Modulated transverse off-plane dust-lattice wavepackets in hexagonal two-dimensional dusty plasma crystals

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Abstract

The propagation of nonlinear dust-lattice waves in a two-dimensional hexagonal crystal is investigated. Transverse (off-plane) dust grain oscillatory motion is considered, in the form of a backward propagating wavepacket whose linear and nonlinear characteristics are investigated. An evolution equation is obtained for the slowly varying amplitude of the first (fundamental) harmonic, by making use of a two-dimensional lattice multiple scales technique. An analysis based on the continuum approximation (spatially extended excitations, compared to the lattice spacing) shows that wavepackets will be modulationally stable and that dark-type envelope solitons (density holes) may occur, in the long wavelength region. Evidence is provided of modulational instability and of the occurrence of bright-type envelopes (pulses) at shorter wavelengths. The role of second neighbor interactions is also investigated, and is shown to be rather weak in determining the modulational stability region. The effect of dissipation, assumed negligible in the algebra throughout the article, is briefly discussed.

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I. INTRODUCTION

Dust crystals are space-periodic strongly coupled dusty plasma configurations (Debye lattices) which occur due to the strong electrostatic interaction between massive, heavily charged, micron-sized dust particulates (dust grains) present in a (dusty, or complex) plasma\textsuperscript{1,2}. The formation and dynamics of dust crystals have been studied in various experiments\textsuperscript{2−7}, where dust particles were essentially created by injecting artificial micro-spheres, which subsequently acquire a high electron charge via inherent dynamic charging mechanisms. Such dust quasi-lattices typically bear a two-dimensional (2D) hexagonal structure, although various two- and three-dimensional configurations have also been reported\textsuperscript{2}. One-dimensional (1D) dust crystals have also been fabricated in experiments, by making use of appropriate confinement potentials\textsuperscript{6,7}.

Dust lattices support a variety of linear modes, of which we single out: longitudinal\textsuperscript{8} (~\(x\), acoustic) and a transverse\textsuperscript{9,10} (~\(y\), shear) in-plane, as well as a transverse (out-of-plane, inverse-optic) dust-lattice mode (DL) wave mode(s) (DLWs). Beyond the linear regime, nonlinear effects may influence or even dominate DL dynamics if wave amplitudes become significant. This may be due to the intrinsincally nonlinear character of electrostatic coupling, to geometric effects (mode coupling), to interaction potential anisotropy\textsuperscript{11} or to sheath potential anharmonicity. The latter is of crucial importance in the vertical (off-plane) mode, to be treated here; indeed, we need to stress the fact that the sheath potential, although generally believed to be parabolic\textsuperscript{12} may take a anharmonic form for low plasma pressure and/or density, as suggested by ab initio calculations\textsuperscript{13} and also by an interpretation of earlier experimental results [See the
discussion in Ref. 14; also Refs. therein.]. A theoretical treatment of the nonlinear aspects of DL modes in 1D Yukawa crystals has been carried out in Ref. 15, where the above aspects are incorporated in an exact nonlinear lattice model.

Recent experimental [5] and numerical (molecular dynamics) [16] investigations have established the occurrence of 2D modulated TDL wavepackets moving at a negative group velocity, i.e. the wave is backward propagating. The observed waves form a 2D analogue of the transverse dust-lattice (TDL) – i.e. vertical - mode (as modelled in 1D crystals [9,10]), obeying similar qualitative physics, yet substantially different dispersion laws in 2D, as discussed in Refs. [5,16]. This “bending mode” was theoretically investigated in Ref. 17 in the linear region, and that linear model has succeeded in the interpretation of earlier experimental and numerical findings. Our study aims at extending those results to the weakly nonlinear regime. The results in Refs. 5, 16 and 17 are naturally recovered in our formulation below.

A well-known nonlinear effect manifested in the dynamics of waves propagating in nonlinear dispersive media is amplitude modulation, referring to the slow space and time variation of the wave’s amplitude, which may potentially be subject to modulational instability. Modulationally unstable wave packets may either collapse, in response to external perturbations, or evolve towards localized structure (envelope soliton) formation, due to a mutual balance between dispersion and nonlinearity. This generic mechanism is well-known in various physical contexts [18,19] to be related to phase harmonic generation and energy localization via the formation of localized excitations (solitons). Analytical theories for the amplitude modulation of DLWs in 1D dust crystals, due to the carrier phase self-interaction, have been furnished for both longitudinal [20] and transverse (off-plane) [21] 1D modes. The nonlinear aspects of in-plane motion
in 2D lattices was covered in Ref. 22. The investigation presented here follows the same methodology, yet for the 2D transverse (off-plane) waves.

Our aim here is to study the amplitude modulation of transverse off-plane DL wavepackets in 2D dusty plasma crystals. We shall investigate the occurrence of modulational instability, which may be viewed as a first stage triggering of the out-of-plane lattice instability observed numerically [16a], presumably leading to the phase transition suggested in that reference. Modulational instability may also be the first stage of the generic (i.e., for any symmetric potential) structural instability suggested in Ref. 23. We must point out, for rigor, that the modulation theory employed here is a mildly nonlinear theory, which claims to model weak vertical displacements. The latter point justifies our choice in neglecting the coupling to in-plane dust grain motion, since we are only interested in the very first stage of the manifestation of nonlinearity in off-plane motion. A more general theory should take into account horizontal-to-vertical motion coupling, and should be the subject of forthcoming work of ours. According to earlier results, will assume interactions between charged dust particles to be of the “standard” screened electrostatic (Debye-Hückel, or Yukawa) type, modeled via a potential 

\[ U(r) = q^2 \exp(-r/\lambda_d) / 4\pi\varepsilon_0 r \]

(here \( \lambda_d \) denotes the effective DP Debye radius, \( q \) is the dust grain charge, assumed constant, and \( \varepsilon_0 \) is the permittivity of vacuum).

