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Modulational Instability of Langmuir Wavepackets in Collisional Plasmas

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Abstract

The conditions for modulational instability of Lagmuir wavepackets propagating in collisional plasmas are investigated, by adopting a phenomenological model where the electrostatic potential ϕ is modelled by a phenomenological nonlinear Klein-Gordon-type equation, involving *ad hoc* terms in account of intrinsic plasma nonlinearity and collisionality. A reduction to a nonlinear Schrodinger type equation is carried out. Explicit criteria for modulational instability are obtained. The role of collisionality is discussed, in comparison with the conservative case.

Introduction. Electron plasma (Langmuir) waves [1], analogous to optical modes in e.g. materials science [2], are characterized by a parabolic-shaped dispersion curve, where the group velocity vanishes and the phase speed diverges, for ultralong wavelenth (small wavenumber) values. The modulational dynamics of plasma wavepackets has been studied in the past, in various contexts [3, 4, 5, 6]. The description is usually amenable to an nonlinear Schrodinger equations (NLSE) for the wavepacket envelope, and deriving criteria for the stability of the latter is straightforward [2]. The NLSE is long known to be an integrable nonlinear equation [2], bearing exact solutions (envelope solitons, breathers) which have naturally been proposed as prototypical models for oscillons, surface waves and freak waves, among others.

In this work, a phenomenological model is adopted for Langmuir wavepackets in plasmas characterized by collisionality (e.g. due to intrinsic electron fluid "viscosity"). We take into account the inherent dynamical nonlinearity mechanisms and collisional effects by introducing suitable *ad hoc* terms in the evolution equation for the electrostatic potential ϕ. Relying on a multiscale perturbation technique [3, 4, 5, 6], we have derived a NLSE for the electric potential amplitude, along with explicit expressions for all relevant harmonic component amplitudes. The exact solutions of the NLSE, in the form of localised envelope pulses [2, 7], are stable in the absence of dissipation, while they decay in the dissipative case as expected [8], as corroborated by numerical simulations.

The meticulous analysis carried out cannot be entirely presented here, and will be reported in a (more detailed) forthcoming work. Our focus in this short report is on deriving explicit analytical criteria for modulational instability to occur, and to investigate the role of collisionality, thus generalizing earlier results on modulational interactions in plasmas [3, 4, 5, 6, 7].

Theoretical Model. We consider an *ad hoc* potential evolution equation in the form:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} + \omega_p{}^2 \phi + \alpha \phi^2 + \beta \phi^3 + \nu \frac{\partial \phi}{\partial t} = 0, \quad (1)$$

where $c = (2k_B T_e/m_e)^{1/2}$ is the electron thermal speed, $\omega_p = (n_{e0}e^2/\varepsilon_0 m_e)^{1/2}$ is the electron plasma frequency, and the ad hoc (real) parameters α and β take into account the effect of non-linearity; finally, a phenomenological damping term $\nu > 0$ is introduced. This has the form of a nonlinear Klein-Gordon-type PDE, here modeling the evolution of the electrostatic potential ϕ.

Nonlinear analysis. We adopt a multiple scales technique [3, 4, 6, 5, 9, 10], by defining a small parameter $\varepsilon \ll 1$ and introducing distinct scales, in order to distinguish the fast (carrier wave related) scales ($X_0 = x$, $T_0 = t$) from the slower (envelope related) scales ($X_1 = \varepsilon x$, $X_0 = \varepsilon^2 x$, ...; $T_1 = \varepsilon t$, $T_0 = \varepsilon^2 t$, ...) . We consider small deviations of the variable ϕ from the equilibrium state $\phi = \phi^{(0)} + \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(0)} + \ldots = \Sigma_{n=1}^{\infty} \varepsilon^n \phi_n$. Harmonic generation is accounted for via the ansatz $\phi_n = \Sigma_{l=-\infty}^{\infty} \phi_l^{(n)}(X,T) e^{il(kx-\omega t)}$. All the perturbed states depend on the fast scales via the phase $\theta_1 = kx - \omega t$ only, while the slow scales only enter the amplitude(s) $\phi_l^{(n)}$.

The dispersion relation obtained in 1st-order prescribed a *complex* frequency $\omega = \omega(k,\nu) = \omega_r - i\nu/2$, with $\omega_r = \sqrt{c^2k^2 - \frac{\nu^2}{4} + \omega_p{}^2}$. The (real) group velocity reads: $v_g = \omega'(k) = c^2k/\omega_r$. The 2nd order equations provide a compatibility condition in the form: $\frac{\partial \phi_1^{(1)}}{\partial T_1} + \omega'(k) \frac{\partial \phi_1^{(1)}}{\partial X_1} = 0$, suggesting that ϕ_1 is a function of $X_1 - v_g T_1$, to leading order. The corresponding 0th-, 1st- and 2nd- harmonic amplitudes (upto order $\sim \varepsilon^2$) - omitted here - are obtained in subsequent steps.

