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RANDOM PARTICLE MOTION IN MAGNETIZED PLASMA

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A multivariate Fokker-Planck-type kinetic equation modeling a test - particle weakly interacting with an electrostatic plasma, in the presence of a magnetic field **B**, is analytically solved in an Ornstein - Uhlenbeck - type approximation. A new set of analytic expressions are obtained for variable moments and particle density as a function of time. The process is diffusive.

1. Introduction - Formulation of the Problem

The kinetic-theoretical treatment of long-range particle interactions ('collisions') in electrostatic plasma is often based on Landau-type equations [1], describing the evolution of a distribution function (*df*) $f(\mathbf{v};\, t)$ in velocity space, in the absence of external force fields. This description needs to be modified in the presence of an external field and/or *df* spatial inhomogeneity, which not only influence the (free) (Liouville) kinetic operator, but also modify the collision term.

A Fokker-Planck-type kinetic equation (*FPE*) was recently derived [2, 3] from first principles for a test-particle (charge q, mass m) weakly interacting with a plasma embedded in a uniform magnetic field $\mathbf{B}$. This equation, describing the evolution of the *df* $f(\mathbf{x}, \mathbf{v};\, t)$ in phase space $\Gamma = \{\mathbf{x}, \mathbf{v}\}$, has the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \Omega\,(\mathbf{v}\times\hat{b})\,\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\right)\Big(D_\perp\, f\Big) + \frac{\partial^2}{\partial v_z^2}(D_\parallel\, f)$$
$$+2\,\Omega^{-1}\left(\frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x}\right)(D_\perp\, f) + \Omega^{-2}\,(Q + D_\perp)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$$
$$-\frac{\partial}{\partial v_x}(\mathcal{F}_x\, f) - \frac{\partial}{\partial v_y}(\mathcal{F}_y\, f) - \frac{\partial}{\partial v_z}(\mathcal{F}_z\, f) + \Omega^{-1}\,\mathcal{F}_y\,\frac{\partial}{\partial x}\, f - \Omega^{-1}\,\mathcal{F}_x\,\frac{\partial}{\partial y}\, f \qquad (1)$$

where $\Omega = qB/mc$ is the cyclotron frequency; $\hat{b} = \mathbf{B}/B$ is the unit vector in the direction of the field $\mathbf{B}$; $\partial f/\partial z = 0$ by assumption. Note the *spatial diffusion* term in the right-hand side (rhs), in fact absent in most previous studies. The lengthy expressions for the coefficients, omitted here, can be found in [3, 4].

In principle, one aims in obtaining an exact solution for $f(\mathbf{x}, \mathbf{v};\, t)$ in order to trace the evolution of variable moments in time, as well as their dependence on physical parameters - the magnetic field $\mathbf{B}$, in particular. However, an exact analytical treatment is not possible, since all coefficients entering the collision term (*rhs*) are complicated functions of particle velocity $\mathbf{v}$; in addition, they explicitly depend on the magnitude of the external magnetic field. Nevertheless, a numerical study of the coefficients in terms of physical parameters shows that there exists a region where the diffusion coefficients $D_\dagger$ are practically *constant* (i.e. independent of $\mathbf{v}$) while friction terms $\mathcal{F}_\dagger$ are *linear* in $\mathbf{v}$ [3], [4b] (throughout this text, $\dagger$ will denote either $\perp$ or $\|$, referring to quantities perpendicular or parallel, respectively, to $\mathbf{B}$). In specific, this is true for low particle velocity value (as compared to the thermal velocity): intuitively speaking, this is close to the standard Langevin picture of a (slow) heavy particle randomly interacting with (faster) light particles surrounding it. This study is devoted to the analytical solution of (1), in the region of validity of this approximation [5].