Our scope lies in the interpretation of dusty plasma experiments in the laboratory. However, going beyond dusty (complex) plasma physics, this work can be viewed as a fundamental investigation of nonlinear transverse motion in hexagonal crystals, which may be of relevance in other physical contexts where Debye crystals structures occur (e.g., ultra-cold plasmas or one-component plasmas), or in lattice theory in general.
The outline of the manuscript is as follows. The model equations of motion are derived in Sec. II, and simplified by adopting a continuum approximation. The derivation of an evolution equation for the modulated wave amplitude is presented in Sec. III, by assuming transverse wave propagation either along a principal axis of the hexagonal structure or perpendicular to it. The effects of second neighbor interaction (SNI) on linear and nonlinear waves are investigated in Sec. IV. The modulational stability in both cases is investigated in Sec. V, and the results are then summarized in Sec. V.

II. ANALYTICAL MODEL AND LINEAR WAVE CHARACTERISTICS

We take into account nearest neighbor interactions only, i.e., each particle \((m,n)\) interact with the three other pairs of particles \((m\pm1,n)\), \((m\pm1/2,n-\sqrt{3}/2)\) and \((m\pm1/2,n+\sqrt{3}/2)\). The physical situation considered is a two-dimensional hexagonal crystal (assumed infinite, for simplicity) consisting of negative dust grains, which are located at equidistant sites \(a\); see Figure 1. If the particles are not at their equilibrium positions, we may define the six length variables \(l_1, l_2, l_3, l_4, l_5, \) and \(l_6\), which represent the distances from the particle \((m,n)\) to the nearest particles, respectively

\[
l_i = \sqrt{a^2 + (\Delta z_i)^2},
\]

where \(\Delta z_i\) (for \(i=1, 2, \ldots, 6\)) denote the displacements of the respective particles from their equilibrium positions in \(z\) direction, and

\[
\begin{align*}
\Delta z_1 &= z_{m+1,n} - z_{m,n}, & \Delta z_2 &= z_{m-1,n} - z_{m,n}, & \Delta z_3 &= z_{m+1/2,n+\sqrt{3}/2} - z_{m,n}, \\
\Delta z_4 &= z_{m-1/2,n-\sqrt{3}/2} - z_{m,n}, & \Delta z_5 &= z_{m+1/2,n-\sqrt{3}/2} - z_{m,n}, & \Delta z_6 &= z_{m-1/2,n+\sqrt{3}/2} - z_{m,n}
\end{align*}
\]
The equation of motion in the $z$–direction is

$$\frac{d^2 z_{m,n}}{dt^2} + \nu \frac{dz_{m,n}}{dt} = \frac{1}{M} (F_e - Mg) + \frac{1}{M} F_z, \quad (3)$$

where the electrostatic binary interaction force in $z$–direction $F_z$ exerted on two grains situated at a distance $r$ is derived from a potential function $U(r)$,

$$F_z = -\frac{\partial U(r)}{\partial z} \quad (4a)$$

$$F_z = -\frac{\partial U}{\partial r} \left( \sum_{i=1}^{6} \frac{\Delta z_i}{l_i} \right) - \frac{1}{2!} \frac{\partial^2 U}{\partial r^2} \left[ 2 \sum_{i=1}^{6} (l_i-a) \frac{\Delta z_i}{l_i} \right] - \frac{1}{3!} \frac{\partial^3 U}{\partial r^3} \left[ 3 \sum_{i=1}^{6} (l_i-a)^2 \frac{\Delta z_i}{l_i} \right] + \ldots \quad (4b)$$

upon defining $G_1 = (\partial U / \partial r)|_{r=a}$, $G_2 = (\partial^2 U / \partial r^2)|_{r=a}$ and $G_3 = \frac{1}{2} (\partial^3 U / \partial r^3)|_{r=a}$, we have calculated the polynomial coefficients $G_1$, $G_2$, and $G_3$ for the Yukawa potential.

$$G_1 = -\frac{q^2}{4\pi \varepsilon_0 \lambda_D^2} \left( 1 + \kappa \right) \frac{\exp(-\kappa)}{\kappa^2}, \quad (5)$$

$$G_2 = \frac{q^2}{4\pi \varepsilon_0 \lambda_D^3} \left( 2 + 2\kappa + \kappa^2 \right) \frac{\exp(-\kappa)}{\kappa^3}, \quad (6)$$

$$G_3 = -\frac{q^2}{4\pi \varepsilon_0 \lambda_D^4} \left( 6 + 6\kappa + 3\kappa^2 + \kappa^3 \right) \frac{\exp(-\kappa)}{\kappa^4} \quad (7)$$

where $\kappa = a / \lambda_D$.

We shall assume a smooth, continuous variation of the field intensity $E$, as well as the grain charge $q$ (which may vary due to charging processes) near the equilibrium position $z_0 = 0$.

Thus, following the method and notation in Ref. 21, we will expand as

$$E(z) \approx E_0 + E_0' z + \frac{1}{2} E_0'' z^2 + \ldots \quad (8)$$

$$q(z) \approx q_0 + q_0' z + \frac{1}{2} q_0'' z^2 + \ldots \quad (9)$$
Where the prime denotes differentiation with respect to \( z \) and subscript “0” denotes evaluation at \( z = z_0 \). Accordingly, the electric force \( F_e = Mg = q(z)E(z) - Mg \) is expressed as

\[
F_e(z) - Mg \approx -Mg + q_0E_0 + (q_0E'_0 + q_0E_0)z + 0.5(q_0E''_0 + 2q'_0E_0 + q''_0E_0)z^2 + \cdots \\
\approx \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \cdots
\]

The zeroth-order term of electric force balances gravity at \( z_0 \), viz., \( q_0E_0 - Mg = 0 \), while the first order \( -\gamma_1 = M\omega_g^2 \) is the effective width of the potential well; the value of the gap frequency \( \omega_g \) may either be evaluated from \textit{ab initio} calculations or determined experimentally. For instance, in Ref. 18, the frequency \( \omega_g \) is typically of order of \( \omega_g/2\pi = 20\text{Hz} \) and \( \gamma_2 = -\gamma_1/2, \quad \gamma_3 = 0.07\gamma_1 \). Now, equation (3) becomes