Annihilating secular terms in ε^3, we obtain a dissipative NLS equation in the form:

$$i\frac{\partial \psi}{\partial \tau} + P \frac{\partial^2 \psi}{\partial \zeta^2} + Q_r \mid \psi \mid^2 \psi + i Q_i \mid \psi \mid^2 \psi = 0, \quad (2)$$

where $\psi = \phi_1^{(1)}$ and the (slow) independent variables are $\zeta = \varepsilon(x - v_g t)$ and $\tau = \varepsilon^2 t$. In eq. (2), $P \in \Re$ and $Q (= Q_r + iQ_i) \in \mathscr{C}$ are dispersion and nonlinearity coefficients, which are given by

$$P = \frac{c^2\left(\omega_p{}^2 - \nu^2/4\right)}{2\omega_r{}^3}, \qquad Q = \frac{-3\beta + 2\alpha^2\left(\frac{2}{\omega_p{}^2} - \frac{1}{3\omega_p{}^2 - i\nu 2\omega_r - \nu^2}\right)}{2\omega_r} = Q_r + iQ_i. \quad (3)$$

The lengthy expressions for Q_r and Q_i are omitted, for brevity. The *loss term* Q_i arises due to the damping process (and, in fact, cancels in the limit $\nu \to 0$).

Modulational instability (MI) analysis. Assuming the existence of a stationary reference state in the form $\psi_0 = \hat{\psi}_0 e^{-i\Omega T}$, we find: $\Omega = -Q|\hat{\psi}_0|^2$. Therefore, $\psi_0 = \hat{\psi}_0 \, e^{iQ_r|\hat{\psi}_0|^2 T} \, e^{-Q_i|\hat{\psi}_0|^2 T}$ is a harmonic solution of Eq. (2), to be adopted as a reference state (in fact, an exponentially damped sinusoidal waveform, since Q is complex).

We proceed by considering a small perturbation around ψ_0, as $\psi_0 = (\hat{\psi}_0 + \varepsilon \psi_1)\, e^{-i\Omega T + \varepsilon i \Theta}$, where $\psi_1 = \psi_1(X,T)$ and $\theta = \theta(X,T)$ are amplitude and phase variations. We take both of these to be harmonic, $\sim e^{\tilde{k}X - \tilde{\omega}T}$, where $\tilde{\omega}$ and $\tilde{k}$ are the perturbation frequency and wavenumber.

Substituting in eq. (2) and linearizing, we obtain a dispersion relation for the perturbation:

$$\tilde{\omega}^2 + b\,\tilde{\omega} + c = 0, \tag{4}$$

where $b = 2Q_i\,|\hat{\psi}_0|^2$ and $c = -P^2\tilde{k}^2\left(\tilde{k}^2 - \frac{2Q_r}{P}\,|\hat{\psi}_0|^2\right) \equiv -\tilde{\omega}_0^2(\tilde{k})$. Note that we have defined the quantity $\tilde{\omega}_0(\tilde{k})$, which represents the functional expression for the unperturbed (un-damped) perturbation frequency [2], here readily recovered if $Q_i = 0$.

Extended MI criteria & threshold(s): the role of ν. Based on (4) above, we have investigated whether the perturbation frequency may possess an imaginary part, hence an instability develops with growth rate Im$\tilde{\omega}$. Explicit criteria are established, to be presented below.

Conservative case / instability criteria. We recall, for reference, that in the "conservative" case $\nu = Q_i = 0$, Eq. (4) reduces to $\tilde{\omega}^2 = \tilde{\omega}_0^2(\tilde{k})$ (as given above). For $PQ < 0$, *no instability* may occur. However, if $PQ > 0$, a purely growing mode develops for long wavelengths, i.e. below a certain wavenumber threshold $\tilde{k} < (2Q_r/P)^{1/2}|\psi_0| \equiv \tilde{k}_{crit,0}$. The window $\tilde{k} \in (0, \tilde{k}_{crit,0})$ thus corresponds to unstable modes, while shorter wavelengths ($\tilde{k} > \tilde{k}_{crit,0}$) will be stable [2].