Setting $D_\dagger = const.$, $\mathcal{F}_\dagger = \gamma_\dagger v_\dagger$, eq. (1) may be cast into the standard form of a multivariate ($6d$) *FPE*:

$$\frac{\partial f}{\partial t} = -\sum_{i,j} A_{ij} \frac{\partial}{\partial y_i}(y_i f) + \sum_{i,j} D_{ij} \frac{\partial^2 f}{\partial y_i \partial y_j} \tag{2}$$

where $f = f(\mathbf{y}; t)$; $\mathbf{y}$ is the position vector $(\mathbf{x}, \mathbf{v})$ in phase space Γ; cf. (VIII.**6.1**) in [6]. The *diffusion* (**D**) and *drift* (**A**) square matrices appearing in (2) are directly derived from (1) via the above assumption and will be omitted here for brevity. Note that **D** is symmetric and positive definite. Retain the equilibrium condition: $\gamma_\dagger = \frac{m}{T_\dagger^{eq}} D_\dagger \equiv 2\beta_\dagger^0 D_\dagger$, which is necessary and sufficient in order for the Maxwellian state: $f_{eq}(\mathbf{v}) = f_{eq}(\mathbf{0})\, e^{-\beta_\perp^0 v_\perp^2}\, e^{-\beta_\|^0 v_\|^2}$ to cancel the *rhs* in (2). Eq. (1) is now approximated by (2), which defines a multi-dimensional *Ornstein-Uhlenbeck* process; it may be solved for $f(t)$ via a Green function method. Furthermore, since it describes a *Gaussian* process, an exact theory exists for the calculation of variable mean values and covariances (see e.g. §VIII.6 in [6]). The calculation, involving multiple integrations in all Γ- space variables ($\{\mathbf{x}, \mathbf{v}\}$), is rather lengthy yet straightforward. This is a brief report of results (exposed in [3] in detail).

2. Exact Solution for a Maxwelian Initial Velocity Distribution

Assuming a Maxwellian initial distribution of the form:

$$f_0(\mathbf{x}, \mathbf{v}) = \frac{1}{\pi^{3/2}}\, \beta_\perp\, \beta_\|^{\,1/2}\, e^{-\beta_\perp v_\perp^2}\, e^{-\beta_\| v_\|^2}\, \delta(\mathbf{x}) \tag{3}$$

($\beta_\dagger = m/2T_\dagger$) we obtain the (time-dependent) *df*:

$$f(\mathbf{x}, \mathbf{v}; t) = \frac{\tilde\beta_\perp\, \tilde\beta_\|^{1/2}\, \tilde\beta_\perp^{(X)}}{\pi^{5/2}}\, e^{-\tilde\beta_\perp v_\perp^2}\, e^{-\tilde\beta_\| v_\|^2}\, e^{-\tilde\beta_\perp^{(X)}\, \Xi^2}\, I_0\left(4 \frac{\tilde\beta_\perp^{(X)}\, \tilde\beta_\perp\, \xi}{\Omega} v_\perp\, \rho\right) \tag{4}$$

where $\rho = x^2 + y^2$, $v_\perp = v_x^2 + v_y^2$ and $v_\| = v_z$; note the definitions:

$$\begin{aligned}
\tilde\beta_\dagger(t) &= \frac{\theta}{(1 - e^{-2\gamma_\dagger t})\,\theta + e^{-2\gamma_\dagger t}}\, \beta_\dagger^0\,, \qquad \Xi(\tau) = \rho^2 + \left(\frac{2\,\tilde\beta_\perp(\tau)\,\xi(\tau)}{\Omega}\, v_\perp\right)^2 \\
\tilde\beta_\perp^{(X)}(\tau) &= \Omega^2\, \beta_\perp^0 \left\{1 - e^{-2\tau} + \frac{(1 - e^{-\tau})^2}{\theta}\right. \\
&\qquad \left. - \frac{1}{\theta}\, \frac{\theta^2 (1 - e^{-2\tau})^2 + e^{-2\tau}(1 - e^{-\tau})^2}{\theta(1 - e^{-2\tau}) + e^{-2\tau}} + 4\beta_\perp^0\, Q\, t\right\}^{-1} \\
\xi(\tau) &= \frac{1}{\beta_\perp^0}\, \frac{1}{2\theta} \left\{\theta^2\, (1 - e^{-2\tau})^2 + e^{-2\tau}\, (1 - e^{-\tau})^2\right\}^{1/2}
\end{aligned} \tag{5}$$