\[
\ddot{z}_{m,n} + v \dot{z}_{m,n} = -\omega_g^2 z_{m,n} - K_1 z_{m,n}^2 - K_2 z_{m,n}^3 \\
+ \Omega^2 \left( 6z_{m,n} - z_{m+1,n} - z_{m-1,n} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} \\
- z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) \\
+ K_3 \left( (z_{m+1,n} - z_{m,n})^3 + (z_{m-1,n} - z_{m,n})^3 + (z_{m+1/2,n+\sqrt{3}/2} - z_{m,n})^3 \\
+ (z_{m+1/2,n-\sqrt{3}/2} - z_{m,n})^3 + (z_{m-1/2,n+\sqrt{3}/2} - z_{m,n})^3 + (z_{m-1/2,n-\sqrt{3}/2} - z_{m,n})^3 \right)
\]

where

\[
K_1 = -\frac{\gamma_2}{M},
\]

\[
K_2 = -\frac{\gamma_3}{M},
\]

\[
K_3 = \frac{G_1}{2Ma^3} - \frac{G_2}{2Ma^2},
\]

\[
\Omega^2 = -\frac{G_1}{Ma} = -\frac{q^2}{4\pi\varepsilon_0 Ma^3} \left( 1 + \kappa \right) \exp(-\kappa).
\]
We note that, upon keeping the single particle contributions (first line only), (11) reduces to the equation of motion suggested in Ref. 25 for dust particle motion in an anharmonic sheath potential. The remaining terms in the \( \text{RHS} \) are due to the electrostatic coupling, including linear (term in \( \Omega^2 \)) and nonlinear (term in \( K_3 \)) contributions.

Linear dispersion relation. Waves can propagate along an arbitrary direction, which is here denoted by an angle \( \theta \), representing the angle between the wavevector \( k \) and a primitive translation vector (along the \( x \) axis), i.e. \( k_x = k \cos \theta \) and \( k_y = k \sin \theta \). Retaining only linear contribution in the form of “phonons” of the type

\[
u_{mn} = u_0 \exp[-i \omega t + i k a (m \cos \theta + n \sin \theta)] + \text{c.c.}
\]

we obtain an inverse-optic-mode like dispersion relation from Eq. (11),

\[
\omega^2 + iv\omega = \omega_g^2 - 4\Omega^2 \left\{ \sin^2 \left( \frac{ka}{2} \cos \theta \right) + \sin^2 \left( \frac{ka}{2} \cos \left( \frac{\pi}{3} - \theta \right) \right) + \sin^2 \left( \frac{ka}{2} \cos \left( \frac{\pi}{3} + \theta \right) \right) \right\}, \quad (16)
\]

In the special cases \( \theta = 0 \) or \( \theta = \pi / 2 \) we obtain,

\[
\theta = 0: \quad \omega^2 + iv\omega = \omega_g^2 - 4\Omega^2 \left\{ \sin^2 \left( \frac{ka}{2} \right) + 2 \sin^2 \left( \frac{ka}{4} \right) \right\}, \quad (17a)
\]

\[
\theta = \pi / 2: \quad \omega^2 + iv\omega = \omega_g^2 - 8\Omega^2 \sin^2 \left( \frac{\sqrt{3}ka}{4} \right). \quad (17b)
\]

The dispersion relation obtained here provides the frequency-wavenumber dependence for TDLW propagation at any direction inside the \( x-y \) plane. This expression is identical to the expression obtained by Vladimirov et. al. [17]. The dispersion relation is an inverse optic-like dispersion, so the frequency at zero wavenumber (infinite wavelength) is finite, and the slope of the curve \( \omega = \omega(k) \) is negative for small \( k \). This information is true for all angles \( \theta \), as obvious
from figures 2a and 3. Note that Fig. 3 here is quasi-identical to Fig. 3 in Ref. 16a (apart from the difference in scaling and notation).

**Group velocity.** The group velocity of TDL waves (for \( \nu = 0 \)) reads,

\[
v_g = - \frac{a \Omega^2}{\omega} \left[ \cos \theta \sin[ka \cos \theta] + \cos(\pi / 3 - \theta) \sin[ka \cos(\pi / 3 - \theta)] \right] + \cos(\pi / 3 + \theta) \sin[ka \cos(\pi / 3 + \theta)]
\]

The dispersion relation presents a negative group velocity for wavenumbers \( k \) below a threshold, say \( k_{\text{critical}} \), and a positive group velocity for \( k > k_{\text{critical}} \). The value of \( k_{\text{critical}} \) depends on direction of wave propagation; see figures 2b and 4. These results are in perfect agreement with results obtained via numerical simulation [16a]. Figure 4 shows a contour plot of the curve \( v_g = 0 \), which thus determines \( k_{\text{critical}} \) -where the group velocity changes sign- as a function of \( \theta \). As discussed in Ref. 16a, the dispersion law in the 2D crystal case differs substantially from the one obtained for 1D crystals. The difference is marked in the form of the dispersion relation (16), and is also in the sign of the group velocity (positive/negative). Our findings recover perfectly those earlier results.

**Continuum approximation.** If the characteristic length scale of the wave form, say \( L \), is much larger than the inter-particle spacing \( a \), then the continuum approximation can be invoked in order to convert the difference equation (11) into a differential equation for \( z_{m,n} \), now expressed as continuous function \( u(x, t) \). We expand \( z_{m\pm 1,n} \) and \( z_{m\pm 1/2,n\pm \sqrt{3}/2} \) around \( z_{m,n} \) in powers of \( a/L \) and retain terms of the order of \( (a/L)^4 \), to obtain

\[
z_{m\pm 1,n} = z_{m,n} + a \frac{\partial u}{\partial x} + \frac{a^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{a^3}{6} \frac{\partial^3 u}{\partial x^3} + \frac{a^4}{24} \frac{\partial^4 u}{\partial x^4} + \cdots
\]
Substituting from (19), (20), and retaining terms of order in $a^4$, the equation of motion (11) takes the form of a differential equation for the particle displacement $u_{m,n}(t) = u(x,y,t)$,

$$
\ddot{u} + \nu \dot{u} = -\omega_g^2 u - K_1 u^2 - K_2 u^3 - \frac{3a^2 \Omega^2}{32} \left[ 16(u_{xx} + u_{yy}) + a^2 (u_{xxxx} + u_{yyyy} + 2u_{xyxy}) \right] + \frac{27a^4 K_3}{24} \left[ 3(u_x)^2 u_{xx} + 3(u_y)^2 u_{yy} + (u_x)^2 u_{yy} + (u_y)^2 u_{xx} + 4u_x u_y u_{xy} \right].
$$

(21)

Note that the friction term introduced on the left-hand-side of the equation of motion (11) and its continuum version (21), leads to the appearance of the damping rate $\nu$ (due to dust-neutral collisions) in the dispersion relations (16)-(18). Dissipation in dusty plasma experiments is admittedly always present, yet may acquire very small values, depending on plasma density and pressure [Ref. 26]. In the following, we shall assume a very small damping rate, and will therefore neglect damping by setting $\nu = 0$ in the nonlinear analysis to follow. This is expected to incur a relative error of the order of $\nu^2 / \omega_g^2$, which is reportedly small in experiments. Our results will later be extended by incorporating dissipation effects here omitted.

