Kinetic theory and diffusion coefficients for plasma in a uniform magnetic field

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Abstract

A test-particle weakly interacting with an electrostatic plasma in equilibrium inside a uniform magnetic field is considered. A markovian Fokker-Planck-type kinetic equation is derived and explicit expressions for the diffusion and drift coefficients, depending on the magnetic field, are presented and calculated in the case of a two-temperature Maxwellian background and a Debye interaction potential.

In the context of plasma kinetic theory, we have undertaken a study of the dynamics of a charged particle interacting with a magnetized background plasma in equilibrium. Starting from first microscopic principles, a markovian Fokker-Planck-type kinetic equation (F.P.E.) was derived in [1]. This new F.P.E., which preserves the positivity of the t.p.'s distribution function (d.f.) (actually shown *not* to be the case in certain "markovianization" techniques proposed in the past), was thus suggested as a basis for the study of the influence of a magnetic field on the kinetic properties of plasma (as compared, that is, to the standard Landau description). In the following, we summarize these results and then carry on by explicitly evaluating the diffusion coefficients for a two-temperature Maxwellian reservoir state and a Debye-type interaction law.

We consider a test-particle (t.p.) Σ (charge $e^{\alpha}_{\Sigma} = e$, mass $m^{\alpha}_{\Sigma} = m$) surrounded by (and weakly coupled to) a homogeneous background plasma (the reservoir 'R': N particles, of species $\alpha' \in \{\alpha_j\} = \{e, i, ...\}, \quad j = 1, 2, \dots, N$). The whole system is subject to a *uniform* stationary magnetic field along $\hat{z}$. The equations of motion for the t.p. read:

$$\dot{\mathbf{x}} = \mathbf{v}\,; \qquad \dot{\mathbf{v}} = \frac{1}{m}\,[\frac{e}{c}(\mathbf{v} \times \mathbf{B}) \,+\, \lambda\,\mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t)] \tag{1}$$

where $\mathbf{X} = (\mathbf{x}, \mathbf{v}) \equiv (\mathbf{x_\Sigma}, \mathbf{v_\Sigma})$ and $\mathbf{X_R} \equiv \{\mathbf{X_j}\} = (\mathbf{x_j}, \mathbf{v_j})$ denote the coordinates of the test- (Σ) and reservoir (R) particles respectively. The *interaction* force $\mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t)$

$= -\frac{\partial}{\partial \mathbf{x}} \sum V(|\mathbf{x} - \mathbf{x_j}|)$, i.e. the sum of random interactions between Σ and the heat bath (assumed in equilibrium), is actually a stationary Gaussian process with zero mean-value.

The zeroth-order (in λ) problem of motion yields the well-known (helicoidal) solution:

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{N}(t)\,\mathbf{v}(0) \qquad\qquad \mathbf{v}(t) = \mathbf{N}'(t)\,\mathbf{v}(0)$$

where

$$\mathbf{N}_j^{\alpha_j}(t) = \Omega^{-1} \begin{pmatrix} \sin\Omega t & s\,(1-\cos\Omega t) & 0 \\ s\,(\cos\Omega t - 1) & \sin\Omega t & 0 \\ 0 & 0 & \Omega t \end{pmatrix} \tag{2}$$

$\Omega = \Omega^{\alpha_j} \equiv \frac{|e_{\alpha_j}|B}{m_{\alpha_j} c}$ is the gyro-frequency of particle j and $s = s_{\alpha_j} = \frac{e_{\alpha_j}}{|e_{\alpha_j}|} = \pm 1$ is the *sign* of e_j (the subscript will be omitted where Σ is understood); $\mathbf{N}'(t) = d\mathbf{N}(t)/dt$.