Dissipative case / instability criteria. The instability of electrostatic wavepackets can be investigated, based on (4), via a tedious algebraic calculation (omitted here for brevity). Summarizing those results, the "traditional" criterion ($PQ_r < 0$ for stability) gives its place to an extended range of criteria (for $Q_i \sim \nu \neq 0$). A number of possibilities arise:

- *Case I (stable)*: If $PQ_r < 0$ (regardless of Q_i), *no instability occurs*: the stable region (as for $Q_i = 0$) remains stable, for all values of the (perturbation) wavenumber $\tilde{k}$.
- *Case II (unstable)*/weak dissipation: if $PQ_r > 0$ *and* $Q_i^2 < Q_r^2 \Leftrightarrow \left|\frac{Q_i}{Q_r}\right| < 1$, then:
  - *Instability window:* MI occurs *in a reduced instability window*, if $\tilde{k} \in \left(\tilde{k}_{-crit}, \tilde{k}_{+crit}\right)$, where
  $$\tilde{k}^2_{\pm crit} = \left(\frac{Q_r}{P_r}\,|\psi_0|^2\right)\left[1 \pm \left(1 - \frac{Q_i^2}{Q_r^2}\right)^{1/2}\right]. \tag{5}$$
  - *Growth rate:* In this case, the complex frequency reads $\tilde{\omega} = -Q_i\,|\psi_0|^2 \pm i\sigma$, where
  $$\sigma = |Im\tilde{\omega}| = \left[-P^2\tilde{k}^2\left(\tilde{k}^2 - \frac{2Q_r}{P}\,|\hat{\psi}_0|^2\right) - \left(PQ_i\,|\hat{\psi}_0|^2\right)^2\right]^{1/2}. \tag{6}$$
- *Case III (stable)*/strong dissipation: if $PQ_r > 0$ *and* $Q_i^2 \geq Q_r^2 \Leftrightarrow \left|\frac{Q_i}{Q_r}\right| \geq 1$, then we obtain *stability* $\forall P, Q_r, \tilde{k}$.

Parametric investigation. The results in the previous section are general. It turns out that the instability profile will depend on the values of the ratios Q_r/P and Q_i/Q_r, for a given problem. For the specific dynamical system described above, these are depicted in Figure 1. We clearly see that, first of all, all possibilities exist, i.,e. small and large values of either of these ratios, depending on the problem's intrinsic parameters.

The growth rate σ is also shown: we see that the ratio Q_i/Q_r increases the instability window, as predicted (see Case II above), and may also enhance the instability growth rate: see the bell-shaped growth rate curves in Figure 1c.

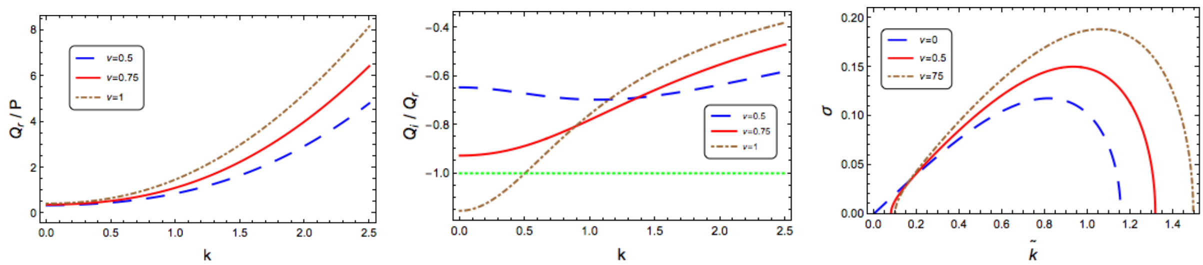

Figure 1: The ratio(s) Q_r/P (left frame) and Q_i/Q_r (middle frame) are shown vs the carrier wavenumber k. The growth rate σ is also shown (right frame) versus the perturbation wavenumber $\tilde{k}$ (for $k=1$, here). All plots are based on the expressions (3), taking $c=\omega_p=\alpha=\beta=1$.

Discussion. Although wavepacket decay cannot be avoided (for $Q_i \neq 0$), as the steady state itself is damped ($\sim e^{-Q_i|\hat{\psi}_0|^2 T}$), it may be slowed down for a significant amount of time by wave growth due to modulational instability occurring in a wide parameter region: see the occurrence of a growth term $e^{+Im\tilde{\omega}T}$ due to onset of the instability. Both these effects may either compete or cooperate (actually depending on various parameter values, affecting the signs of Q_i and $Im\tilde{\omega}$) against the decay of the carrier wave due to damping (for $\nu \neq 0$), i.e. $\sim e^{-\nu t/2}$.

Conclusion. We have studied the occurrence of modulational instability in Langmuir waves, and have actually also established explicit generic criteria for MI in dissipative media.

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