III. AMPLITUDE MODULATION AND DYNAMICS - DERIVATION OF A NONLINEAR SCHRÖDINGER (NLS) EQUATION

We assume that the transverse wave propagates in-plane in an arbitrary direction, given by the general form $\hat{x}\cos\theta + \hat{y}\sin\theta$. We shall employ the standard lattice version of the reductive perturbation technique\textsuperscript{20,21} in the quasi-continuum limit.
A 2D lattice perturbation scheme.

Allowing for a weak departure from the small-amplitude (linear) theory, we consider

\[ u = u_1 + \epsilon^2 u_2 + \cdots, \] (22)

where \( \epsilon \) \((<<1)\) is a small (real) parameter characterizing the strength of the nonlinearity. The function \( u_j \) at each order is assumed to be a sum of \( l \)th order harmonics, viz.

\[ u_j = u_{j0} + \sum_{l=1}^{\infty} \left[ u_{jl} \exp\left[ il(mk \cos \theta + nk \sin \theta - \omega t) \right] + cc \right], \] (23)

where \( cc \) denotes the complex conjugate.

The amplitudes \( u_{jl} \) are assumed to be slowly varying function of time and space, via the set of independent stretched variables

\[ \tilde{\zeta} = (\hat{x}_i + \hat{y}_i - \nu_0 t), \quad \text{and} \quad \tau = t_2 = c t_1. \] (24)

The analytical expression for the propagation velocity \( \nu_0 \) is anticipated as a compatibility constraint; the outcome is, in fact, expected to yield the group velocity \( \nu_g = \omega(k) \), in the continuum approximation. We shall now substitute these expansions into the equation of motion (21) and collect the contributions appearing in each power in \( \epsilon \).

1st order – linear dynamics in the continuum limit.

At first order, we obtain a linear equation which is solved for the first harmonic solution [cf. (23) for \( l=m=1 \); the zeroth-order amplitude vanishes]. The dispersion relation reads

\[ \omega^2 = \omega_g^2 - \frac{3k^2 a^2}{2} \Omega^2 \left( 1 - \frac{k^2 a^2}{16} \right), \] (25)
which coincides with the dispersion relation (16) in the limit $ka << 1$; see Fig. 5.

We note that, quite surprisingly, the angle-dependence disappears in the dispersion law, once the continuum limit $ka << 1$ is considered. It is straightforward to verify (upon a simple McLaurin expansion near zero $k$) that the angle vanishes in the first (five) terms in a small $k$ expansion, and thus won’t appear in any of the quantities to follow, herein. Still, we add for rigor that the angle does appear in the algebraic evaluation of the coefficients $P$ and $Q$ below, yet only through combinations of terms which all vanish (upon making use of appropriate trigonometric identities).

2nd order – group dispersion.

In the second order, considering the annihilation of secular terms, we obtain the following expression for the propagation velocity $v_0$:

$$v_0 = -\frac{3ka^2}{2\omega} \Omega^2 \left(1 - \frac{k^2 a^2}{8}\right).$$

(26)

It is easy to verify that $v_0 = v_{g,\text{cont}} = d\omega/dk$, as physically expected. Eq. (26), here obtained as a condition for secular term annihilation, can therefore also be derived either from (25), or as the continuum expansion of (18) above. The first harmonic $u_{11}$ therefore propagates at the group velocity in an (arbitrary) direction, as suggested by the functional dependence $u_{11} = u_{11}(\xi, ...)$, where $\xi = \epsilon(x \cos \theta + y \sin \theta - v_g t)$ here determines the slowly varying amplitude reference frame.

The solution obtained up to this order is given by:

$$u_j = \epsilon (u_{11} \exp i\phi + cc) + \epsilon^2 [u_{20} + (u_{21} \exp i\phi + cc) + u_{22} \exp 2i\phi + cc] + O(\epsilon^3)$$

(27)
where the fundamental carrier (1st harmonic) phase was denoted by $\phi = k(m \cos \theta + n \sin \theta) - \omega t$.

The harmonic amplitudes are given by:

$$u_{20} = \frac{-2K_1|u_{11}|^2}{\omega_g^2},$$

$$u_{22} = \frac{K_1u_{11}^2}{4\omega^2 - \omega_g^2 + 6k^2a^2\Omega^2\left(1 - k^2a^2/4\right)},$$

Notice the generation of second- and zeroth- harmonics, which is entirely due to the sheath potential anharmonicity; see that the harmonics only involve (the quadratic force - or cubic sheath potential - nonlinearity coefficient) $K_1$, defined in (12) above; higher-order nonlinearity only affects amplitude dynamics in higher orders; see below. We also note that the zeroth harmonic was found to vanish ($u_{10} = 0$), in qualitative agreement with the 1D transverse wave case; see Ref. 21 (and, in fact, in contrast with the 1D longitudinal wave case; see Ref. 20). A detailed qualitative discussion of these matters is carried out in Ref. 15. The above formulation provides a direct tool for harmonic generation related diagnostics, to be used in experiments.

**3rd order – NLS equation.**

In third order in epsilon, the condition for annihilation of secular terms leads to the nonlinear Schrödinger (NLS) equation

$$i \frac{\partial U}{\partial \tau} + P \frac{\partial^2 U}{\partial \xi^2} + QU|U|^2 = 0,$$

which describes the evolution of the fundamental (carrier) harmonic amplitude $u_{11} = u_{11}(\xi, \tau)$ ($\tau = \varepsilon^2 t$ is a slow time scale; the 1st order space variable $\xi$ was defined above).