The test-particle's reduced distribution function is $f(\mathbf{x}, \mathbf{v}; t) = (I,\, \rho)_R \equiv \int_{\Gamma_R} d\mathbf{X_R}\, \rho$, ($\rho = \rho(\{\mathbf{X}, \mathbf{X_R}\}; t)$ denotes the total phase-space d.f., normalized to unity: $\int d\mathbf{X}\, \rho = 1$). By assuming interactions to be weak, the BBGKY hierarchy of equations is truncated to 2nd order in λ; neglecting initial correlations, f is found to obey a Non-Markovian Master Equation. Following an approach developed in the past in the theory of open statistical mechanical systems [2], the latter was shown [1] to lead to the equation:

$$\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\frac{\partial f}{\partial \mathbf{v}} &= \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\Big)\Big[D_\perp(\mathbf{v}) f\Big] + \frac{\partial^2}{\partial v_z^2}[D_\parallel(\mathbf{v}) f] \\
&+ 2\,s\,\Omega^{-1}\Big[\frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x}\Big]\Big[D_\perp(\mathbf{v}) f\Big] + \Omega^{-2}\,[D_\perp^{(XX)}(\mathbf{v})]\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big) f \\
&- \frac{\partial}{\partial v_x}\Big[\mathcal{F}_x(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_y}\Big[\mathcal{F}_y(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_z}\Big[\mathcal{F}_z(\mathbf{v})\, f\Big] \\
&+ s\,\Omega^{-1}\,\mathcal{F}_y(\mathbf{v})\,\frac{\partial}{\partial x} f - s\,\Omega^{-1}\,\mathcal{F}_x(\mathbf{v})\,\frac{\partial}{\partial y} f
\end{aligned} \tag{3}$$

where $f = f(\mathbf{x}, \mathbf{v}; t)$ [3]. Note that, by integrating over $\{\mathbf{x}\}$, one recovers a reduced F.P.E., describing the evolution of $f(\mathbf{v}; t)$, which has appeared in earlier works [4].

The diffusion coefficients in (3) are defined by:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} \frac{1}{m_\alpha^2} \int_0^t d\tau \left\{ \begin{array}{c} \left\{ C_\perp^{\alpha,\alpha'} \right\} \\ C_\parallel^{\alpha,\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\Omega^\alpha \tau \\ (-s^\alpha)\,\frac{1}{2}\sin\Omega^\alpha\tau \\ (1 + \frac{1}{2}\cos\Omega^\alpha\tau) \end{array} \right\} \\ 1 \end{array} \right\} \tag{4}$$

where $C_{\{\perp,\parallel\}}^{\alpha,\alpha'}(v_\perp, v_\perp; \Omega)$ are (diagonal) elements of the force-correlation matrix $\mathbf{C}(\tau) = \langle \mathbf{F_{int}}(t)\ \mathbf{F_{int}}(t-\tau)\rangle_R$; they come out to be:

$$C_* = n_{\alpha'}(2\pi)^3 \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\, k_*^2 \tag{5}$$

($* \in \{\perp, \|\}$; a summation over n, m is understood) where v_i $(v_{1,i})$, $i = 1, 2, 3$ denote the velocity coordinates of the test- (R-) particle and $\tilde{V}_k$ stands for the Fourier transform of $V(r)$; remember that $V = V(|\mathbf{r}|) = V(r)$ implies $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$. The dynamical friction terms in (3) are given by:

$$\begin{aligned} \mathcal{F}_x &= (1+\mu)\,(\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y}) \qquad\qquad \mathcal{F}_y = (1+\mu)\,(-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y}) \\ \mathcal{F}_z &= (1+\mu)\,\frac{\partial D_\|}{\partial v_z} \qquad\qquad \mu = \frac{m_\alpha}{m_{\alpha'}} \end{aligned} \tag{6}$$

Note the explicit dependence on the magnetic field as well as on the form of the reservoir equilibrium d.f. $\phi_{eq} = \phi_{eq}(v_\perp, v_\|)$ and the interaction potential $V(r)$.

The v_1- integration in (4) can be carried out at this stage, once one assumes an analytic form for ϕ_{eq}. Here, it will be explicitly taken to be a Maxwellian of the form:

$$\phi^{\alpha'}_{Max}(v_1) = \prod_{i=1,2,3} \phi_0^{(i,\alpha')}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \tag{7}$$

