.

1. Introduction

Methods from non-equilibrium statistical mechanics have often been used in the past to derive a kinetic equation for magnetized plasma. The starting point of such studies has been either the BBGKY hierarchy of equations for reduced distribution functions (rdf) or formal projection-operator methods. In a generic manner, both approaches rely on a ('non-markovian') generalized master equation (GME). A Fokker-Planck-type equation is derived from the GME as a "markovian" approximation. Such an equation does not preserve the positivity of the distribution function $f(\mathbf{x}, \mathbf{v}; t)$. In this work, this problem is exposed in the simple case of a uniform external field, and an alternative approach is considered.

We consider a test-particle (t.p.) Σ (charge $e_\Sigma = e$, mass $m_\Sigma = m$) surrounded by (and weakly coupled to) a homogeneous background plasma (N particles, of species α_j) (the reservoir 'R'). The whole system is subject to a uniform external magnetic field.

The equations of motion for the test-particle are:

$$\dot{\mathbf{x}} = \mathbf{v} \; ; \qquad \dot{\mathbf{v}} = \frac{1}{m}\,[\frac{e}{c}(\mathbf{v} \times \mathbf{B}) \,+\, \lambda\,\mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t)] \tag{1}$$

$\mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t)$, which is due to interactions between the t.p. and the reservoir particles surrounding it may be viewed as a 'stochastic' forces, given that the reservoir will be assumed to be in statistical equilibrium state. As obvious, $\mathbf{X} = (\mathbf{x}, \mathbf{v}) \equiv (\mathbf{x_\Sigma}, \mathbf{v_\Sigma})$ and $\mathbf{X_R} \equiv \{\mathbf{X_j}\} = \{(\mathbf{x_j}, \mathbf{v_j})\}$, $(j = 1, 2, \ldots, N)$ denote the coordinates of the test- (Σ-) and reservoir ('R'-) particles respectively. The zeroth-order (in λ) problem of motion yields the well-known (helicoidal) solution.

The test-particle's rdf $f(\mathbf{x}, \mathbf{v}; t) = (I, \rho)_R \equiv \int_{\Gamma_R} d\mathbf{X_R}\, \rho(\{\mathbf{X}, \mathbf{X_R}\}; t)$ is defined through a projection:

$$I\!\!E\, \rho = \sigma_R\, f$$

(note that $I\!\!E = I\!\!E^2$) where $\rho = \rho(\mathbf{X}, \mathbf{X_R})$ ($\sigma = \sigma(\mathbf{X_R})$) denotes the total (reservoir) phase-space distribution function, which is normalized to unity: $\int d\mathbf{X}\, \rho = 1$ ($\int d\mathbf{X_R}\, \sigma = 1$). Let us point out that $\mathbf{F_{int}}$ thus comes out to be described by a stationary Gaussian process, determined by a vanishing mean-value. By introducing the Fourier transformation of the interaction potential $V(r)$, the force correlation matrix takes an explicit form in terms of the solution of the dynamical problem; in principle, that is, the external field explicitly enters the 'collision part' of the evolution equation for f.

The equation of continuity in phase space reads:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{v}}(\frac{1}{m}\mathbf{F}\, f) \;=\; 0 \qquad (2)$$

(i.e. $\partial_t f = L_0\, f \;+\; \lambda\, L_{int}\, f \equiv Lf$; cf. (1)). In the weak-coupling approximation (i.e. $\lambda \ll 1$) a (non-markovian) generalized master equation (GME) is obtained to order λ^2 (note that the Vlasov term, in λ^1, disappears once the reservoir is taken to be homogeneous). The standard 'markovianization' method consists in substituting with the zeroth-order solution, assuming that $f(t-\tau) \approx e^{-L_0 \tau}\, f(t) \;\equiv\; U(-\tau)\, f(t)$, and then evaluating the kernel asymptotically. In the homogeneous case (i.e. $f = f(\mathbf{v};\, t)$) one thus obtains the 2nd order PDE [1]:

