Higher-order nonlinear contributions to ion-acoustic waves in a plasma consisting of adiabatic warm ions, non-isothermal electrons and a weakly relativistic electron beam

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Abstract. Ion-acoustic solitary waves in a collisionless plasma consisting of adiabatic warm ions, a weakly relativistic electron beam and non-isothermal electrons, are studied by using the reductive perturbation method. The basic set of model equations is reduced to a modified Korteweg-de Vries equation (mKdV) for the first order electric potential correction, and to a linear inhomogeneous equation for its second-order counterpart. It is remarked that the effect of higher order nonlinearity results in an increased soliton amplitude for one of these modes, in addition to deforming the soliton's shape (from a simple positive pulse to a W-shaped excitation), while in the case of the three other modes, the effect of higher-order nonlinearity is to increase the soliton's amplitude without deforming its shape (a simple pulse is obtained). The effects of the electron beam density and the ion temperature on the existence and propagation of solitary waves are also briefly studied.

Keywords: Ion-acoustic waves, Relativistic electron beam, Two fluid plasma, Modified Korteweg-de Vries equation, Higher-order nonlinearity
PACS: 52.35.Mw,52.40.Mj,52.30.Ex

THE MODEL EQUATIONS

We assume a collisionless plasma consisting of warm ions, non-isothermal electrons and weak (for the sake of simplicity) relativistic electron beam. We shall consider the lowest-order ion-acoustic solitary waves, anticipating the effect of higher-order nonlinearity on them. A reductive perturbation method will be employed. Wave propagation parallel to the external magnetic field is considered. The basic set of fluid equations for ions and the electron beam [1, 2] are as follows:

$$\frac{\partial n}{\partial t}+(1+\alpha)\frac{\partial v}{\partial x}+\frac{\partial (nv)}{\partial x}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{3\sigma}{1+\alpha}\frac{\partial n}{\partial x}+\frac{3\sigma}{(1+\alpha)^2}n\frac{\partial n}{\partial x}+\frac{\partial \varphi}{\partial x}=0, \tag{2}$$

$$\frac{\partial n_b}{\partial t}+v_{b_0}\frac{\partial n_b}{\partial x}+\alpha\frac{\partial v_b}{\partial x}+\frac{\partial (n_b v_b)}{\partial x}=0, \tag{3}$$

$$\frac{\partial v_b}{\partial t}+v_{b_0}\frac{\partial v_b}{\partial x}+v_b\frac{\partial v_b}{\partial x}-\frac{1}{\mu_i}\big(1-\frac{3}{2}\frac{v_{b_0}^2}{c^2}\big)\frac{\partial \phi}{\partial x}+\frac{3v_{b_0}v_b}{\mu_i c^2}\frac{\partial \phi}{\partial x}=0, \tag{4}$$

$$\frac{\partial^2 \varphi}{\partial x^2}=n_e+n_b-n-1. \tag{5}$$

in which all quantities have been normalized as in [1], except for $T_e \rightarrow T_{ef}$. It is supposed that the expression for the electron density [3, 4] is given by

$$n_e=1+\varphi-\frac{4}{3}b\varphi^{\frac{3}{2}}+\frac{1}{2}\varphi^2+\dots; \qquad b=\frac{(1-\beta)}{\pi^{\frac{1}{2}}}, \qquad \beta=\frac{T_{ef}}{T_{et}}, \tag{6}$$

in which T_{ef} denotes the constant temperature of the free electrons and T_{et} that of the trapped electrons. The third term in relation (2) introduces the contribution of the resonant electrons (both free and trapped) to the electron density.

REDUCTIVE PERTURBATION ANALYSIS

Anticipating the derivation of a KdV-type equation describing the propagation of nonlinear ion-acoustic waves from the basic system of Eqs (1)-(6), we expand all state variables around the equilibrium state, by defining a smallness parameter ε, viz.

