A modified Schamel equation for ion acoustic waves in superthermal plasmas

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The presence of energetic particles in plasmas, resulting in long-tailed distributions, is an intrinsic element in many space and laboratory plasma observations. Another commonly observed phenomenon in both space and laboratory plasmas is that of particle trapping, whereby some of the plasma particles are confined to a finite region of phase space. This nonlinear effect was first included in analytical models of electrostatic structures by Bernstein, Greene and Kruskal (BGK) [1], and later Schamel [2] developed a pseudopotential method for the construction of equilibrium solutions, and also derived a KdV equation for ion acoustic waves which is modified due to resonant (trapped) electrons.

Motivated by these observations, we have undertaken an investigation of the propagation of ion acoustic waves in nonthermal plasmas in the presence of trapped electrons. An unmagnetized collisionless electron-ion plasma is considered, featuring a superthermal (non-Maxwellian) electron distribution, which is modelled by a (kappa) distribution function [3]. We have considered the effect of particle trapping, deriving an expression for the electron density. We have used reductive perturbation theory to construct a modified Schamel equation, and examine its dynamics. A solitary wave solution is presented and its dynamics discussed. The chief effect of modification due to the degree of particle trapping is stronger nonlinearity, while superthermality affects the amplitude and width of solitons adversely.

The hybrid Schamel-Kappa (κ) model

The kappa electron distribution in one dimension [3] is given by:

$$f_e^\kappa(v) = \frac{N_0}{(\pi\kappa\theta^2)^{1/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})} \left(1+\frac{v^2}{\kappa\theta^2}\right)^{-\kappa}, \tag{1}$$

where N_0 is the species equilibrium number density and the effective thermal speed is

$$\theta = [(\kappa-3/2)/\kappa]^{1/2}(2k_BT/m)^{1/2},$$

which requires $\kappa > 3/2$ to be physically realistic.

Normalizing such that $\int_{-\infty}^{+\infty} f_e^\kappa(v)dv = 1$, and using the energy conservation relation ($m_e v_e^2/2 - e\phi = m_e V^2/2$), where $e\phi$ is the increase in potential energy, and V is the velocity of the particles in the initial equilibrium state), scaling v by k_BT/m, and ϕ by k_BT/e, Eqn. (1) can be written as:

$$f_e^\kappa(v,\phi) = \frac{1}{\sqrt{2\pi}(\kappa-\frac{3}{2})^{\frac{1}{2}}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}\left(1+\frac{\frac{v^2}{2}-\phi}{\kappa-\frac{3}{2}}\right)^{-\kappa}. \tag{2}$$

Schamel [2] introduced the concept of a separatrix to the Maxwellian distribution which separates free electrons from trapped ones. The energy of the electrons is defined as $E_e = v(x)^2/2 - \phi(x)$, and the energy separatrix occurs at the point where the energy equals zero, that is, $E_{es} := v(x)^2/2 - \phi(x) = 0$. When $E_e > 0$ the distribution is described by the Maxwellian. For $E_e < 0$ Schamel defines the distribution as:

$$f_{e,t}^{Max}(v,\phi) = \frac{1}{\sqrt{2\pi}}\exp[-\beta(v^2/2-\phi)] \qquad \text{for } E_e \le 0, \tag{3}$$

where β is a parameter which determines the efficiency of electron trapping. At $\beta = 1$, the Maxwellian is recovered. Applying the same argument for the separatrix to the kappa distribution, a trapped electron κ distribution can be written as:

$$f_e^\kappa(v,\phi) = \frac{1}{\sqrt{2\pi}(\kappa-\frac{3}{2})^{\frac{1}{2}}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}\left(1+\beta\frac{\frac{v^2}{2}-\phi}{\kappa-\frac{3}{2}}\right)^{-\kappa} \qquad \text{for } E_e \le 0. \tag{4}$$

Equation (4) recovers Eqn. (3) as $\kappa \to \infty$, and Eqn. (2) as $\beta \to 1$.

The electron density n_e is found by integrating the Schamel κ distribution (4) over all v and Taylor expanding:

$$n_e = 1 + p\phi + q\phi^{3/2} + ..., \tag{5}$$

with $p = \frac{2\kappa-1}{2\kappa-3}$, $\qquad q = \frac{8\sqrt{2/\pi}(\beta-1) \quad \kappa \quad \Gamma(\kappa)}{3(2\kappa-3)^{3/2}\Gamma(\kappa-1/2)}$.

The Fluid Model. We shall now consider ion acoustic waves propagating in a plasma consisting of cold ions ($T_i = 0$) and electrons with a Schamel-kappa distribution. As is usual for ion acoustic structures, we require that the wave phase speed lies between the ion and electron thermal speeds, that is, $v_{ti} \ll v_{ph} \ll v_{te}$, to avoid Landau damping [6]. The one dimensional system of normalized fluid equations for the ions, together with Poisson's equation are:

$$\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{7}$$

$$\frac{\partial^2\phi}{\partial x^2} = n_e - n \simeq -(n-1) + p\phi + q\phi^{3/2} + r\phi^2, \tag{8}$$

where n and u represent the ion density and velocity respectively, and ϕ is the electrostatic potential. We assume charge neutrality at equilibrium, that is, $n_e = Zn_0$, where Zn_0 is the equilibrium ion charge density. We have employed the following normalizations: lengths are normalized by a characteristic Debye length $\lambda_D = \left(\frac{\varepsilon_0 k_B T_e}{n_0 Z e^2}\right)^{\frac{1}{2}}$, time by the inverse plasma frequency

$\omega_p = \left(\frac{n_0 Z^2 e^2}{\varepsilon_0 m}\right)^{\frac{1}{2}}$, number density by the equilibrium ion density n_0, electrostatic potential by $\left(\frac{k_B T_e}{e}\right)$, and velocities by a characteristic species sound speed $c_s = \left(\frac{Z k_B T_e}{m}\right)^{\frac{1}{2}}$.