$$\frac{\partial f}{\partial t} \;+\; \frac{1}{m}(\mathbf{v} \times \mathbf{B})\frac{\partial f}{\partial \mathbf{v}} \;=\; \frac{\partial}{\partial \mathbf{v}}[\mathbf{A}(\mathbf{v})\frac{\partial}{\partial \mathbf{v}} \;+\; \mu\, \mathbf{a}(\mathbf{v})]\, f \qquad (3)$$

where $A_{11} = A_{22} \equiv D_\perp(\mathbf{v})$, $A_{12} = -A_{21} \equiv D_\angle(\mathbf{v})$, $A_{33} \equiv D_\|(\mathbf{v})$, $A_{i3} = A_{3i} = 0 \quad (i = 1, 2)$ and $\mathbf{a}$ is a 3d vector whose form will be omitted (the field was assumed to lie along the z-direction); note that the rhs can be re-arranged into the form of a 'Landau-Fokker-Planck'-type equation [2]:

$$rhs(3) \;=\; \frac{\partial^2}{\partial v^r\, \partial v^s}\, [A_{rs}(\mathbf{v})\, f] \;-\; \frac{\partial}{\partial v^r}\, [\mathcal{F}_r(\mathbf{v})\, f]$$

In the general case ($f = f(\mathbf{x}, \mathbf{v};\, t)$) we find:

$$\frac{\partial f}{\partial t} \;+\; \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} \;+\; \frac{1}{m}(\mathbf{v} \times \mathbf{B})\frac{\partial f}{\partial \mathbf{v}} \;=\; \frac{\partial}{\partial \mathbf{v}}[\mathbf{A}(\mathbf{v})\frac{\partial}{\partial \mathbf{v}} \;+\; \mathbf{C}(\mathbf{v})\frac{\partial}{\partial \mathbf{x}} \;+\; \mu\, \mathbf{a}(\mathbf{v})]\, f \qquad (4)$$

(cf.[3]); $\mu \equiv m/m_1^\alpha$. The exact expressions for the coefficients in eqs. (3), (4) are too lengthy to report here; in fact, they can be found in previous studies (cf. [1], [3]; our results are in full agreement with expressions therein). The point we want to make is that equation (4) does **not** preserve the positivity of the d.f., as the 2nd order matrix is not positive definite (because of the second term in the rhs [4]).

In search for a correct markovian approximation, we have considered the averaging operator:

$$\mathcal{A}_{t'} \cdot \; = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \, U(-t') \, \cdot \, U(t') \tag{5}$$

which was applied to the rhs of Eq. (4). The (markovian) evolution operator thus defined was first introduced in the theory of quantum open systems [5] and was later implemented in classical systems [6]. As a matter of fact, the implementation of this operator seems to be well defined for classical subsystems possessing a discrete spectrum of eigenvalues of the corresponding Liouville operator (cf.[6]); yet, this is not the case for free particle motion. It was therefore expected (and indeed verified) that a problem would probably arise in the z-direction as the magnetic field does not confine motion along z (the Lorentz force yields no component along the field). For this reason, we shall only consider distribution functions which do not depend on z (actually looking into the plane $\perp \mathbf{B}$).

The result in the homogeneous case coincides with Eq. (3). In the general case, however, the change is rather dramatic; for a single-species plasma we find the equation:

$$\begin{aligned}
\frac{\partial f}{\partial t} \;&+\; \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc}\,(\mathbf{v}\times\mathbf{B})\,\frac{\partial f}{\partial \mathbf{v}} \;= \\
&= \; \left[\left(\frac{\partial}{\partial v_x} + s\Omega^{-1}\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial v_y} - s\Omega^{-1}\frac{\partial}{\partial x}\right)^2\right]\left[D_\perp(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)\right] \\
&\quad + \frac{\partial^2}{\partial v_z^2}[D_\parallel(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)] + \Omega^{-2}\, Q(\mathbf{v})(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f(\mathbf{x},\mathbf{v};t) \\
&\quad - \left(\frac{\partial}{\partial v_x} + s\Omega^{-1}\frac{\partial}{\partial y}\right)\left[\mathcal{F}_x(\mathbf{v})\, f(\mathbf{x},\mathbf{v};t)\right] - \left(\frac{\partial}{\partial v_y} - s\Omega^{-1}\frac{\partial}{\partial x}\right)\left[\mathcal{F}_y(\mathbf{v})\, f(\mathbf{x},\mathbf{v};t)\right] \\
&\quad - \frac{\partial}{\partial v_z}\left[\mathcal{F}_z(\mathbf{v})\, f(\mathbf{x},\mathbf{v};t)\right]
\end{aligned} \tag{6}$$