$$\begin{aligned}
\xi = \varepsilon^{\frac{1}{4}}(x - St)\,, \qquad \tau = \varepsilon^{\frac{3}{4}}t\,; \qquad n &= \varepsilon n_1 + \varepsilon^{\frac{3}{2}} n_2 + \varepsilon^2 n_3 + \varepsilon^3 n_4 + \dots\,, \\
n_b = \varepsilon n_{b_1} + \varepsilon^{\frac{3}{2}} n_{b_2} + \varepsilon^2 n_{b_3} + \varepsilon^3 n_{b_4} + \dots\,; \qquad u &= \varepsilon u_1 + \varepsilon^{\frac{3}{2}} u_2 + \varepsilon^2 u_3 + \varepsilon^3 u_4 + \dots\,, \\
v_b = \varepsilon v_{b_1} + \varepsilon^{\frac{3}{2}} v_{b_2} + \varepsilon^2 v_{b_3} + \varepsilon^3 v_{b_4} + \dots\,; \qquad \varphi &= \varepsilon \varphi_1 + \varepsilon^{\frac{3}{2}} \varphi_2 + \varepsilon^2 \varphi_3 + \varepsilon^3 \varphi_4 + \dots\,.
\end{aligned} \tag{7}$$

To the lowest power of ε we obtain

$$v_1 = \frac{S}{S^2 - 3\sigma}\varphi_1\,, \qquad n_1 = \frac{1+\alpha}{S^2 - 3\sigma}\varphi_1\,, \qquad v_{b_1} = \frac{1 - \frac{3}{2}\frac{v_0^2}{c^2}}{(v_0 - S)^2}\varphi_1\,.$$

We also obtain the dispersion relation

$$\frac{\alpha}{\mu_i}\frac{[1 - \frac{3}{2}\frac{v_0^2}{c^2}]}{(v_0 - S)^2} + \frac{1+\alpha}{S^2 - 3\sigma} = 1. \tag{8}$$

which is quartic in S, so that four distinct modes can propagate in this plasma system. It must be stressed that $S \neq v_0$ and $S \neq 3\sigma$. For $v_0 = 0$, the dispersion relation is quadratic in S^2, so $S_1^2 = \frac{1}{2}(\frac{\alpha}{\mu} + \alpha + 3\sigma + 1) - \frac{1}{2}\big[(\frac{\alpha}{\mu} + \alpha + 3\sigma + 1)^2 - \frac{12\alpha\sigma}{\mu}\big]^{1/2}$ for the slow ion-acoustic mode, and $S_2^2 = \frac{1}{2}(\frac{\alpha}{\mu} + \alpha + 3\sigma + 1) + \frac{1}{2}\big[(\frac{\alpha}{\mu} + \alpha + 3\sigma + 1)^2 - \frac{12\alpha\sigma}{\mu}\big]^{1/2}$, for the fast ion-acoustic one.

From the next-order equations in ε, we obtain the modified *KdV* equation

$$\varphi_{1\tau} + bA\varphi_1^{\frac{1}{2}}\varphi_{1\xi} + \frac{A}{2}\varphi_{1\xi\xi\xi} = 0\,, \tag{9}$$

with

$$A = \frac{(S^2 - 3\sigma)^2(v_0 - S)^3}{S[S(1+\alpha)(v_0 - S)^3 - \frac{\alpha}{\mu_i}(S^2 - 3\sigma)^2(1 - \frac{3}{2}\frac{v_0^2}{c^2})]}$$

(to be compared to the *KdV* equation obtained in the isothermal case [2]). For $S = v_0$ and $S = 3\sigma$, the coefficient A becomes zero and thus ion-acoustic (mKdV) solitons do not exist at sonic velocities.

A better agreement between theory and experiments is expected by taking into account higher-order nonlinearity effects [6, 7]. Therefore, proceeding to the next order in ε, we obtain

$$\varphi_{2\tau} + bA(\varphi_1^{\frac{1}{2}}\varphi_2)_\xi + \frac{A}{2}\varphi_{2\xi\xi\xi} = \frac{A}{2}[P - bAQ]\varphi_1\varphi_{1\xi} - \frac{A^2Q}{4}\varphi_1^{\frac{1}{2}}\varphi_{1\xi\xi\xi} - \frac{A^2Rb}{2}(\varphi_1^{\frac{1}{2}}\varphi_{1\xi})_{\xi\xi} - \frac{A^2R}{4}\varphi_{1\xi\xi\xi\xi\xi}\,. \tag{10}$$

