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Envelope localized modes in electrostatic plasma waves

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Abstract

A generic methodology is proposed for the study of the amplitude modulation of electrostatic plasma modes via a collisionless (unmagnetized) one fluid plasma decsription. An explicit analytical framework is provided for the investigation of the wave's modulational stability profile and the occurrence of localized envelope structures, whose explicit forms are presented and discussed.

1. Introduction. In a general manner, several known electrostatic plasma modes [1] consist of propagating oscillations of one dynamical plasma constituent, say α (mass m_α, charge $q_\alpha \equiv s_\alpha Z_\alpha e$; e is the absolute electron charge; $s = s_\alpha = q_\alpha/|q_\alpha| = \pm 1$ is the charge *sign*), against a background of one (or more) constituent(s) α' (mass $m_{\alpha'}$, charge $q_{\alpha'} \equiv s_{\alpha'} Z_{\alpha'} e$, similarly); the latter is (are) often assumed to obey a known distribution, e.g. being in a fixed (uniform) or in a thermalized (Maxwellian) state, for simplicity, depending on the particular aspects (e.g. frequency scales) of the physical system considered. For instance, the *ion-acoustic* (IA) mode refers to ions ($\alpha = i$) oscillating against a Maxwellian electron background ($\alpha' = e$) [2], the *electron-acoustic* (EA) mode [3] refers to electron oscillations ($\alpha = e$) against a fixed ion background ($\alpha' = i$), and so forth [1] The purpose of this brief paper is to provide a generic methodological framework for the study of the nonlinear (self-)modulation of the amplitude of such electrostatic modes, a mechanism known to be associated with harmonic generation and the formation of localized envelope modulated wave packets, such as the ones abundantly observed during laboratory experiments and satellite observations, e.g. in the Earth's magnetosphere.

2. The model. The standard (single) fluid model, for the inertial species α reads:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{u}) = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -s\,\nabla\phi - \frac{\sigma}{n}\nabla p\,, \tag{2}$$

$$\frac{\partial p}{dt} + \mathbf{u} \cdot \nabla p = -\gamma\,p\,\nabla \cdot \mathbf{u}\,, \tag{3}$$

where the particle density n_α, mean fluid velocity $\mathbf{u}_\alpha$, pressure p_α and electric potential Φ are scaled as: $n = n_\alpha/n_{\alpha,0}$, $\mathbf{u} = \mathbf{u}_\alpha/c_*$, $p = p_\alpha/n_{\alpha,0} k_B T_\alpha$, and $\phi = |q_\alpha|\Phi/(k_B T_*)$, where $n_{\alpha,0}$ is the equilibrium density and $c_* = (k_B T_*/m_\alpha)^{1/2}$ is a characteristic (e.g. sound) velocity. Time and space are scaled over appropriately chosen scales t_0 [e.g. $\omega_{p,\alpha}^{-1} = (4\pi n_{\alpha,0} q_\alpha^2/m_\alpha)^{-1/2}$] and $r_0 = c_* t_0$; $\gamma = c_P/c_V = 1 + 2/f$ is the specific heat ratio; T_α is the fluid temperature, and T_* is an effective temperature (related to the background considered), to be determined for each problem under consideration (k_B is Boltzmann's constant). The temperature ratio T_α/T_* is denoted by σ (the so-called *cold model* is recovered for $\sigma = 0$). The Lorentz force term was omitted, since wave propagation along the external magnetic field ($\sim \hat{x}$) is considered. The system is closed by Poisson's equation $\nabla^2\Phi = -4\pi \sum_{\Sigma=\alpha,\{\alpha'\}} n_\Sigma\, q_\Sigma$. Overall neutrality is assumed at equilibrium, i.e. $\sum_{\Sigma=\alpha,\{\alpha'\}} n_{\Sigma,0}\, q_\Sigma = 0$.

3. Weakly nonlinear oscillation regime. What follows is essentially an implementation of the long known *reductive perturbation* technique, which was first applied in the study of electron plasma [4] and electron-cyclotron [5] waves, three decades ago. The above system of evolution equations for the state vector $\mathbf{S} = \{n, \mathbf{u}, \phi\}$ describes electrostatic harmonic waves in the form $\mathbf{S} = \mathbf{S_0} \exp[i(k\mathbf{r} - \omega t)] + \text{c.c.}$. In order to study the modulational stability profile of these electrostatic waves and model the nonlinear harmonic generation mechanism entering into play when their amplitude becomes non-negligible, we consider small deviations from the equilibrium state $\mathbf{S}^{(0)} = (1, \mathbf{0}, 0)^T$, viz. $\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}^{(1)} + \epsilon^2 \mathbf{S}^{(2)} + ...$, where $\epsilon \ll 1$ is a smallness parameter. We have assumed that $Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(X, T)\, e^{il(k\mathbf{r}-\omega t)}$ (for $j = 1, 2, ...$; $S_{j,-l}^{(n)} = S_{j,l}^{(n)*}$, for reality), thus allowing the wave amplitude to depend on the stretched (*slow*) coordinates $X = \epsilon(x - \tilde{v}_g\, t)$, $T = \epsilon^2\, t$ [where $\tilde{v}_g = \partial\omega(k)/\partial k_x$ is the wave group velocity along the modulation direction x]. The amplitude modulation direction is assumed *oblique* with respect to the (arbitrary) propagation direction, expressed by the wave vector $\mathbf{k} = (k_x, k_y) = (k \cos\theta, k \sin\theta)$. Accordingly, we set: $\partial/\partial t \to \partial/\partial t - \epsilon\tilde{v}_g \partial/\partial X + \epsilon^2 \partial/\partial T$ and $\partial/\partial x \to \partial/\partial x + \epsilon\partial/\partial X$ (while $\partial/\partial y$ remains unchanged). By expanding near $\phi \approx 0$, Poisson's eq. may formally be cast in the form