The dispersion coefficient $P$ is given by...
\[ P = \frac{-v_g^2}{2\omega} - \frac{3a^2 \Omega^2}{4\omega} \left(1 - \frac{3k^2a^2}{8}\right). \]  

Note that \( P \) is related to the curvature of the dispersion curve as \( P = \frac{d^2\omega}{2dk^2} \), as expected.

The cubic nonlinearity coefficient is due to the nonlinearity induced by the sheath “substrate” potential via \( K_1 \) and \( K_2 \) and by the electrostatic coupling via \( K_3 \). It is given by

\[ Q = \frac{1}{2\omega} \left[ \frac{4K_1^2}{\omega_g^2} - \frac{2K_1^2}{4\omega^2 - \omega_g^2 + 6\Omega^2k^2a^2(1 - k^2a^2/4)} - 3K_2 - \frac{27k^4a^4K_3}{8} \right]. \]

Recapitulating, the dynamics of the wave fundamental harmonic amplitude \( u_{11} = U(\xi, \tau) \) is governed by the NLS Eq. (30), and is thus essentially dynamics by the interplay among the dispersion and nonlinearity coefficients \( P \) and \( Q \). Their analytical behavior on relevant parameters will be investigated below, once their role in the dynamics is briefly summarized in what follows.

**IV. EFFECTS OF SECOND-NEIGHBOR INTERACTION**

In the previous Sections, we have made the choice to keep only nearest neighbor interactions (NNI), hence neglecting longer-range effects. For first principles, this appears to be justified by the fact that the lattice constant is of order of magnitude comparable to the Debye radius, which measures the range of inter-particle interactions. On the other hand, electrostatic interactions (albeit screened) are characterized by their long-range of action, so one might wonder whether the contribution of longest, e.g. second-order, neighbors would play a significant role. Therefore, it appears appropriate to investigate the strength of second-neighbor interactions (SNI). Below, we shall show that the addition of SNI in the model certainly provides a small qualitative, yet no major quantitative modification in the dynamics, thus in principle
confirming our FNI result above. We have chosen to dedicate a separate brief section to second-
neighbor interactions, in order to trace their influence in a transparent manner. Once the
analytical derivation, nevertheless, has led us to the anticipated evolution equation for the
amplitude [e.g. (30) above], the analysis will be carried out in the next Section in parallel, i.e.
comparing among the FNI and SNI models.

We shall assume that each particle \((m,n)\) interacts with three pairs of particles, which are
located at the sites \((m,n\pm \sqrt{3})\), \((m\pm 3/2, n-\sqrt{3}/2)\) and \((m\pm 3/2, n+\sqrt{3}/2)\). The distance between the
central particle and the second neighbors are all \(\sqrt{3}a\), in the directions \(\theta = \pi / 6, 3\pi / 6, 5\pi / 6, 7\pi / 6, 9\pi / 6, 11\pi / 6\). The electrostatic binary interaction force in
\(z\) – direction \(F_z\) exerted on two grains situated at a distance \(r\) is derived from a potential
function \(U(r)\); see Eqs. (4) above. By defining \(G'_{1}=(\partial U / \partial r)\big|_{r=\sqrt{3}a}\), and \(G'_{2}=(\partial^2 U / \partial r^2)\big|_{r=\sqrt{3}a}\),
we can compute the polynomial coefficients \(G'_{1}\) and \(G'_{2}\) for the Yukawa potential. These are

\[
G'_{1} = -\frac{q^2}{4\pi \varepsilon_0 \varepsilon_D^2} \left(1 + \sqrt{3}\kappa\right) \exp(-\sqrt{3}\kappa) \left(\sqrt{3}\kappa\right)^2,
\]

(33 a)

and

\[
G'_{2} = \frac{q^2}{4\pi \varepsilon_0 \varepsilon_D^3} \left(2 + 2(\sqrt{3}\kappa) + (\sqrt{3}\kappa)^2\right) \exp(-\sqrt{3}\kappa) \left(\sqrt{3}\kappa\right)^3
\]

(33 b)

Now, the discrete lattice equation of motion becomes

\[
\ddot{z}_{m,n} + \nu \dot{z}_{m,n} = -\alpha \sigma^2 \dot{z}_{m,n} - K_1 \dot{z}_{m,n} - K_2 \dot{z}_{m,n} + \Omega^2 \left\{6\dot{z}_{m,n} - z_{m+1,n} - z_{m-1,n} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right\}
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \beta \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n-\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right)
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \frac{1}{2} \left\{z_{m,n} + \sqrt{3}\dot{z}_{m,n} \right\} + \frac{1}{2} \left\{z_{m,n} - \sqrt{3}\dot{z}_{m,n} \right\}
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \beta \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n-\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right)
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \frac{1}{2} \left\{z_{m,n} + \sqrt{3}\dot{z}_{m,n} \right\} + \frac{1}{2} \left\{z_{m,n} - \sqrt{3}\dot{z}_{m,n} \right\}
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \beta \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n-\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right)
\]

\[
+ \left(6\dot{z}_{m,n} + \dot{z}_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n+\sqrt{3}/2} - z_{m+1/2,n-\sqrt{3}/2} - z_{m-1/2,n+\sqrt{3}/2} - z_{m-1/2,n-\sqrt{3}/2} \right) + \frac{1}{2} \left\{z_{m,n} + \sqrt{3}\dot{z}_{m,n} \right\} + \frac{1}{2} \left\{z_{m,n} - \sqrt{3}\dot{z}_{m,n} \right\}
\]

(34)
where the parameters $\alpha = G'_1 / \sqrt{3} G_1$ and $\beta = \left( G'_1 - (\sqrt{3} a) G'_2 \right) / \left( 3 \sqrt{3} (G_1 - G_2 a) \right)$ represent the second neighbor contribution. Combining with (33, 34) above, the SNI parameters are given by:

$$\alpha = \frac{1}{3 \sqrt{3}} \left( 1 + \frac{\sqrt{3} \kappa}{1 + \kappa} \right) \exp[- \kappa (\sqrt{3} - 1)] \quad (35 \text{a})$$

and

$$\beta = \frac{1}{3 \sqrt{3}} \left( 1 + \frac{\sqrt{3} \kappa + \kappa^2}{3 + 3 \kappa + \kappa^2} \right) \exp[- \kappa (\sqrt{3} - 1)] \quad (35 \text{b})$$

Obviously, the FNI expressions above are recovered in the limit $\alpha = \beta = 0$. Figure 6 shows the parameters $\alpha$ and $\beta$ as a function of $\kappa$. We stress the fact that they both take small values (of the order of 0.1 or less, approximately) in the region of experimental interest ($\kappa$ values near unity) a priori suggesting a small contribution by second-neighbor interactions.