Expressions for the quantities P, Q and R are given in the Appendix. In order to solve Eqs. (9) and (10), we use the renormalization method of Kodama and Taniuti [5], according to which Eqs. (9) and (10) are modified as

$$\frac{\partial\tilde{\varphi}_1}{\partial\tau} + bA\tilde{\varphi}_1^{\frac{1}{2}}\frac{\partial\tilde{\varphi}_1}{\partial\xi} + \frac{A}{2}\frac{\partial^3\tilde{\varphi}_1}{\partial\xi^3} + \delta\lambda\frac{\partial\tilde{\varphi}_1}{\partial\xi} = 0, \tag{11}$$

and

$$\frac{\partial\tilde{\varphi}_2}{\partial\tau} + bA\frac{\partial(\tilde{\varphi}_1^{\frac{1}{2}}\tilde{\varphi}_2)}{\partial\xi} + \frac{A}{2}\frac{\partial^3\tilde{\varphi}_2}{\partial\xi^3} + \delta\lambda\frac{\partial\tilde{\varphi}_2}{\partial\xi} = s(\tilde{\varphi}_1) + \delta\lambda\frac{\partial\tilde{\varphi}_1}{\partial\xi}, \tag{12}$$

where $s(\tilde{\varphi}_1)$ is the right hand side of Eq. (10). In Eqs. (11) and (12), the parameter $\delta\lambda$ is introduced in such a way that the resonant term in $s(\tilde{\varphi}_1)$ is cancelled by the term $\delta\lambda\frac{\partial\tilde{\varphi}_1}{\partial\xi}$ in (12). In order to obtain stationary profile travelling wave solutions, we defining the new variable η as

$$\eta = \xi - (\lambda + \delta\lambda)\tau\,. \tag{13}$$

Under this transformation, and using the boundary conditions $\tilde{\varphi}_1 = \tilde{\varphi}_2 = \frac{\partial \tilde{\varphi}_2}{\partial \eta} = \frac{\partial^2 \tilde{\varphi}_2}{\partial \eta^2} = 0$, as $|\eta| \to \infty$, we can integrate Eqs. (11) and (12) and obtain

$$\tilde{\varphi}_1 = \varphi_0 sech^4(\frac{\eta}{D}), \qquad \varphi_0 = \frac{225\lambda^2}{(8bA)^2}, \qquad D = (\frac{8A}{\lambda})^{\frac{1}{2}}. \tag{14}$$

and

$$\frac{\partial^2 \tilde{\varphi}_2}{\partial \eta^2} + 2(b\tilde{\varphi}_1^{\frac{1}{2}} - \frac{\lambda}{A})\tilde{\varphi}_2 = \{\frac{1}{2}(P - bAQ)\varphi_0^2 + [\frac{15}{2}\frac{AQ}{D^2} + \frac{42}{3}\frac{bAR}{D^2}]\varphi_0^{\frac{3}{2}} - \frac{420}{D^4}AR\varphi_0\}sech^8\frac{\eta}{D} + \\ \{-(\frac{16AQ}{3D^2} + \frac{12bAR}{D^2})\varphi_0^{\frac{3}{2}} + \frac{520AR}{D^4}\varphi_0\}sech^6\frac{\eta}{D} + (\frac{2}{A}\delta\lambda - \frac{16^2AR}{2D^4})\varphi_0 sech^4\frac{\eta}{D}. \tag{15}$$

Now, in order to cancel the secular terms in $s(\tilde{\varphi}_1)$, we set the second expression in the second line of the right-hand side of Eq. (15) equal to zero. We therefore immediately get

$$\delta\lambda = \frac{64A^2R}{D^4} = R\lambda^2. \tag{16}$$

In order to solve Eq. (15), we introduce a new variable $\mu = \tanh(\frac{\eta}{D})$ and use the method of variation of parameters. We therefore obtain the complementary and particular solutions as

$$\tilde{\varphi}_{2c} = c_1 P_5^4(\mu) + c_2 Q_5^4(\mu). \tag{17}$$

$$\tilde{\varphi_2}_p(\eta) = (\frac{1}{6}Z_1 + \frac{1}{10}Z_2)\cosh^4(\frac{\eta}{D}) - \frac{Z_1}{12}\cosh^6(\frac{\eta}{D}). \tag{18}$$

