# Freak Waves and Modulational Dynamics in Plasmas with Negative Ions

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## Abstract

The nonlinear dynamics of modulated electrostatic wavepackets propagating in negative-ion plasmas is investigated from first principles. A nonlinear Schrödinger equation is derived by adopting a multiscale technique. The stability of breather- like (bright envelope soliton) structures, considered as a precursor to *freak wave* (*rogue wave*) formation, is investigated and then tested via numerical simulations.

**Introduction.** Plasmas incorporating negative ions, in addition to positive ions and electrons exist in the laboratory [1, 2] and also in Space environments [3, 4, 5]. Such multicomponent plasmas support a variety of nonlinear modes, related with various instabilities and relevant phenomena [6, 7, 8]. The propagation of ion-acoustic (IA) waves in negative-ion plasmas (NIP) was studied theoretically within a nonlinear Schrödinger equation (NLSE) framework by Saito *et al* [9], who discussed modulational instability as a precursor for envelope modes in such a system. That work was based on the modified Korteweg-de Vries (mKdV) formalism (applicable in a "special" plasma configuration [10], where the nonlinearity coefficient in the ordinary KdV equation vanishes and higher-order nonlinearity dominates), and was thus limited to the weak-amplitude superacoustic region. The problem was later investigated experimentally by Bailung and coworkers [11, 12], who reported the observation of *freak-wave* (*rogue wave*) type structures, and adopted the interpretation provided by Saito *et al* [9] for their interpretation.

In this study, we adopt a multiscale perturbation technique to derive a nonlinear Schrodinger equation for modulated ion-acoustic wavepackets in negative ion plasma. Explicit conditions and instability thresholds for envelope soliton formation and modulational instability are determined, in terms of the negative-to-positive density and mass ratio(s). The stability of the solutions obtained has been investigated numerically via a Crank-Nicolson method. The main aim of our study is to extend the criteria (existence region) for freak waves in negative-ion plasmas, beyond the simplifying assumptions (weak-amplitude, weakly superacoustic regime) implicitly adopted in [11, 12]. In particular, we have shown, as anticipated, that the occurrence of freak wave structures is not necessarily due to the "special" plasma configuration that leads to the mKdV description [9], but may be observed in an extended region of parameter values.

**Theoretical model.** We consider a three component collisionless unmagnetized plasma consisting of electrons (mass $m_e$, charge $e$), positive ions (mass $m_+$, charge $q_+ = Z_+e$) and negative ions (mass $m_-$, charge $q_- = -Z_-e$), and electrons (mass $m_e$, charge $e$). A reduced (dimensionless) fluid model for this configuration, in one-dimensional (1D) geometry, reads

$$\partial_t n_+ + \partial_x(n_+u_+) = 0, \tag{1}$$

$$\partial_t u_+ + u_+\partial_x u_+ = -\nabla\phi, \tag{2}$$

$$\partial_t n_n + \partial_x(n_-u_-) = 0, \tag{3}$$

$$\partial_t u_- + u_-\partial_x u_- = \delta\nabla\phi, \tag{4}$$

$$\partial_{xx}\phi = -n_+ + \beta n_- + (1-\beta)\exp(\phi), \tag{5}$$

The fluid velocity ($u_\pm$), density ($n_\pm$) and electrostatic potential ($\phi$) variables are scaled by $V_0 = (Z_+ k_B T_e/m_+)^{1/2}$, $n_{0,+}$ and $k_B T_e/e$, respectively. Space and time are respectively scaled by $\omega_{p,+}^{-1} = (4\pi e^2 n_{+,0} Z_+^2/m_+)^{-1/2}$ and $\lambda_{D,+} = (k_B T_e/4\pi Z_+ e^2 n_{+,0})^{1/2}$. We have defined the quantities $\beta = n_{-,0}Z_-/(n_{+,0}Z_+)$ and $\delta = (Z_-/m_-)/(Z_+/m_+)$. Here, $\omega_{p,+}$ is the positive-ion plasma frequency, $T_e$ is the electron temperature and $k_B$ is the Boltzmann constant.