where $\theta = \beta_\dagger/\beta_\dagger^0 = T_\dagger^{eq}/T_\dagger$, $\tau = \gamma_\perp t$. We see that the velocity distribution will relax to the equilibrium state anticipated above, as physically expected, since: $\lim_{\tau\to\infty} \tilde{\beta}_\dagger(\tau) = \frac{\gamma_\dagger}{2D_\dagger} \equiv \beta_\dagger^0 = \frac{m}{2T_\dagger^{eq}}$, while spatial distribution will exhibit a *classical* diffusive behaviour, under the influence of collisions; check that, for $\tau \gg \gamma_\dagger^{-1}$: $\tilde{\beta}_\perp^{(X)}(\tau)$ behaves as $\approx \Omega^2 \beta_\perp^0/(1+4\beta_\perp^0\, Q\, t)$. A similar qualitative behaviour is also obtained for a Maxwellian (i.e. not localized, cf. (3)) spatial distribution at $t = 0$, and also for an initial velocity distribution of the type: $f_\perp(\mathbf{v};\, t=0) = \delta(\mathbf{v}_\perp - \mathbf{v}_\parallel^0)\, \delta(v_\parallel - v_\parallel^0)$ (cf. fig. 1b); details are omitted here, for lack of space [3].

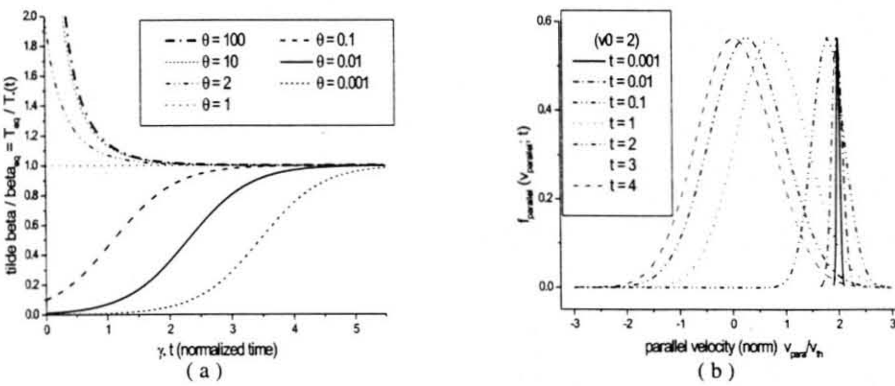

Fig. 1. (a) The inverse temperature $\tilde{\beta}_\dagger(t)$ (scaled over $\tilde{\beta}_\dagger(\infty) = \beta_\dagger^0$) versus time $\tau = \gamma_\dagger t$ for different values of $\theta = \beta/\beta^0 = T_{eq}/T(0)$. Collisions appear more efficient in accelerating initially slow particles ($\theta > 1$) than slowing down fast ones ($\theta < 1$).
(b) The evolution $f_\perp(v_\parallel;\, t)$ of an initial $\delta(v_\parallel - v_\parallel^0)$ state (sharp solid line) ($\parallel$ −part; see in §2) versus velocity $v_\parallel$ ($v_\parallel \in \Re$) (normalized over the thermal velocity ${\beta_\parallel^0}^{-1/2}$) for an initial velocity value of $2v_{th}$. The initial sharp profile spreads fast and attains the final zero-averaged thermalized state within a few time constants.

3. Density Profile - Moments

The mean particle density: $n(\mathbf{x},\, t) = \int d\mathbf{v}\;\, f(\mathbf{x}, \mathbf{v};\, t)$ may now be calculated. Considering the initial condition: $f_0(\mathbf{x}, \mathbf{v}) = \delta(\mathbf{x} - \mathbf{x_0})\, \delta(\mathbf{v} - \mathbf{v_0})$, we obtain:

$$n(\rho,\, t) = \frac{1}{2\pi}\frac{1}{L^2(t)}\, e^{-\rho^2/2L^2(t)}\, e^{-R_0^2(t)/2L^2(t)}\, I_0\!\left(\frac{R_0(t)\,\rho}{L^2(t)}\right) \tag{6}$$