respectively, where Z_1 and Z_2 are given in appendix. The first term in Eq. (17) is the secular one, which can be eliminated by renormalizing the amplitude. Also, $c_2 = 0$ as a result of vanishing boundary conditions for $\tilde{\varphi}_2(\eta)$ as $|\eta| \to \infty$. Therefore, only the particular solution (18), contributes to $\tilde{\varphi}_2(\eta)$ so that the stationary solution for the potential for ion-acoustic wave is given by

$$\tilde{\varphi}(\eta) = \tilde{\varphi}_1(\eta) + \tilde{\varphi}_2(\eta) = \phi_0\cosh^4(\frac{\eta}{D}) + (\frac{1}{6}Z_1 + \frac{1}{10}Z_2)\cosh^4(\frac{\eta}{D}) - \frac{Z_1}{12}\cosh^6(\frac{\eta}{D}). \tag{19}$$

NUMERICAL ANALYSIS AND CONCLUSION

The dispersion relation (8) is quartic in S, and thus shows that the inclusion of a finite ion temperature, non-isothermal electrons and relativistic electron beam gives rise to four ion-acoustic modes propagating with different phase velocities S_1, S_2, S_3 and S_4. For instance, the dispersion relation has only two real roots S_1, S_2 for $v_0 = 1.8$, $c = 30$, $\mu = 1/1836$, $\lambda = 0.5$, $b = -0.99$; $\alpha = 0.0001$ and $\sigma = 0.2 - 0.3$. Figures (1)-(2) show curves for the lowest-order modified KdV soliton (8) and for the second-order one soliton solution (19). It is remarked that the effect of higher order nonlinearity results in an increased soliton amplitude for one of these modes S_1, while deforming the soliton's shape (from a simple positive pulse to a W-shaped excitation), whilst in the case of the second mode S_2, the effect of higher-order nonlinearity is to increase the soliton's amplitude without deforming its shape (a simple pulse is obtained). In addition, increasing the value of σ increases (decreases) the soliton's amplitude in the case of mode S_1 (S_2), for fixed α. Generally, the analysis reveals that four distinct ion-acoustic modes, which propagate at different phase velocities, may occur in this plasma system. Two of these modes exist for all values of the electron beam to background electron density ratio α and ion to free electron temperature ratio σ . The amplitude of ($mKdV$) solitons decreases (increases) as α (σ, respectively) increases, while the inverse behavior is witnessed by the soliton width, for these two modes. On the other hand, the two remaining modes exist only for small values of α and high values of σ; in fact, the phase speed of these modes becomes complex at some range of values of α and σ. The amplitude and width associated to these modes decreases (increases) as α(σ, respectively) increases. Finally, it is remarked that the effect of higher order nonlinearity results in an increased soliton amplitude for one of these modes, while deforming the soliton's shape (from a simple positive pulse to a W-shaped excitation), while in the case of the (three) remaining modes, the effect of higher-order nonlinearity is to increase the soliton's amplitude without deforming its shape (i.e. a simple pulse is obtained).