$$\nabla^2\phi = \phi - \alpha\,\phi^2 + \alpha'\,\phi^3 - s\,\beta\,(n-1)\,, \tag{4}$$

where the exact form of the coefficients α, α' and β, which should be determined exactly for any specific problem, contain all the essential dependence on the plasma parameters. Note that the right-hand side in Eq. (4) cancels at equilibrium. Substituting into Eqs. (1) - (4), one is then left with the task of isolating orders in ϵ^n (i.e. $n = 1, 2, ...$) and successively solve for the harmonic amplitudes $S_{j,l}^{(n)}$. The calculation, particularly lengthy yet straightforward, can be found e.g. in [2] for IA ($s = +1$) and in [3] for EA waves ($s = -1$) (also see [6, 7] for details on the method).

The first harmonic amplitudes are determined (to order $\sim \epsilon^1$) as

$$n_1^{(1)} = s[(1+k^2)/\beta]\,\psi\,, \quad u_{1,x}^{(1)} = (\omega/k)\cos\theta\, n_1^{(1)}\,, \quad u_{1,y}^{(1)} = (\omega/k)\sin\theta\, n_1^{(1)}\,, \quad p_1^{(1)} = \gamma n_1^{(1)}$$

in terms of the potential correction $\phi_1^{(1)} \equiv \psi$, along with the dispersion relation $\omega^2 = \beta k^2/(k^2+1)$. Furthermore, the amplitudes of the 2nd and 0th (constant) harmonic corrections are obtained in $\sim \epsilon^2$; the lengthy expressions are omitted for brevity†.

4. The envelope evolution equation. The potential correction ψ is found to obey a compatibility condition in the form of a *nonlinear Schrödinger–type equation* (NLSE)

$$i\,\frac{\partial\psi}{\partial T} + P\,\frac{\partial^2\psi}{\partial X^2} + Q\,|\psi|^2\,\psi = 0\,. \tag{5}$$

Both the *dispersion coefficient* P, in fact related to the curvature of the dispersion curve as $P = \partial^2\omega/2\partial k_x^2 = [\omega''(k)\cos^2\theta + \omega'(k)\sin^2\theta/k]/2$, and the *nonlinearity coefficient*

Q, which is due to carrier wave self-interaction, are functions of k, θ and β, as expected (in addition to α, α', for Q). The exact general expressions thus obtained (omitted here[††] [8]) may be *tailor fit* to any given electrostatic plasma wave problem (via the form of the parameters α, α', β) in view of a numerical investigation of the wave's amplitude dynamics (e.g. stability profile, wave localization; see in the following).

5. Stability profile – envelope excitations. It is known that the evolution of a wave whose amplitude obeys Eq. (5) depends on the coefficient product PQ, which may be investigated in terms of the physical parameters involved. The resulting modulated wave here is of the form $\psi = \epsilon\psi_0 \cos(kx - \omega t + \Theta) + \mathcal{O}(\epsilon^2)$, where the slowly varying amplitude $\psi_0(\epsilon x, \epsilon t)$ and phase correction $\Theta(\epsilon x, \epsilon t)$ are determined by (solving) Eq. (5) for $\psi = \psi_0 \exp(i\Theta)$ (see [8] for details; also [2, 3, 6, 7] for a summary).

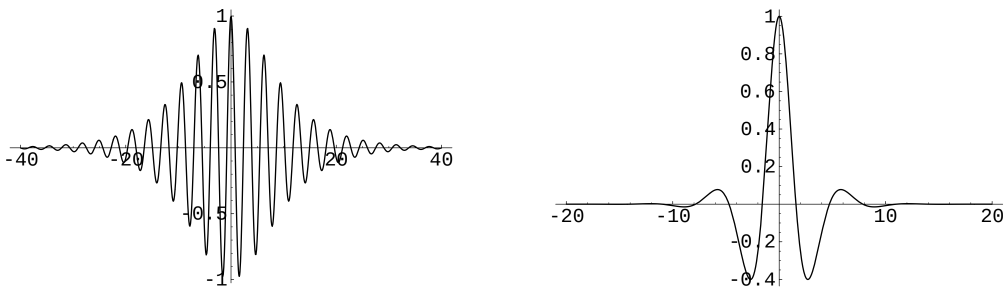

Figure 1: *Bright* wavepacket, for two different (arbitrary) sets of parameter values.