(for simplicity the direction $\parallel$ to the field was neglected and the gyro-phase ϕ was averaged out); $I_0(x)$ denotes the modified Bessel function; the *mean square displacement* L^2 reads:

$$L^2(t) \;=\; \Omega^{-2}\left[\frac{T_\perp^{eq}}{m}(1 - e^{-2\gamma_\perp t}) + 4\,Q\,t\right] \tag{7}$$

and $R_0(t) = (1 + e^{-2\gamma_\perp t} - 2e^{-\gamma_\perp t}\cos\Omega t)^{1/2}\, v_\perp^0/\Omega$; see that $R_0 \approx \frac{v_\perp^0}{\Omega} \equiv \rho_L^0$ after a while. For zero initial velocity, $R_0 = 0$ so a purely Gaussian profile is recovered. For higher $v_\perp^0$, the distribution spreads in space, as expected. See that $L^2(t)$ asymptotically behaves as $\sim t$, hence the classical diffusion mechanism encountered in §2. Earlier results predicting diffusion as $\sim B^{-2}$ are thus confirmed [7].

Finally, the (symmetric) covariance matrix $\langle\langle y_k\, y_l\rangle\rangle = \langle y_k\, y_l\rangle - \langle y_k\rangle\,\langle y_l\rangle$ (where $\mathbf{y} = (x, y, z;\, v_x, v_y, v_z)$) can be evaluated via the same formalism. The results for all

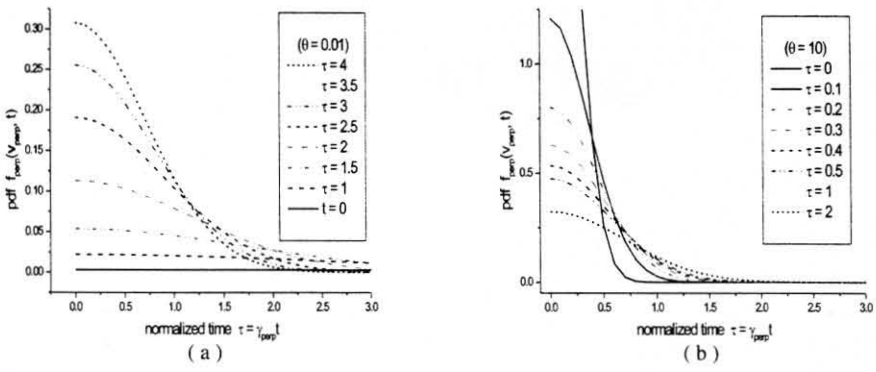

Fig. 2. The evolution $f_\perp(v_\perp;\, t)$ of an initial Maxwellian state ($\perp$ –part, as given by (5), (6)) versus velocity $v_\perp$ (normalized over the thermal velocity ${\beta_\perp^0}^{-1/2}$) for two values of $\theta = \beta_*/\beta_*^0 = T_{eq}/T_*(0)$: (a) $\theta = 0.01$ (high initial plasma temperature, see black solid line at the bottom) and (b) $\theta = 10$ (low initial plasma temperature, see black solid line on top). In the second case, the initial distribution relaxes faster to the final equilibrium state (practically attained after 2 time constants) in agreement with Fig. 1a.

elements are obtained in [3] and briefly exposed in [5]. For velocity covariances we obtain: $\langle\langle v_x\, v_x\rangle\rangle = \langle\langle v_y\, v_y\rangle\rangle = \frac{T_\perp^{eq}}{m}\,(1 - e^{-2\gamma_\perp t})$ (also for $\langle\langle v_z\, v_z\rangle\rangle$, upon $\perp \rightarrow \|$). In the plane $\perp$ **B**: $\langle\langle x\, x\rangle\rangle = \langle\langle y\, y\rangle\rangle) = L^2$; cf. (7) above.

In conclusion, the analytical treatment of random electrostatic interactions modelled by the Fokker-Planck-type equation (1) in an Ornstein-type approximation reveals a classical diffusive behaviour in space and allows for an exact study of plasma relaxation as well as moment evolution in time.

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