Reductive Perturbation Theory. Using a reductive perturbation technique [4], independent variables are stretched as follows: $\zeta = \varepsilon^{\frac{1}{4}}(x - \mathbb{V}t)$, $\tau = \varepsilon^{\frac{3}{4}}t$, where $\mathbb{V}$ is the phase velocity of the excitation and ε is a smallness parameter. The second order terms yield compatibility conditions which lead to a KdV-like Schamel equation [5]:

$$\frac{\partial \phi_1}{\partial \tau} + A\, \phi_1^{1/2} \frac{\partial \phi_1}{\partial \zeta} + B \frac{\partial^3 \phi_1}{\partial \zeta^3} = 0, \tag{9}$$

where $A = \frac{(1-\beta)}{\sqrt{\pi}\sqrt{\kappa - \frac{1}{2}}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}$, $B = \frac{1}{2\left(1+\frac{1}{\kappa-\frac{3}{2}}\right)^{3/2}}$. A solitary wave solution of Eqn. (9) [5] reads:

$$\phi_1 = \phi_m \,\mathrm{sech}^4\left((\zeta - u_0\tau)/\Delta\right), \text{where} \quad \phi_m = (15u_0/8A)^2, \quad \Delta = \sqrt{16B/u_0}. \tag{10}$$

ϕ_m and Δ are the height and width of the solitary waves, respectively, moving with speed u_0. We have carried out an extensive parametric investigation of the solitary wave characteristics. Details are omitted here for brevity, but the main points are as follows.

Effect of Superthermality on Soliton Structure. In plasmas with higher proportions of superthermal electrons (that is, plasmas with a lower κ value) the nonlinearity and width parameters are affected, producing solitons with decreased amplitude and width with more localised electric fields. This can be seen in Figure 1.

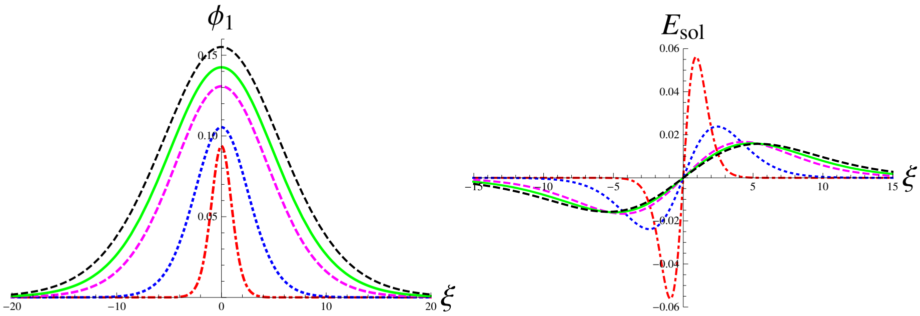

Figure 1: Left plot: solitary wave variation with κ, for $\beta = 0.5$ and $u_0 = 0.06$. Right plot: electrostatic potential (pulse) variation for the same values. The red line is $\kappa = 1.6$, blue $\kappa = 2$, pink $\kappa = 4$, green $\kappa = 7$, and black $\kappa = 30$.

Effect of Particle Trapping on Soliton Structure. Solitary waves in plasmas containing higher proportions of free electrons, signified by a lower β value, have decreased amplitude while their width is unaffected, as shown in Figure 2.

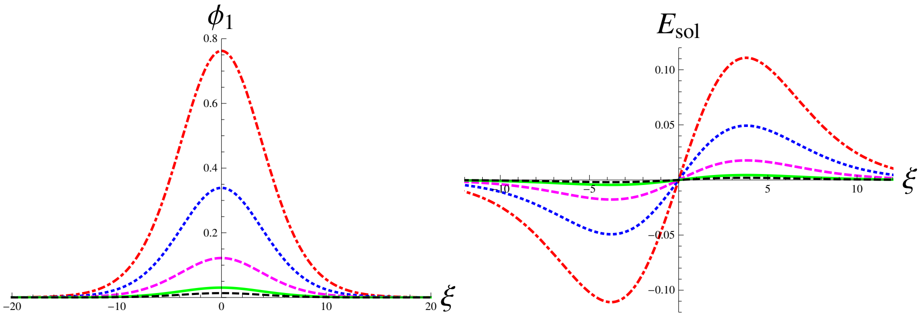

Figure 2: Left plot: solitary wave variation with β, for $\kappa = 3$ and $u_0 = 0.06$. Right plot: electrostatic potential (pulse) variation for the same values. The red line is $\beta = 0.8$, blue $\beta = 0.7$, pink $\beta = 0.5$, green $\beta = 0$, and black $\beta = -0.5$

Conclusion. We have investigated solitary ion acoustic wave propagation in the presence of electron trapping and superthermality. A physically meaningful Schamel-like κ distribution has been developed. Using reductive perturbation theory, we have derived a Schamel equation, and its corresponding solitary wave solution.

At higher levels of superthermality the solitary wave amplitude decreases and wave structures become narrower. The corresponding electric field becomes more localized with much sharper peaks in conditions of high superthermality.

With higher proportions of free electrons, that is, a reduction in the value of the β parameter, the solitary wave amplitude decreases, becoming almost negligible for negative values of β; however, the width of the wave remains unchanged.

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