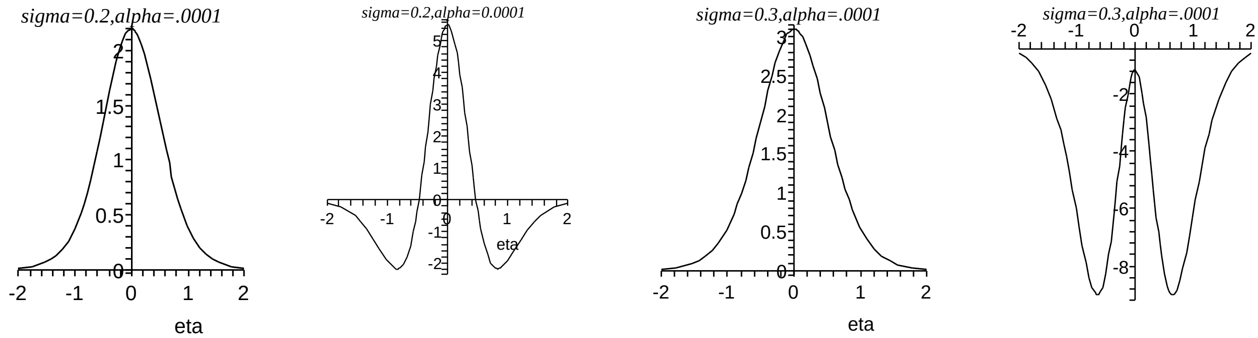

FIGURE 1. From left to right: curves 1 and 3 show steady one-soliton solutions $\tilde{\varphi}_1$ [Eq. (14)] versus $\frac{\eta}{D}$; curves 2 and 4 show second-order one-soliton $\tilde{\varphi}$ [Eq. (19)] versus $\frac{\eta}{D}$, for the first mode S_1.

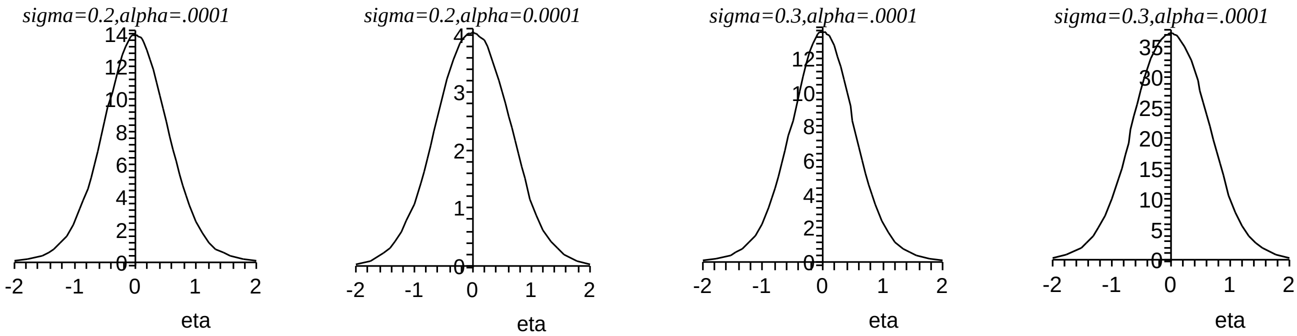

FIGURE 2. From left to right: curves 1 and 3 (1/10000 scale) show steady one-soliton solutions $\tilde{\varphi}_1$ [Eq. (14)] versus $\frac{\eta}{D}$; curves 2 and 4 (1/10000 scale) show second-order one-soliton $\tilde{\varphi}$ (Eq(19)) versus $\frac{\eta}{D}$, for mode S_2.

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APPENDIX

$$P = 1 + \frac{\alpha(1-\frac{3}{2}\frac{v_0^2}{c^2})}{\mu_i^2(v_0-S)^3}\left(\frac{2}{v_0-S} + \frac{3v_0}{c^2}\right) - \frac{3(1+\alpha)}{(S^2-3\sigma)^3}(S^2+\sigma),$$

$$Q = -\frac{\alpha b A S^2}{\mu_i}\frac{(1-\frac{3}{2}\frac{v_0^2}{c^2})}{(v_0-S)^4} - \frac{4Sb}{(v_0-S)}\left[1 - A\frac{S^2(1+\alpha)}{(S^2-3\sigma)^2}\right] + \frac{S^2 bA(1+\alpha)}{(S^2-3\sigma)^4}(S^4+2S^2-9\sigma^2),$$

$$R = -\frac{\alpha A S^2}{2\mu_i}\frac{(1-\frac{3}{2}\frac{v_0^2}{c^2})}{(v_0-S)^4} - \frac{2S}{(v_0-S)}\left[1 - A\frac{S^2(1+\alpha)}{(S^2-3\sigma)^2}\right] + \frac{3}{2}(S^2+\sigma)\frac{AS^2(1+\alpha)}{(S^2-3\sigma)^3},$$

$$Z_1 = \frac{D^2}{2}(P - bAQ)\phi_0^2 + A\left(\frac{15}{2}Q + \frac{42}{2}bR\right)\phi_0^{\frac{3}{2}} - 420\frac{AR}{D^2}\phi_0,$$

$$Z_2 = -A\left(\frac{16}{3}Q + 12bR\right)\phi_0^{\frac{3}{2}} + 520\frac{AR}{D^2}\phi_0.$$