A Mathieu equation for dust charge dynamics in multi-component dusty plasmas

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Abstract

The parametric excitation of dust acoustic oscillations due to dust grain charge fluctuations in a four-component plasma consisting of positive and negative inertial dust grains and a thermalized (Maxwellian) background of electrons and ions is investigated. Employing a two-fluid plasma description, and assuming a periodic fluctuation of the dust charge Q, a Mathieu-type ordinary differential equation is obtained for the dust number density, and analyzed via an averaging technique.

1. Introduction. Complex (dusty) plasmas (DPs) are characterized by the presence of massive mesoscopic (micron-sized, typically) dust particulates, whose charge may vary in time via a plethora of charging processes [1]. Although the electric charge residing on the dust grains is mostly negative, due to the high electron mobility, positive and negative dust coexistence is also witnessed in space and laboratory plasmas [2].

Here, we investigate the parametric excitation of dust acoustic oscillations due to charge fluctuations in a DP featuring a coexistence of negative and positive dust charge.

2. The model. We consider an unmagnetized collisionless dusty plasma, consisting of *two* dust grain species, of positive and negative charge $q_{d\pm} = \pm Q$ and constant mass $m_{d\pm} = M$, in addition to a thermalized (Maxwellian) background of electrons (mass m_e, charge $q_e = -e$) and ions (mass m_i, charge $q_i = +Z_i e$); e is the (absolute) electron charge. The dust grain charge is considered to be a time-dependent variable, i.e. $Q(t) = Z_d(t)e$.

The (two) cold inertial dust fluids are described by the density evolution equation

$$\frac{\partial n_{d\pm}}{\partial t} + n_{d\pm}\frac{\partial u_{d\pm}}{\partial z} = 0\,, \tag{1}$$

where t and z are independent time and (one-dimensional) space variables and $n_{d\pm}$ and $v_{d\pm}$ are density and velocity variables, respectively. The momentum equation(s) read

$$\frac{\partial u_{d\pm}}{\partial t} = \mp\frac{Q}{M}\frac{\partial \phi}{\partial z}\,, \tag{2}$$

where the convective term was omitted. The electric potential ϕ obeys Poisson's equation

$$\frac{\partial^2 \phi}{\partial z^2} = -4\pi \sum_{s(pecies)=1}^{4} n_s q_s = 4\pi e[(Z_d(n_{d-} - n_{d+}) + n_e - Z_i n_i]\,, \tag{3}$$

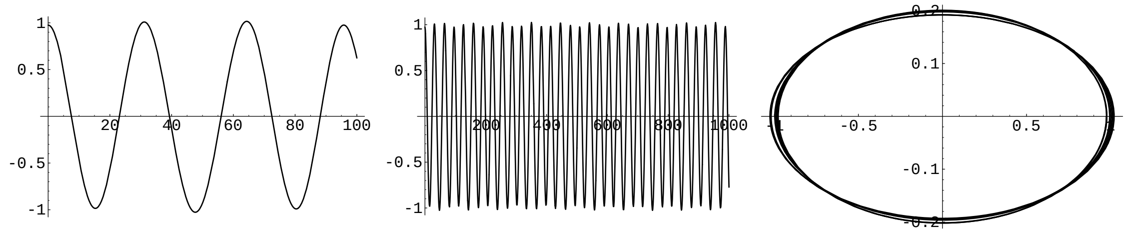

Figure 1: (a) and (b) $x(t)$ vs. t; (c) phase space $x(t)$-$x'(t)$; $\omega_0 = 100\gamma = 2\pi$ and $h = 0.1$.

where n_i and n_e denote the ion and electron number density, respectively. The right-hand side of Eq. (3) cancels at equilibrium, thanks to the charge neutrality condition $Z_i n_{i0} - n_{e0} + Z_{d,0}(n_{d+,0} - n_{d-,0}) = 0$, where n_{s0}, for $s = i, e, d^{+/-}$, denotes the ion, electron and dust $(+/-)$ particle number density at equilibrium, respectively. We assume a harmonic potential variation in space, characterized by a wave length $\lambda \equiv 2\pi/k$ (and a wave number k), i.e. $\phi(z,t) = \hat{\phi}(t)\exp(ikz)$. Being much lighter than dust particles, both electrons and ions are assumed to be in local thermodynamic equilibrium, so their number densities, n_e and n_i, obey a Boltzmann distribution, viz. $n_e = n_{e0}\exp(e\phi/k_B T_e)$ and $n_i = n_{i0}\exp(-Z_i e\phi/k_B T_i)$.