**Amplitude modulation modelling.** Let us define the state (column) vector $\mathbf{S} = (n_+, u_+, n_-, u_-, \phi)^T$ and the corresponding equilibrium state $\mathbf{S}^{(0)} = (1,0,1,0,0)^T$. We shall consider small deviations from equilibrium by taking $\mathbf{S} = \mathbf{S}^{(0)} + \varepsilon\mathbf{S}^{(1)} + \varepsilon^2\mathbf{S}^{(2)} + ... = \mathbf{S}^{(0)} + \sum_{n=1}^{\infty}\varepsilon^n\mathbf{S}^{(n)} = \mathbf{S}^{(0)} + \sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-n}^{n}\mathbf{S}_{\mathbf{j},\mathbf{l}}^{(n)}\exp[il(kx-\omega t)]$, where $\varepsilon \ll 1$ is a small real parameter. We define the stretched (slow) space and time variables $T_r = \varepsilon^r t$ and $X_r = \varepsilon^r x$ (for $r = 0,1,2,3,...$). The fast scales ($r = 0$) affect the dynamics via the phase $\theta = kX - \omega T$ (only), while the slow scales ($r \geq 1$) enter the argument of the $l$−th harmonic amplitude $\mathbf{S}_l^{(n)}$, allowed to vary along $x$.

*Linear dynamics*: The first-order (linear) expressions lead: $n_1 = \frac{\phi_1}{v_{ph}^2}$, $u_1 = \frac{\phi_1}{v_{ph}}$, along with the *(dispersion relation)*: $\omega^2 = \frac{(1+\delta\beta)k^2}{k^2+(1-\beta)}$. This provides (in the long-wavelength limit) the expression $v_{ph} = \omega/k \simeq \left(\frac{1+\delta\beta}{1-\beta}\right)^{1/2}$ for the true sound speed (affected by the negative ions ia the parameters $\beta$ and $\delta$). Note that an e-i plasma is recovered upon setting $\beta = 0$.

The 2nd-order equations provide a set of expressions (here omitted for brevity) for the amplitude(s) of the 2nd, 1st and 0th harmonic corrections at this order. Annihilation of secular terms prescribes the group velocity as $v_g = \frac{d\omega}{dk} = \frac{(1-\beta)(1+\delta\beta)^{1/2}}{[k^2+(1-\beta)]^{3/2}}$, viz. $S_{j,1}^{(1)} = S_{j,1}^{(1)}(X_1 - v_g T_1)$.

Proceeding to order $\sim \varepsilon^3$, one obtains the Nonlinear Schrödinger Equation (NLSE)

$$i\partial_\tau\Psi + P\partial_{\xi\xi}\Psi + Q|\Psi|^2\Psi = 0 \tag{6}$$

where $\tau = T_2 = \varepsilon^2 t$ and $\xi = X_1 - v_g T_1 = \varepsilon(x - v_g t)$. Note that the *dispersion coefficient* $P =$
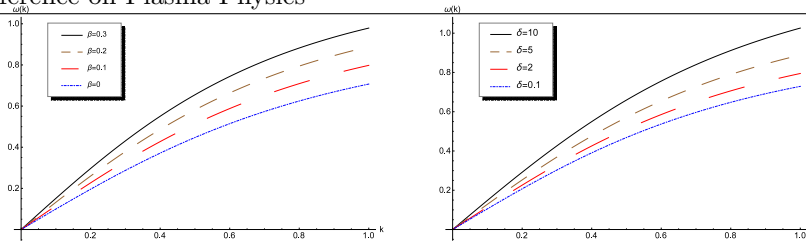
Figure 1: (Color online) The angular frequency $\omega$ is depicted versus the wavenumber $k$, for different values of $\beta$ and $\delta$.

$\omega''(k)/2$ is negative, while the *nonlinearity coefficient Q*, due to carrier wave self-interaction, is given as a perplex function $Q = Q(\{\omega; \beta, \delta, ...\})$ (here omitted).

**Freak waves as breather-type solutions of the NLSE.** Various solutions of the NLSE (6) have been proposed as prototypical forms of freak (rogue) waves. These are summarized, e.g., in [13] and thus need not be presented here.
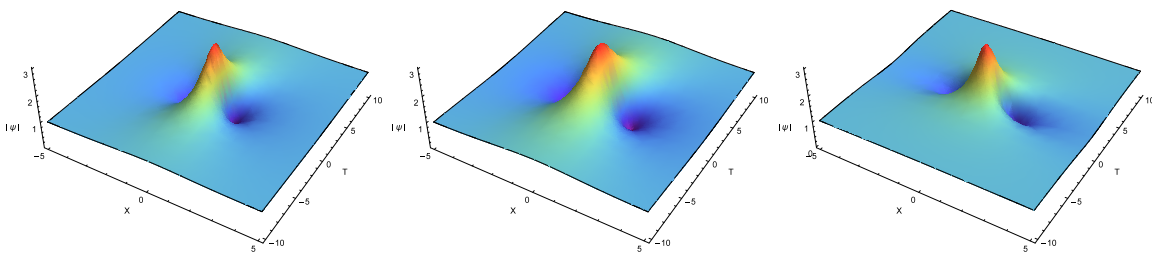


Figure 2: (Color online) The *Peregrine soliton* is depicted for different values of $\beta$ (= $0.2, 0.4, 0.6$), with $k = 0.1$ and $\delta = 1$ (e.g. for $H^+/H^-$ plasma).