The linear dispersion relation [cf. (16) above] now becomes

$$\omega^2 + i \nu \omega = \omega_0^2 - 4 \Omega^2 \left[ \sin^2 \left( \frac{ka}{2} \cos \theta \right) + \sin^2 \left( \frac{ka}{2} \cos \left( \frac{\pi}{3} - \theta \right) \right) + \sin^2 \left( \frac{ka}{2} \cos \left( \frac{\pi}{3} + \theta \right) \right) \right]$$

$$- 4 \Omega^2 \alpha \left[ \sin^2 \left( \frac{\sqrt{3} ka}{2} \sin \theta \right) + \sin^2 \left( \frac{\sqrt{3} ka}{2} \cos \left( \frac{\pi}{6} - \theta \right) \right) + \sin^2 \left( \frac{\sqrt{3} ka}{2} \cos \left( \frac{\pi}{6} + \theta \right) \right) \right] \quad (36)$$

Note that only the linear SNI contribution, via $\alpha$ enters the dispersion relation. Setting $\nu = 0$ to continue (see discussion above), we now advance to the TDLW group velocity, to find

$$v_g = - \frac{a \Omega^2}{\omega} \left[ \cos \theta \sin [ka \cos \theta] + \cos (\pi / 3 - \theta) \sin [ka \cos (\pi / 3 - \theta)] + \cos (\pi / 3 + \theta) \sin [ka \cos (\pi / 3 + \theta)] \right]$$

$$- \frac{\sqrt{3} a \Omega^2}{\omega} \alpha \left[ \sin \theta \sin [\sqrt{3} ka \sin \theta] + \cos (\pi / 6 - \theta) \sin [\sqrt{3} ka \cos (\pi / 6 - \theta)] \right] + \cos (\pi / 6 + \theta) \sin [\sqrt{3} ka \cos (\pi / 6 + \theta)]. \quad (37)$$

Again, as expected, expression (18) is recovered in the appropriate limit. On the other hand, in the long-wavelength region (for $ka \ll 1$) one obtains an approximate expression which recovers expression (26) above for $\alpha = 0$. This continuum approximation of (37) needn’t be
stated here, as it is exactly identical to the expression (40) for the propagation velocity $v_0$ derived below as a compatibility constraint.

Recall that the continuum approximation allowed us to pass from (11) to (21) above. The same procedure now yields the SNI-modified continuous equation of motion

$$\ddot{u} + v \dot{u} = -6\rho u - K_1 u^2 - K_2 u^3 - \frac{3a^2\Omega^2}{32} \left[ 16(u_{xx} + u_{yy})(1+3\alpha) + a^2(u_{xxxx} + u_{yyyy} + 2u_{xyyy})(1+9\alpha) \right]$$

$$+ \frac{27a^4K_3^4}{24} \left[ 3(u_x)^2 u_{xx} + 3(u_y)^2 u_{yy} + (u_x)^2 u_{yy} + (u_y)^2 u_{xx} + 4u_xu_yu_{xy} \right],$$

where $K_3 = K_3(1+9\beta)$. Eq. (38) is to be compared with Eq. (21), which is indeed recovered in the appropriate limit. We note that the modification due to SNI being taken into account is only quantitative (slight modification of coefficients) rather than qualitative (no structural modification in (38), as compared to (21)), and is expected to be rather weak (as indeed confirmed by the plots below).

At first order, we obtain a linear equation which is solved for the first harmonic solution. The dispersion relation reads

$$\omega^2 = \omega_0^2 - \frac{3k^2a^2}{2} - \frac{k^2a^2}{16} \left[ (1+3\alpha) - \frac{k^2a^2}{16}(1+9\alpha) \right],$$

which coincides with the dispersion relation (36) in the limit $ka \ll 1$ (and switching off damping therein).

In the second order, imposing the annihilation of secular terms, we obtain the following expression for the propagation velocity $v_0$:

$$v_0 = -\frac{3ka^2}{2\omega_0} \Omega^2 \left[ 1+3\alpha - \frac{k^2a^2}{8}(1+9\alpha) \right].$$
In third order in epsilon, the condition for annihilation of secular terms leads exactly to the nonlinear Schrödinger equation (30) above. We recall that it describes the evolution of the envelope (of the fundamental harmonic amplitude) \( u_{11} = u_{11}(\xi, \tau) \). The difference now, SNI being taken into account, lies in the modification of the form of the coefficients appearing in (30). The modified dispersion coefficient \( P \) is now given by

\[
P = \frac{-\nu^2}{2\omega} - \frac{3a^2\Omega^2}{4\omega} \left( 1 + 3\alpha - \frac{3k^2a^2}{8}(1+9\alpha) \right).
\]  

(41)

Note that \( P \) is again related to the curvature of the dispersion curve as \( P = \frac{d^2\omega}{2dk^2} \), as expected [and indeed verified, upon a double differentiation of (39) or of (36) in the continuum limit]. The cubic nonlinearity coefficient with the effect of SNI is now given by

\[
Q = \frac{1}{2\omega} \left[ \frac{4K_1^2}{\omega_k^2} - \frac{2K_1^2}{\omega_k^2 + 6\Omega^2 k^2 a^2(1+3\alpha-k^2 a^2(1+9\alpha)/4)} - 3K_2 - \frac{27k^4 a^4 K_3^3}{8} \right],
\]  

(42)

where the SNI influence is manifested via the appearance of \( \alpha \) and \( \beta \) - the latter via \( K_3' = K_3(1+9\beta) \) (as defined above).