3. Derivation of a Mathieu equation for dust fluctuations. Assuming a weak potential value $\phi \ll \{k_B T_e/e,\, k_B T_i/Z_i e\}$, and considering a harmonic dust charge fluctuation, i.e. $Z_d = Z_{d0}(1 + h\cos\gamma t)^{1/2}$ (where the real parameter $h \ll 1$ and the frequency γ are real constants), Eqs. (1) to (3) are combined into a closed evolution equation for the dust density, in the form

$$\frac{d^2x}{dt^2} + \omega_0^2(1 + h\cos\gamma t)\,x = 0\,, \tag{4}$$

where we have defined the dimensionless parameter $x = (n_{d+} - n_{d-})/(n_{d+,0} - n_{d+,0}) - 1$, and the characteristic oscillation frequency $\omega_0 = \sqrt{2}\omega_{pd}\,k/(k^2 + k_D^2)^{1/2}$; the dust plasma frequency reads $\omega_{pd} = (4\pi n_{d0} Z_{d0}^2 e^2/M)^{1/2}$. The Debye wave number k_D is defined via the effective Debye length $\lambda_{Deff} = (\lambda_{De}^{-2} + \lambda_{Di}^{-2})^{-1/2}$, where $\lambda_{De} = (k_B T_e/4\pi n_{e0} e^2)^{\frac{1}{2}}$ and $\lambda_{Di} = (k_B T_i/4\pi Z_i n_{i0} e^2)^{\frac{1}{2}}$ are the electron and ion Debye radii, respectively. See that the inertialess electrons and ions affect the dust acoustic oscillations via a dynamical charge balance. In the limit $k \gg k_D$, $\omega_0 \approx \omega_{pd}\sqrt{2}$. Dust density gradients are neglected.

Upon formally setting $\omega_0^2 \to \alpha > 0$, $h \to -2q/\omega_0^2$ and $\gamma t \to 2y$, the ordinary differential equation (ODE) (4) is cast into the *canonical form* of the MATHIEU EQUATION:

$$\frac{d^2x}{dt^2} + (\alpha - 2q\cos 2y)\,x = 0\,, \tag{5}$$

which is well known to describe parametric oscillations [3-5]. The Mathieu Equation has been extensively studied in the past [6, 7] and its nonlinear behavior is more or less known. Some of this know how may now be applied in the present case, in view of elucidating dust charge dynamics in complex plasmas.

For given α, q and y parameter values, Eq. (5) possesses a solution in terms of (combinations of) the odd and even parity *Mathieu functions* $C(\alpha,q,y)$ and $S(\alpha,q,y)$, respectively [6, 7]. Naturally, the harmonic solutions $C(\alpha,0,y) = \cos\sqrt{a}y$ and $S(\alpha,0,y) =$

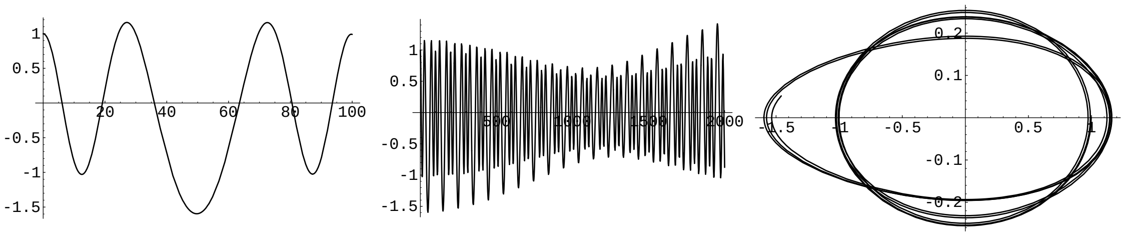

Figure 2: (a) and (b) $x(t)$ vs. t; (c) phase space $x(t)$-$x'(t)$; $\omega_0 = 100\gamma = 2\pi$ and $h = 0.7$.

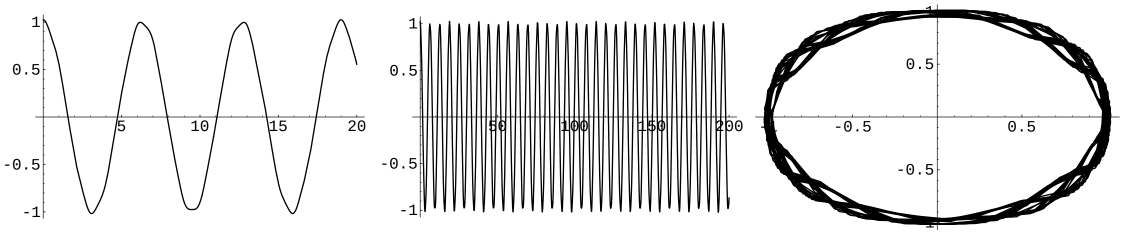

Figure 3: (a) and (b) $x(t)$ vs. t; (c) phase space x-x'; $\omega_0 = \gamma/20 = 2\pi/20$ and $h = 0.9$.

