Two-dimensional envelope structures in quantum plasmas

M. McKerr[1] and I. Kourakis[1]

[1] *Centre for Plasma Physics, Department of Physics and Astronomy, Queen's University Belfast, BT7 1NN Northern Ireland, UK*

The dynamics of electrostatic excitations is investigated in a quantum plasma, with degenerate electrons and positrons. A fluid model is obtained by considering a three-component (electron-positron-ion) plasma.The classical case has been studied in ([1], [2], [3]). In each of these works, a coupled set of nonlinear partial differential equations is derived, which describes the motion of a two-dimensional structure, directly related to the evolution of measurable quantities such as the electric potential and ion density.

Quantum mechanics becomes important in the limits of high density and low temperature. Since we deal with spin-$\frac{1}{2}$ particles, the high-density, low-temperature limit involves a significant pressure due to the Pauli exclusion principle. After the work of Manfredi and Haas ([4], [5]), much effort has focused on developing models which account for quantum effects [6] (e.g. the Bohm-Pines diffraction term). Manfredi and Haas approached quantum plasmas in a manner analogous to classical kinetic theory, taking moments of the Wigner function and truncating at a suitable level to obtain dynamical equations, eventually closing the system with Poisson's Equation.

Presented in this work is a classical framework with quantum-mechanical considerations. Following closely the work of [7] we assume the Bohm-Pines diffraction term is small enough to be ignored. A modulated state (wavepacket) (n_i, ϕ, $\vec{v}$) is described by slowly-evolving pulse, moving at the group velocity, expressed via the electric potential, $\psi(\varepsilon(\vec{X}-\vec{v}_g t), \varepsilon^2 t)$, which is a solution of a Davey-Stewartson system [8].

The fluid model reads:

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_i) = 0$$
$$\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i = -\frac{q_i}{m_i}\nabla\phi - \frac{1}{m_i n_i}\nabla P_i$$
$$\nabla^2 \phi = \frac{e}{\varepsilon_0}(n_e - n_i Z_i - n_p) \tag{1}$$

where the indices 'e', 'i' and 'p' denote electrons, ions and positrons respectively, while the variables n and $\vec{v}$ denote the ion fluid density and velocity (ϕ is the electric potential).

The inertia of the electrons and positrons is assumed to be negligible, so that [7]

$$n_{e,p} = -\left(\frac{mK_B T_{e,p}}{2^{\frac{1}{3}}\pi\hbar^2}\right)^{\frac{3}{2}} Li_{\frac{3}{2}}\left(-e^{\frac{\mu_{0e,p}\pm e\phi}{K_B T_{e,p}}}\right) \tag{2}$$

where $Li_s(x)$ is the polylogarithm function $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$ and $\mu_{0e,p}$ is the electron (positron) chemical potential at equilibrium. Finally we consider substitute$P_i = \frac{2}{5} E_{F_i} n_{i0} \left(\frac{n_i}{n_{i0}}\right)^{\frac{5}{3}}$.

Defining $\beta = \frac{n_{p0}}{Z_i n_{i0}}$ in (1), expanding $n_{e,p}$ in ϕ near the equilibrium value, $\phi_0 = 0$ and finally rescaling time, distance, number densities and electrostatic potential, we obtain

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_i) = 0$$
$$\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i = -a \nabla \phi - g n_i^{-\frac{1}{3}} \nabla n_i$$
$$\nabla^2 \phi \approx b(1 - n_i) + c_1 \phi + c_2 \phi^2 + c_3 \phi^3 \tag{3}$$

where $g = \frac{2T_0 \mu_{i0}}{3m_i V_0 L_0} = \frac{2\mu_{i0}}{3m_i V_0^2}$, $a = \frac{Z_i e \phi_0 T_0}{m_i V_0 L_0} = \frac{Z_i e \phi_0}{m_i V_0^2}$, $b = \frac{n_{i0} Z_i e L_0^2}{\varepsilon_0 \phi_s}$ and

$$\begin{aligned} c_j \;=\; & \frac{L_0^2}{\phi_0} \frac{e Z_i n_{io}}{\varepsilon_0} \frac{1+\beta}{j! Li_{\frac{3}{2}}(-e^{\frac{\mu_{e0}}{K_B T_e}})} \left(\frac{e\phi_0}{K_B T_e}\right)^j \frac{d^j}{d\mu^j} Li_{\frac{3}{2}}(-e^{\mu})|_{\mu = \frac{\mu_{e0}}{K_B T_e}} \\ + \; & \frac{L_0^2}{\phi_0} \frac{e Z_i n_{io}}{\varepsilon_0} \frac{(-1)^{j-1}\beta}{j! Li_{\frac{3}{2}}(-e^{\frac{\mu_{p0}}{K_B T_p}})} \left(\frac{e\phi_0}{K_B T_p}\right)^j \frac{d^j}{d\mu^j} Li_{\frac{3}{2}}(-e^{\mu})|_{\mu = \frac{\mu_{p0}}{K_B T_p}} \end{aligned} \tag{4}$$

In the above, the subscript '0' denotes appropriate scaling quantities, to be determined later.