V. MODULATIONAL INSTABILITY – 2D ENVELOPE EXCITATIONS

The amplitude dynamics of a TDL wavepacket was shown to be governed by the NLS equation (30) above. Two physical phenomena which are generally modeled via this formulation are wave collapse via modulational instability and the formation of envelope excitations. Without reproducing the whole of the existing theory, which may be found e.g. in Ref. 27 (also in Refs. 21-22), we shall provide the basic information needed to understand our findings in what follows.
The detailed analysis of the NLS Eq. (30) above\textsuperscript{18,19,26} reveals that a modulated wavepacket whose amplitude obeys the NLSE (30) is modulationally unstable for $PQ > 0$, and stable for $PQ < 0$. Assuming a perturbation of amplitude $\Psi_0$ and characteristic wavenumber $\tilde{k}$, the perturbation grows $PQ > 0$ leading to wave blowup. The maximum growth rate $\sigma = Q |\Psi_0|^2$ is attained for a perturbation wavenumber $\tilde{k} = \left(\frac{2Q}{P}\right)^{1/2} |\Psi_0|$. The coefficients $P$ and $Q$ therefore determine the occurrence and first stage evolution of the instability.

Only the first evolution stage of the instability, outlined above, can be described analytically. The further evolution of the instability can only be modeled numerically. It is known that energy occurs, via the formation of localized envelope structures (envelope solitons). In the case $PQ > 0$, bright-type solitons are formed: these model localized envelope pulses, which confine the fast carrier wave and move at –or near- the group velocity, and are formally equivalent to bright pulses in nonlinear fiber optics. On the other hand, for $PQ < 0$, modulated wavepackets may propagate in the form of dark/grey envelope solitons, modeling localized voids amidst constant values everywhere else).

In an attempt to go one step beyond the continuum approximation adopted above, we shall consider in our analysis, for the sake of rigor, two versions of the dispersion coefficient $P$, namely $P_1 = P_{\text{cont}}$ as defined (in the continuum limit) in Eq. (31), and the exact (discrete) expression $P_0 = P_{\text{disc}} = d^2\omega / 2d\kappa^2$, as obtained from the accurate dispersion relation Eq. (16). Figure 7a (solid line) show the variation of the dispersion coefficient $P$ in the discrete description, as obtained from the dispersion relation (for $\nu = 0$) for FNI (Eq. 16) on one hand
(solid curves; notice the $4\pi$-periodicity), and separately for SNI – from Eq. (36) – on the other (dashed curves). The continuous and discrete descriptions obviously coincide in the low wavenumber $k$ limit, yet diverge strongly for larger $k$ (shorter wavelengths); compare Fig. 7a to Fig. 9 near $k \approx 0$ to see this. This twofold analysis (continuous versus discrete and FNI vs. SNI) is meant to give a flavor of what should be an accurate discrete-system investigation, which is to follow in future work.

The behavior of the coefficients $P$ and $Q$, are depicted in Fig. 7. The product $PQ$, and also the ratio $Q/P$ are depicted in Fig. 8. Recalling that the sign of the product $PQ$ determines the stability profile of the wave, we see that the occurrence of modulational instability is prescribed, since both $P$ and $Q$ are negative (hence $PQ>0$). Stable bright-type envelope structures should therefore be sustained in the system. Indeed, this prediction seems to have been confirmed already, in the laboratory, where the observation of backward propagating wavepackets is reported and their characteristics are tested against a linear theory, which thus confirmed in the 2D picture. It must be added, for rigor, that the sign of $P$ (and, presumably, $Q$) may change by taking into account SNI, thus affecting the stability profile of modulated wavepackets, and the type of envelope solitons susceptible to occur.

Continuum vs. discrete. It must be stated, for rigor, that the above findings are only true for low $k$ (within the continuum approximation, here) i.e. for large wavelengths, say $\lambda >> r_0$ (the region below $k \approx r_0^{-1}$ seems to be satisfactorily covered by the continuum approximation, within an error of 10%); see Fig. 7a). For higher $k$, the dispersion coefficient $P$ (in fact, $4\pi$-periodic) changes sign, and $Q$ may presumably also do the same. Properly speaking, $Q$ (see Fig. 7b) does become positive above $k \approx 1.1 r_0^{-1}$ (for FNI) and $k \approx 0.85 r_0^{-1}$ (for SNI), yet this is expected to
change if a discrete analysis were to be undertaken. In general, it seems safe to assert that a more complex stability profile will be predicted by a more accurate discrete analysis; in particular, wavepackets will be stable (and dark-type envelope excitations will occur) at wavenumbers higher than $k \approx n_0^{-1}$, as suggested by Fig. 7. Concluding therefore, we stress that one needs to go into a fully discrete description of dust crystal dynamics, in order to obtain a valid expression for the nonlinearity coefficient $Q$ ($P$ on the other hand is readily obtained from the dispersion law, as explained above). This is anticipated as part of a future investigation, yet goes beyond the continuum approximation adopted here.

*FNI vs. SNI.* Summarizing the overall effect of taking second neighbor interactions into account, we note that it appears to be small, yet rather non-negligible. The dispersion laws certainly undergo a modification: the frequency is reduced by (near or less than) 20% (see in Fig. 3), while a small effect is also observed in the critical wavenumber threshold where the slope changes from negative (backward wave) to positive (forward propagation): note the positions of the minima in Fig. 3. This obviously also affects the group velocity and group velocity dispersion terms (see e.g. Fig. 9 for $P$); nevertheless this remark is rather not relevant in the continuous (small $k$) region (modeled by our NLS Eq. here), as explained above. Interestingly, for propagation parallel to the lattice principal axes, second neighbor interactions may account for slowing down a wave with a short wavelength ($k$ higher than 4, roughly) and thus the backward wave character may be modified in a critical manner; compare the bottom two curves (in green, online) to see this: the slope of the dashed one essentially goes horizontally after $ka=4$ approximately. To be stated again, this enters a region (of high $k$ values) which goes beyond the continuum approximation adopted here. This is entirely legitimate in the linear regime, yet no
prediction is in principle to be made in the nonlinear region (involving the NLS Eq. coefficients $P$ and $Q$, say).