$\sin\sqrt{a}y$ are recovered in the limit $q \sim h \to 0$. For $q \sim h \neq 0$, the Mathieu functions are periodic (in $y \sim t$) *only* for specific values (eigenvalues) of α, often denoted as a_r and b_r (where r is a rational number), for which C and S give rise to the *elliptic cosine* and *elliptic sine* functions, $ce_r(y,q)$ and $se_r(y,q)$, respectively. For arbitrary values of $\alpha \sim \omega_0^2$ (other than the eigenvalues a_r and b_r), the Mathieu functions may be rather complicated non-periodic functions of time.

The (even) solution of Eq. (4) is depicted in Figs. 1-3, for a set of representative parameters values: $\omega_0 = 100\gamma = 2\pi$ (so that $T_0 \equiv 2\pi/\omega_0 = 0.01 \times 2\pi/\omega_0 \equiv 0.01T_{ch} \ll T_{ch}$). Furthermore, in Fig. 1 we have taken $h = 0.1$, which results to a phase portrait which is qualitatively reminiscent of the harmonic behavior (for $h = 0$). A higher value of $h = 0.7$ results in a more complex, bi-periodic behavior; see Fig. 2. The opposite limit $T_{ch} \ll T_0$ is considered in Fig. 3.

Interestingly, for certain (realistic) parameter values, Mathieu functions may possess a finite imaginary part; as a consequence, the solutions of Eq. (4) (physically defined via the dust species' number densities; see above) may undergo a *damping* effect. Physically, this is a different facet of the Melandsø collisionless damping mechanism [1], well known to dominate dust-acoustic oscillations due to dust grain dynamical charging.

4. Parametric resonance and fixed point stability analysis. The stability of dust charge oscillations near parametric resonance may be studied via an averaging technique proposed by Landau (see §27 in Ref. 3). Since parametric resonance is stronger for frequencies near $\omega(t) \approx 2\omega_0$, we shall consider the parametric excitation frequency to be $\gamma = 2\omega_0 + \epsilon$, where $\epsilon \in \Re$ (and h here) is (are) assumed to be small.

We shall assume a solution given by the ansatz

$$x = a(t)\,\cos\Big(\omega_0 + \frac{\epsilon}{2}\Big)t + b(t)\,\sin\Big(\omega_0 + \frac{\epsilon}{2}\Big)t\,, \qquad (6)$$

where the (real) coefficients a and b vary *slowly* with time. Substituting Eq. (6) into (4) and keeping only first order terms ϵ and h, we obtain the system of equations

$$\frac{da}{dt} = -\frac{b}{2}\left(\epsilon + \frac{h\omega_0}{2}\right) \equiv f(a,b)\ , \qquad \frac{db}{dt} = +\frac{a}{2}\left(\epsilon - \frac{h\omega_0}{2}\right) \equiv g(a,b)\,. \tag{7}$$

See that $a = b = 0$ (i.e. $x = 0$) determines an equilibrium state, i.e. a *fixed point* in phase space (a,b).

Parametric resonance: Taking $a,\, b \sim e^{st}$, one obtains $s^2 = \frac{1}{4}[(h\omega_0/2)^2 - \epsilon^2]$; the reality condition $-\frac{h\omega_0}{2} < \epsilon < \frac{h\omega_0}{2}$ delimits the range where parametric resonance occurs, in the vicinity of ω_0. For ϵ sufficiently small, any deviation from equilibrium leads to a rapidly increasing displacement $x(t)$.

Linear stability: The stability of the fixed point is determined by the eigenvalues of the Jacobian matrix $\mathbf{J}$ of the vector fields in Eqs. (7). Here, $a_{11} = \frac{\partial f}{\partial a} = 0$, $a_{12} = \frac{\partial f}{\partial b} = -\frac{1}{2}(\epsilon + \frac{h\omega_0}{2})$, $a_{21} = \frac{\partial g}{\partial a} = \frac{1}{2}(\epsilon - \frac{h\omega_0}{2})$ and $a_{22} = \frac{\partial g}{\partial b} = 0$. The eigenvalues are determined by the characteristic polynomial $p(\lambda) = \lambda^2 + (\epsilon^2 - h^2\omega_0^2/4)/4$, which possesses two (conjugate) imaginary roots $\lambda_{1,2} = \pm i(\epsilon^2 - h^2\omega_0^2/4)^{1/2}/2$, provided that $|\epsilon| > |h\omega_0/2|$. Thus, sufficiently far from resonance, the origin is the center of a (stable) oscillatory orbit. This is in agreement with the phase portrait obtained numerically in Figs. 1-3.

5. Discussion. Concluding, we have investigated the parametric excitation of dust acoustic oscillation due to dust grain charge variation in time. A two-fluid model led to a Mathieu-type ODE, describing nonlinear oscillations of the dust number density in time. Further investigation may allow to predict parameter regions where dust charging may lead to instability due to parametric resonance or damping.

It may be added, for rigor, that a similar analysis was employed in the past to model dust dynamics in the presence of dynamical source and sink mechanisms [8, 9], leading to a hybrid Van der Pol-Mathieu Equation for dust dynamics.

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