For *positive* PQ, the carrier wave is modulationally unstable and may thus either *collapse*, due to external perturbations, or lead to the formation of *bright* envelope modulated wavepackets, i.e. localized envelope *pulses* confining the carrier (see Fig. 1):

$$\psi_0 = \left(\frac{2P}{QL^2}\right)^{1/2} \operatorname{sech}\left(\frac{X - v_e T}{L}\right), \quad \Theta = \frac{1}{2P}\left[v_e X + \left(\Omega - \frac{v_e^2}{2}\right)T\right] \tag{6}$$

[8], where v_e is the envelope velocity; L and Ω represent the pulse's spatial width and oscillation frequency, respectively. We note that L and ψ_0 satisfy $L\psi_0 = (2P/Q)^{1/2} =$ constant [in contrast with Korteweg-deVries (KdV) solitons, where $L^2\psi_0 =$ const. instead]. Also, the maximum amplitude ψ_0 is independent from the velocity v_e here.

For *negative* PQ, the carrier wave is modulationally *stable* and may propagate as a *dark/grey* envelope wavepackets, i.e. a propagating localized envelope *hole* (a *void*) amidst a uniform wave energy region. The exact expression for *dark* envelopes reads:

$$\psi_0 = \pm{\psi'}_0 \tanh\left(\frac{X - v_e T}{L'}\right), \qquad \Theta = \frac{1}{2P}[v_e X + (2PQ{\psi'}_0{}^2 - \frac{v_e^2}{2})T] \tag{7}$$

(see Fig. 2a); again, $L'{\psi'}_0 = (2|P/Q|)^{1/2}$ (=cst.). The *grey* envelope reads [8]:

$$\psi_0 \;=\; {\psi''}_0 \, \{1 - d^2 \operatorname{sech}^2\{[X - v_e T]/L''\}\}^{1/2} \quad , \tag{8}$$

$$\Theta = \frac{1}{2P}\left[V_0 X \;-\; \left(\frac{1}{2}V_0^2 - 2PQ{\psi''}_0^2\right) T + \Theta_0\right] \;-\; S \,\sin^{-1} \frac{d\, \tanh(\frac{X - v_e T}{L''})}{\left[1 - d^2 \operatorname{sech}^2\left(\frac{X - v_e T}{L''}\right)\right]^{1/2}} \,. \tag{9}$$

Here Θ_0 is a constant phase; S denotes the product $S = \text{sign}(P) \times \text{sign}(v_e - V_0)$. The pulse width $L'' = (|P/Q|)^{1/2}/(d\psi''_0)$ now also depends on the real parameter d, given by: $d^2 = 1 + (v_e - V_0)^2/(2PQ\psi''^2_0) \leq 1$. $V_0 = \text{const.} \in \Re$ satisfies [8]: $V_0 - \sqrt{2|PQ|\,\psi''^2_0} \leq v_e \leq V_0 + \sqrt{2|PQ|\,\psi''^2_0}$. For $d = 1$ (thus $V_0 = v_e$), one recovers the *dark* envelope soliton.

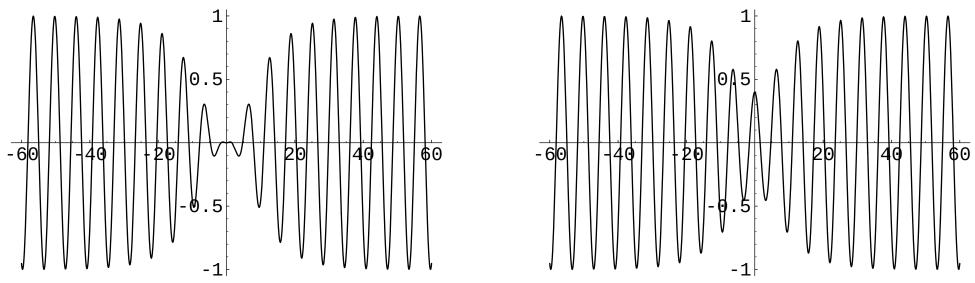

Figure 2: *Dark* (left) and *grey* (right) type modulated wavepacket (for $PQ < 0$). See that the amplitude never reaches zero in the latter case.

6. Conclusion. The envelope modulated localized electrostatic structures (wave packets) which are widely observed during satellite missions and laboratory experiments, may be efficiently modeled by making use of the standard reductive perturbation method [4, 5]. Explicit criteria for the excitation type (and the carrier wave stability) are thus obtained, allowing for an analysis of the nonlinear profile of a (any) given electrostatic mode under consideration. An extended report is under way and will be reported soon [7].

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† The expressions (10) - (42) in [6] hold exactly here (upon an index shift, $d \to \alpha$, therein).
†† See Eq. (29) in [6], for P, as well as Eqs. (31), (40) - (42), for Q.

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