**V. CONCLUSIONS**

The amplitude modulation of transverse off-plane DL wavepackets in 2D hexagonal dusty plasma crystals has been investigated. The modulational instability predicted by our findings may be seen as a first stage of the out-of-plane lattice instability observed numerically [16], and might presumably lead to the phase transition suggested in the latter reference. Modulational instability may also be the first stage of the generic (i.e., for any symmetric potential) structural instability suggested in Ref. 23. We need to point out, for rigor, that the modulation theory employed here is a mildly nonlinear theory, which is only valid for weak vertical displacements. The latter point justifies our choice in neglecting the coupling to in-plane dust grain motion, since we are only interested on the very first stage of the manifestation of nonlinearity in off-plane motion. A more general theory should take into account horizontal-to-vertical motion coupling, and should be the subject of forthcoming work of ours.

The linear dispersion characteristics of transverse DL waves were studied, including the dispersion relation, group velocity, and an evolution equation for the modulated amplitude of the first harmonic was derived. The dispersion relation shows a negative group velocity of the wave for $k < k_{critical}$ and a positive group velocity for $k > k_{critical}$. The value of $k_{critical}$ depends on the direction of wave propagation. These results are in excellent agreement with earlier numerical [16], experimental [5] and theoretical [17] results.
We have relied on a two-dimensional lattice multiple scale theory to separate the slow envelope evolution scale from the fast carrier space/time scales, and investigate the amplitude dynamics. We have shown that transverse wavepackets will in principle be stable in the long wavelength region, although modulational instability for shorter wavelengths is in principle not to be excluded (yet to be covered by a discrete version of the model, to come). Furthermore, we predict the formation of both bright and dark-type envelope solitons, in regions, similar to the bright envelope structures observed in laboratory experiments [5]. Admittedly, our study was limited within the continuum approximation, thus our results are valid in the long-wavelength limit. Therefore, rigorously speaking, only the small (wavenumber) \( k \) region of our plots should be retained in a strictly quantitative interpretation. Nevertheless, we can anticipate a discrete version of the theory, which would incorporate a discrete form of the NLS coefficients \( P \) (known, from the discrete dispersion relation) and \( Q \) (unknown, to be determined). Our graphs seem to suggest that both coefficient may change sign for higher \( k \) (shorter wavelength), hence allowing for a richer dynamical profile (beyond the continuum limit).

Furthermore, in an attempt to determine the region of validity of our study, in as much rigor as possible, we have investigated the role of the interaction among grains located at second-neighbor-sites. We have shown that this effect is rather small (compared to first neighbor interactions), as more or less expected physically; recall that the lattice spacing is of comparable order of magnitude to the Debye sphere in these strongly-coupled configurations. However, longer-than-nearest-neighbor interactions may give rise to interesting phenomena, as in particular a modification of the system’s behavior from the backward to the forward wave regime, as discussed in the text (see Fig. 3). Finally, we may add that although energy dissipation was neglected in this investigation, it may be added at a later stage. Physical effects thus
predicted are quite distinct, as shown by preliminary studies, so our aim was to pin-point the difference by addressing the damped wave case separately, in forthcoming work.

Our work is of relevance in dusty plasma crystal experiments in the laboratory, where our predictions for the type and stability of modulated wavepackets can be tested and will hopefully be confirmed. Beyond dusty (complex) plasma physics, we view this work as a fundamental investigation of nonlinear transverse motion in hexagonal crystals, of potential relevance (either currently or in the future) in other physical contexts where electrostatic-interaction-sustained crystalline structures occur (such as ultra-cold plasmas or one-component plasmas), or in lattice theory and in discrete dynamical systems, where pulse formation and wavepacket localization occurs.

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REFERENCES


[26] Dr D Samsonov, University of Liverpool, UK; private communication.

FIGURE CAPTIONS

Figure 1:
(Color online) Hexagonal lattice geometry: a) elementary cell: the six nearest particles around particle \( n,m \); b) the six nearest particles and the six second neighbors around particle are depicted as empty circles (forming the central hexagon) and as solid circles (forming the external hexagon), respectively.

Figure 2:
(Color online) Surface plot of a) the normalized frequency \( \omega/\omega_g \), b) the normalized group velocity \( v_g = \omega'(k) \), and c) the normalized group-velocity dispersion coefficient \( P = \omega''(k)/2 \) as a function of the (normalized) wave number and the angle \( \theta \). Here, we have taken \( \Omega/\omega_g = 0.305 \) (as in Ref. 17). Only FNI are taken into account here.

Figure 3:
(Color online) The dispersion relation (36) is plotted (in the collisionless case) for different values of the propagation angle. The solid curves are for FNI (first neighbor interactions), while the dashed ones are with SNI (second neighbor interactions). The angle of propagation is as follows. Upper curves: \( \theta = 30^\circ \); middle curves: \( \theta = 15^\circ \); bottom curves: \( \theta = 0 \). Note that the solid curves agree perfectly with Fig. 3 in Ref. 16a.

Figure 4:
(Color online) Contour plot of \( v_g = 0 \) up to FNI. The normalized \( k_{critical} \) as a function of \( \theta \).
The contour plot for SNI is practically superposed on that for FNI.
Figure 5:
(Color online) a) Linear dispersion curve for $\nu = 0$ and $\theta = 0$: compared the discrete result (dashed line) from Eq. (17a), to the result of the continuum approximation (solid line), from Eq. (25). b) Same, for SNI.

Figure 6:
(Color online) The parameters $\alpha$ and $\beta$ (effect of SNI) as a function of $\kappa$.

Figure 7:
(Color online) Variation of a) the coefficient $P$ for FNI (solid line) and for (dashed lines) in the continuous model as a function of $ka$; b) the coefficient $Q$ (continuous) as a function of $ka$.

Figure 8:
(Color online) Variation of a) the product of coefficients $PQ$ (continuous) for FNI (solid lines) and for SNI (dashed lines) as a function of $ka$; b) the ratio of coefficients $Q/P$ (continuous) as a function of $ka$ (solid lines for FNI and dashed lines for SNI).

Figure 9:
(Color online) Variation of the dispersion coefficient $P$ in the discrete description as a function of $ka$, (solid line) obtained from the dispersion relation up to FNI (Eq. 16, for $\nu = 0$) and (dashed line) up to SNI (Eq. 36). Notice the $4\pi$-periodicity.
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9