**Numerical analysis.** In order to test our predictions for the nature and stability of bright envelope solitons, we have integrated the NLSE via an implicit integration method, in different cases. The known bright soliton solution was adopted as initial condition (IC) in the numerical integration of the NLSE (6). In some runs, the plasma parameters were taken to differ between IC and numerical integrator, with the aim to assess the impact of the variation of e.g., $\beta$ on a given energy lump injected in the system as initial condition. In brief, a bright pulse changing shape (but retaining its stability) in the bright ($PQ > 0$) region, while it decays and spreads in the dark ($PQ < 0$) region.

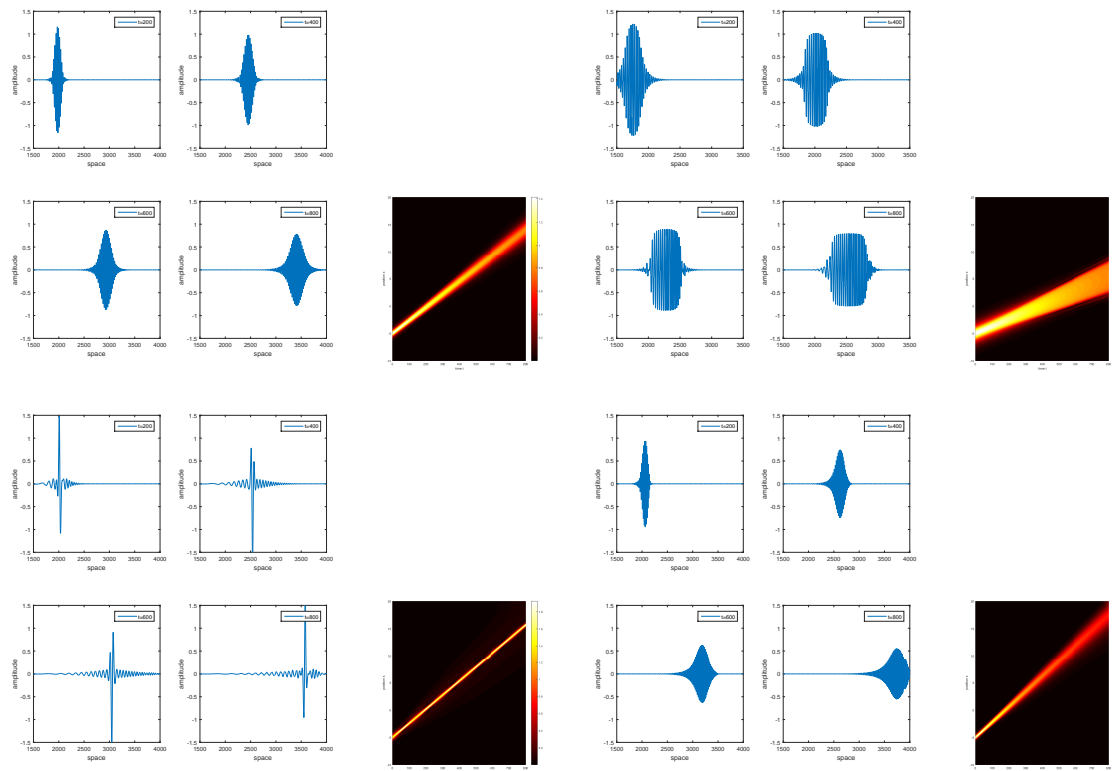Our results will be reported in full detail elsewhere.

Figure 3: (Color online) Time evolution and propagation of a breather (bright soliton) on the space-time plane. Top left panel: IC ($\beta = 0.1, \delta = 1, k = 0.1$, $PQ > 0$), NLSE ($\beta = 0.2, \delta = 1, k = 0.1$, $PQ > 0$). Top right panel: IC ($\beta = 0.2, \delta = 1, k = 0.1$, $PQ > 0$), NLSE ($\beta = 0, \delta = 1, k = 0.1$, $PQ < 0$). Bottom left panel: IC ($\beta = 0.2, \delta = 1, k = 0.1$, $PQ > 0$), NLSE ($\beta = 0.4, \delta = 2.1, k = 0.7$, $PQ > 0$). Bottom right panel: IC ($\beta = 0, \delta = 1, k = 0.1$, $PQ < 0$), NLSE ($\beta = 0.2, \delta = 1, k = 0.1$, $PQ > 0$).

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