To first order, assuming harmonic waves: $\phi = \psi e^{i(\vec{k} \cdot \vec{X}_0 - \omega T_0)} + c.c.$, we obtain $n_1 = \frac{c_1 + k^2}{b} \phi$ and $\vec{v}_1 = \frac{\omega}{k^2} \frac{c_1 + k^2}{b} \vec{k} \phi$, along with the dispersion relation

$$\omega^2 = \frac{abk^2}{c_1 + k^2} + gk^2 \tag{5}$$

To proceed, we assume that the wavepacket envelope varies slowly in space and time. Adopting a multiscale technique [9], we define $T_j = \varepsilon^j t$, $X_j = \varepsilon^j x$ and $Y_j = \varepsilon^j y$. To second order, we obtain: $\frac{\partial \psi}{\partial T_1} + \vec{v}_g \cdot \nabla_1 \psi = 0$, where $\vec{v}_g = \nabla_k \psi$, along with specific expressions for the second-order amplitudes of the zeroth, first and second harmonics [10]. In particular, the zeroth harmonics read: $\phi_2^{(0)} = \mathrm{C}_{2_3}^0 |\psi|^2 + \gamma_\phi Y$, $n_2^{(0)} = \mathrm{C}_{2_1}^0 |\psi|^2 + \gamma_n Y$, $\vec{v}_{2x}^{(0)} = \mathrm{C}_{2_2}^0 |\psi|^2 + \gamma_u Y$ and

$$\vec{v}_{2y}^{(0)} = \int dY_1 \frac{\partial Y}{\partial X_1} \tag{6}$$

The compatibility condition arising from the first harmonic amplitudes and the consistency requirement for $\vec{v}_2^{(0)}$, $\phi_2^{(0)}$ and $n_2^{(0)}$ combine into a Davey-Stewartson system (x, y and t now represent the stretched variables εx, εy and $\varepsilon^2 t$):

$$i\psi_t + P_1 \psi_{xx} + P_2 \psi_{yy} + Q_1 |\psi|^2 \psi + Q_2 \psi Y = 0 \tag{7}$$

$$P_3 Y_{xx} + P_4 Y_{yy} + Q_3 |\psi|^2_{yy} = 0 \tag{8}$$

Possible solutions include multidimensional generalizations of the envelope soliton form (e.g. dromions) [11]. The coefficients in the above expressions are defined and analyzed elsewhere [10]. In particular, we note that $P_{1,2}$ are related to the dispersion relation as $P_{1,2} = \frac{1}{2} \frac{\partial^2 \omega}{\partial k_{x,y}^2}$.

Modulational Instability Analysis. A monochromatic plane wave, $a_0 e^{iQ_0 t}$, and $Y = 0$ are taken to form a reference solution of (7, 8). The amplitude and phase are disturbed by real functions $a(x,y,t)$ and $b(x,y,t)$ respectively. After substituting into the linearized equations, then choosing $a,b,Y \propto e^{i(x\kappa\cos\theta + y\kappa\sin\theta - \Omega t)}$ and separating real from imaginary parts, one arrives at the dispersion relation for the perturbation of the amplitude

$$\Omega^2 = (P\kappa)^2\left(1 - \frac{2a_0^2 Q}{P\kappa^2}\right) \tag{9}$$

where

$$P = P_1\cos^2\theta + P_2\sin^2\theta\,, \qquad Q = Q_1 + \frac{Q_2 Q_3 \sin^2\theta}{P_3\cos^2\theta + P_4\sin^2\theta} \tag{10}$$

The frequency becomes imaginary for $Q \cdot P > 0$, which denotes a growing or decaying wave for $0 < \kappa < \kappa_{crit} = a_0\sqrt{\frac{2Q}{P}}$. The growth rate attains its maximum value of $a_0^2 Q$ when $\kappa = a_0\sqrt{\frac{Q}{P}}$.

We proceed by choosing

$$\begin{array}{ccccccc} t & \mapsto & \omega_{p_i} t & x & \mapsto & \frac{x}{\lambda_{TF}} \\ 3m_i V_0^2 & = & 2E_{F_i} & \phi_s & = & \frac{2E_{F_i}}{3Z_i e} \end{array} \tag{11}$$

so that $a = b = g = 1$, in order to investigate the parametric dependence of instability-related quantities on the density ratio β and on the angle θ.