($\phi_0^{(i)} = (\frac{m_{\alpha'}}{2\pi T^{(i)}_{\alpha'}})^{1/2} \equiv \frac{1}{\sqrt{\pi \sigma_i^{\alpha'}}}$; $\sigma_i^{\alpha'} \equiv 2\, {v_{i,th}^{\alpha'}}^2 \equiv \frac{2T_i^{\alpha'}}{m_{\alpha'}}$ $\forall i \in \{1,2,3\} \equiv \{x,y,z\}$; For a two-temperature *single-species* plasma (i.e. $\sigma_1 = \sigma_2 = \sigma_\perp$, $\sigma_3 = \sigma_\|$) we get:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\| \end{array} \right\} = \frac{n}{m^2}(2\pi)^4 e^{-v_\|^2/\sigma_\|} \int_0^t d\tau \int_0^\infty dk_\perp \Big[\int_{-\infty}^{\infty} dk_\|\, k_\|^{\{0,2\}} e^{-\sigma_\| \left(k_\| \tau - i\frac{2v_\|}{\sigma_\|}\right)^2/4}\, \tilde{V}_k^2 \Big]$$

$$k_\perp^{\{3,1\}}\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} J_O(2\frac{k_\perp v_\perp}{\Omega} \sin\frac{\Omega\tau}{2}) \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\Omega\tau \\ (-s)\,\frac{1}{2}\sin\Omega\tau \\ 1 + \frac{1}{2}\cos\Omega\tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{8}$$

Obviously, m (n) in $\{m, n\}$ correspond to the upper (lower) i.e. $\perp$ ($\|$) parts respectively.

In fact, relation (8) holds as it stands for *any* particular form of (long-range) central interaction potential $V(r)$. Let us now explicitly consider a Debye potential: $V(r) = e^2 \frac{e^{-k_D r}}{r}$ i.e. $\tilde{V}_k = \frac{e^2}{2\pi}\frac{1}{k^2+k_D^2} = \frac{\tilde{V}_0^2}{k^2+k_D^2}$ ($\lambda_D = k_D^{-1}$ is the Debye length [5]; obviously $k^2 = k_\perp^2 + k_\|^2$). The coefficients in (4) (actually functions of $\{v_\perp, v_\|, t;\, \sigma_\perp, \sigma_\|, \Omega\}$) now become:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\| \end{array} \right\} = \frac{n_\alpha}{m^2}(2\pi)^4\, \tilde{V}_0^2 \int_0^t d\tau \int_0^\infty dk_\perp\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}}\, J_O(2\frac{k_\perp v_\perp}{\Omega} \sin\frac{\Omega\tau}{2})$$

$$\Big(1 - \frac{k_D^2}{k_D^2 + k_\perp^2}\Big)^{\{3/2,1/2\}} \left\{ \begin{array}{c} \left\{ \begin{array}{c} F_\perp \end{array} \right\} \\ F_\parallel \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\Omega\tau \\ \frac{-s}{2}\sin\Omega\tau \\ 1 + \frac{1}{2}\cos\Omega\tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{9}$$

where the functions $F = F_{\{\perp,\parallel\}}(k_\perp, v_\parallel, \tau; \sigma_\parallel)$ are given by:

$$F_{\{\perp,\parallel\}} = \pm\frac{\sqrt{\pi}}{2}\sqrt{\sigma_\parallel}\,\tilde{k}_\perp\,\tau\,e^{-v_\parallel^2/\sigma_\parallel}$$
$$+\frac{\pi}{4}\,e^{\sigma_\parallel \tilde{k}_\perp^2\tau^2/4}\sum_{s=+1,-1}\Big[e^{s\tilde{k}_\perp v_\parallel \tau}\,(1 \mp \sigma_\parallel \tilde{k}_\perp^2\tau^2/2 \mp s\tilde{k}_\perp v_\parallel \tau)\,Erfc(\frac{1}{2}\sqrt{\sigma_\parallel}\tilde{k}_\perp\tau + s\frac{v_\parallel}{\sqrt{\sigma_\parallel}})\Big] \tag{10}$$

the upper (lower) signs corresponding to the $\perp$ ($\parallel$)- parts respectively; $\tilde{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$. $Erfc(x)$ is the *complementary* error function: $Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

Note that the integrand vanishes at infinity i.e. at $\tilde{k}_\perp \to \infty$ (and also at $\tau \to \infty$). Futhermore, the limit of the integrands at $\tilde{k}_\perp \to 0$ is finite (and the same holds for $\tau \to 0$).

In conclusion, we have reported eq.(3) as a correct kinetic description, from first principles, of the dynamics of magnetized plasma. In the homogeneous d.f. case, the previous result [4] is obtained; furthermore, in the absence of external field, the Landau equation is recovered. Once some insight considering the role of the magnetic field on transport properties is gained with this model, more realistic fields and/or geometries will be taken into account; work in this direction is in progress.

References

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