A_{ij}, $\mathcal{F}_i$ are the same as in (3), (4); the expression for $Q(\mathbf{v})$ is too lengthy to present here. All coefficients, actually functions of $\{v_\perp,\, v_\parallel;\, \Omega\}$ $(a_\perp \equiv (a_x^2 + a_y^2)^{1/2}$, $\hat{a} \equiv \frac{1}{|\mathbf{a}|}\mathbf{a}$ $\quad \forall \mathbf{a} \in \Re^3$; $\Omega \equiv \frac{eB}{mc}$), can be analytically evaluated in a convenient reference frame e.g. $\{\hat{e}_1,\, \hat{e}_2,\, \hat{e}_3\} = \{\hat{b},\, \hat{v}_\perp,\, \hat{b}\times\hat{v}_\perp\}$.

Preservation of the positivity of f(**x**,**v**; t) by Eq. (6) can be readily verified [7].

2. Conclusions

In conclusion, Eq. (6) provides a correct kinetic description, from first principles, of the dynamics of magnetized plasma, at least up to second order in the (weak) interaction. In the homogeneous case, the well-known previous result is obtained; furthermore, in the absence of external field, the Landau equation is recovered. Realistic generalizations, taking into account field-inhomogeneities and/or geometry, are definitely imposed and work in this direction is in progress.

Acknowledgements

The author would like to acknowledge a number of fruitful discussions with Prof. Alkis Grecos, who actually suggested the subject and supervised most of this work.
Part of this work was funded by an EU grant (Fusion Programme).

References

[1] P.P.J.M. Schram: Physica **45** (1969) 165; D. Montgomery, L. Turner and G. Joyce: Phys. Fluids **17** (5) (1974) 954.

[2] R. Balescu: Statistical Mechanics of Charged Particles. Wiley, 1963.

[3] *see for instance in* A.H. Øien: J. Plasma Phys. **53** (part 1) (1995) 31; P. Ghendrih: PhD Thesis, 1987, Université de Paris-Sud (Orsay).

[4] In order to see this, express the second order terms in the rhs as: $\frac{\partial}{\partial \mathbf{q}} \underline{\underline{\mathbf{D}}} \frac{\partial f}{\partial \mathbf{q}}$, where $\mathbf{q} \equiv \{\mathbf{x}, \mathbf{v}\} \in \Re^6$ and the 6×6 2nd-order matrix is $\underline{\underline{\mathbf{D}}} \equiv \begin{pmatrix} \underline{\underline{\mathbf{0}}} & \frac{1}{2}\underline{\underline{\mathbf{C}}}^{\mathbf{T}} \\ \frac{1}{2}\underline{\underline{\mathbf{C}}} & \underline{\underline{\mathbf{A}}} \end{pmatrix}$.

[5] E.B. Davies: Comm. Math. Physics **39** (1974) 91; note that the operator defined in (5) is called *'the Davies device'* in N.G. Van Kampen: *Stochastic Processes in Physics and Chemistry*, North-Holland 1992; yet, curiously enough, the existence of the problem in the **classical** case is not addressed therein.

[6] A.P. Grecos and C. Tzanakis: Physica A 151 (1988), 61-89; C. Tzanakis: PhD thesis, 1987, U.L.B. Université Libre de Bruxelles.

[7] Note that $A_{ii}, \quad i = 1, 2, 3$, are positive quantities, so the corresponding diffusion matrix comes out to be positive definite.