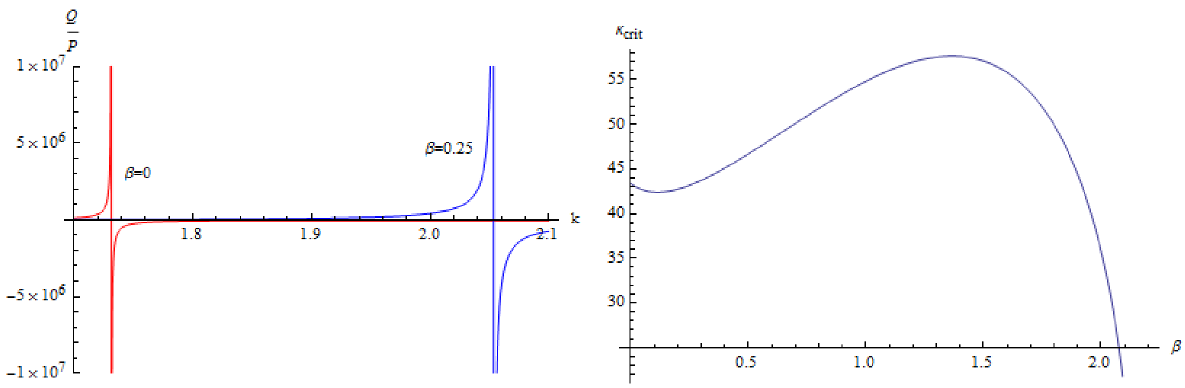

Figure 1: For longitudinal perturbation, an increase in β leads to an increase in the range of carrier wavenumber, k, for which instability arises. However, increasing β with all other variables fixed results in an initial increase in κ_{crit}, followed by a decline and ultimate disappearance of any instability. On the right, $k = 1$, $\frac{e\phi_0}{K_B T_e} = 1$, $\theta = 0$. For sufficiently large β, the perturbation is stable.

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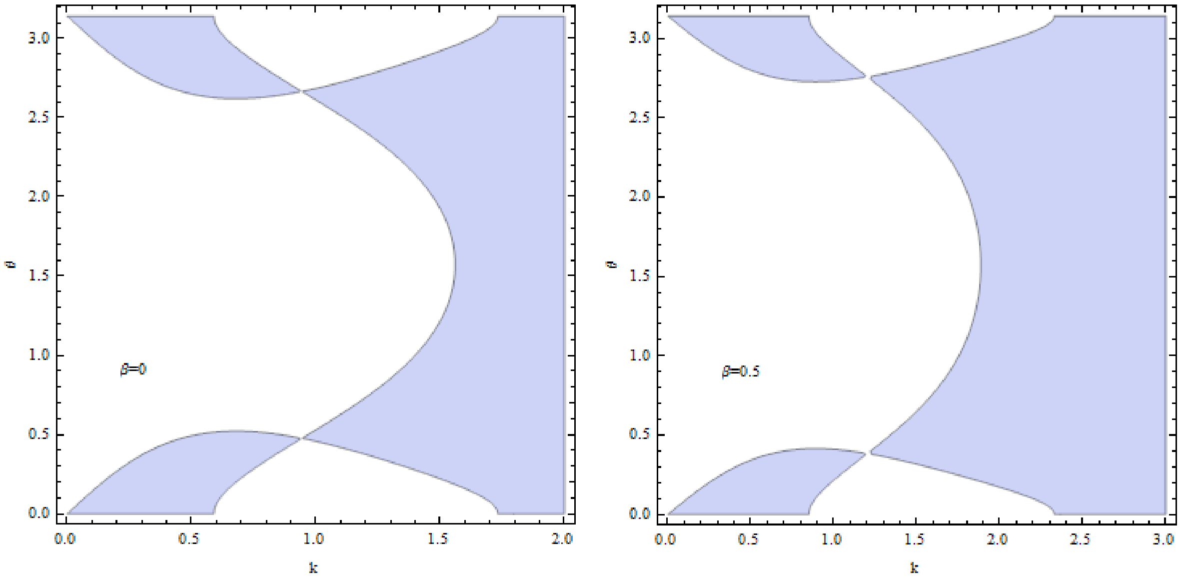

Figure 2: Two plots showing how changing the angle of the disturbance, θ, affects the range of wavenumber, k, that admits instability. Unshaded regions denote areas where $\Omega^2 < 0$ and each horizontal slice shows unstable intervals of k for a particular θ. The graph on the left depicts the effect for on electron-ion plasma, whereas the graph on the right describes the effect on an electron-ion-positron plasma where positrons account for half of the positive charge at equilibrium.

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