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Théorie cinétique pour une particule témoin faiblement couplée à un grand réservoir à l'équilibre - application au plasma magnétisé

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Théorie cinétique pour une particule témoin faiblement couplée à un grand réservoir à l'équilibre - application au plasma magnétisé

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Résumé

A partir des premiers principes de la Mécanique Statistique de Non-Equilibre, nous avons entrepris la dérivation d'une équation cinétique pour une particule (-`témoin') en interaction faible avec un grand réservoir à l'équilibre thermique. Les deux sous-systèmes sont soumis à un champ de force externe. Ce texte commence par une présentation des méthodes de la Mécanique Statistique permettant d'obtenir l'équation cinétique. La théorie de perturbation standard fournit, à partir de l'équation de Liouville, une *équation maîtresse généralisée* (*'Generalized Master Equation'*); dès lors, une *équation du type Fokker-Planck* suit dans l'approximation "markovienne". Des formules explicites générales sont présentées pour les coefficients de cette équation en fonction des paramètres physiques et notamment du champ extérieur.

Le formalisme est alors appliqué au cas particulier d'un plasma électrostatique soumis à un champ magnétique extérieur **B**, supposé uniforme. Des nouvelles expressions analytiques sont obtenues pour les coefficients de diffusion et de dérive, faisant explicitement apparaître le champ magnétique ainsi que le potentiel d'interaction électrostatique (à longue portée).

Finalement, le calcul analytique est avancé en considérant de manière explicite un potentiel d'interaction de type Debye et un état Maxwellien en arrière plan. Nous obtenons donc des nouvelles expressions non-dimensionelles pour les corrélations de forces d'interaction $C_{\perp,//}(\mathbf{v}, \tau; \mathbf{B})$ ainsi que pour les coefficients de diffusion $D_{\perp,\angle,//}(\mathbf{v}; \mathbf{B})$. Finalement, nous discutons leur variation en fonction des paramètres physiques du problème (à savoir: le temps, la grandeur du champ magnétique et la vitesse).

Kinetic theory for a test-particle weakly coupled to a large heat-bath in equilibrium - application to magnetized plasma *

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Abstract

We have considered the derivation from first principles of a kinetic equation for a test-particle weakly interacting with a large heat-bath in thermal equilibrium. Both subsystems are subject to an external force field. This text starts with an outline of the statistical-mechanical methods leading to the kinetic equation. The Liouville equation leads to a generalized master equation to 2nd order in the "weak" interaction; a *Fokker-Planck-type equation* then follows as a "markovian" approximation. Generic *field-dependent* formulae for the coefficients in the *collision term* are presented.

The formalism is then applied to the model case of a charged test-particle in an electrostatic plasma in a *uniform* magnetic field. Explicit expressions for the diffusion and drift coefficients are obtained in terms of the (long-range) interaction potential and the magnetic field.

Finally, we advance the analytical computation by considering Debye-type interactions and a Maxwellian state in the background. Re-scaled (non-dimensional) expressions are then derived for the two-time force correlations $C_{\perp,\|}(\tau; field)$ and the diffusion coefficients $D_{\perp,\angle,\|}$ and briefly discussed in terms of their parameters (namely: time, the magnitude of the magnetic field, velocity).

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1 Introduction

A wealth of physical phenomena have been elucidated since non-equilibrium statistical mechanics originally provided the necessary framework in an attempt to understand the role of microscopic collisions on the (irreversible) macroscopic dynamics of matter. In particular, charged matter (*plasma*) has always been thought of as a challenging test-bed for transport theories. Furthermore, in the last few decades, the aim of controlling thermonuclear fusion for man-oriented applications and the big amount of experimental research run on fusion devices has provided a vast field of application for plasma transport theories, as well as a real challenge in terms of demand for theoretical ground for prediction of measurable quantities (transport coefficients).

However, the kinetic description of plasma is a long-standing, open problem, since the *long-range* character of electrostatic interactions makes the standard neutral gas formalism, i.e. through Boltzmann-type theories, inappropriate. A different approach is therefore needed. Since the original work of Landau a plethora of works have focused on the kinetic description of plasma, as a step towards theoretical explanation (as well as experimental manipulation) of plasma transport phenomena. Outstanding contributions have been the works of Vlasov, who was the first to include the reciprocal interaction between a particle and the mean surrounding field in a "self-consistent" description and Balescu-Lenard-Guernsey, who went one step further by including collective (three-body) effects in an elegant description [1]. The so-called Brussels' school has then provided a new formal framework and microscopic theories of dissipative phenomena were often associated to master equations.

A typical paradigm of such a theory is the relaxation of a small subsystem close to (but not at) equilibrium in (weak) interaction with a heat bath. The evolution of the system is described by a (*non-Markovian*) generalized master equation. The standard procedure consists in evaluating the kernel of the master equation, by adopting some 'markovianization' assumption. This is not an easy task, as one has to take into account inter-particle interactions (*collisions*) on one hand, and also the inevitable *influence of external force fields*, if such are present, on particle trajectories between collisions. As the latter has often been neglected in collisions terms presented in literature, we have undertaken this work in order to stress the influence of the magnetic field on the transport properties of plasma.

The system considered in our work consists of a (or 'a few') test-particle(s) injected in a neutral background heat-bath (*'reservoir'* 'R') in thermal equilibrium. Both subsystems are embedded in an external field; they are assumed to be initially decorrelated and interactions are taken to be "weak". In a generic manner, the formalism presented in the text applies to *any* dynamical problem[1] obeying the above description, that is for

[1] provided that a given analytic solution of the linear (or linearized) zeroth-order (single-particle)

a specific choise of (a) problem of motion (in a force field)[1] and (b) inter-particle (long-range) interaction law. In this work we will:
(i) outline the derivation of a kinetic equation, from first principles, describing the evolution in time of the test-particle distribution function $f(\mathbf{v}; t)$,
(ii) present analytic expressions for the coefficients in the collision term, in terms of the external field, the interaction potential $V(r)$ and the field-particle equilibrium distribution function $\phi_{eq}(\mathbf{v_1})$ and
(iii) explicitly evaluate the diffusion coefficients, in the case of plasma embedded in an external magnetic field, taken to be uniform for simplicity. Considering Debye-type interactions and a Maxwellian distribution for the heat bath, the diffusion coefficients are evaluated and studied in terms of the magnitude of the external field.

Let us remark that the system considered here will be taken to be spatially homogeneous, so that $f = f(\mathbf{v})$. The generalization of the formalism to a *non-uniform plasma* in the presence of an external field has been considered elsewhere [2]; it is the object of current research work which will be presented later [3].

The figures are attached in the end of the text.

2 The model

We consider a test-particle (t.p.) 'Σ' surrounded by (and weakly coupled to) a homogeneous reservoir $R \equiv \{1, 2, ..., N\}$. $\mathbf{X} = (\mathbf{x}, \mathbf{v}) \equiv (\mathbf{x_\Sigma}(t), \mathbf{v_\Sigma}(t))$ and $\mathbf{X_R} \equiv \{\mathbf{X_j}\} = (\mathbf{x_j}(t), \mathbf{v_j}(t))$ will denote the coordinates of the test- ($\Sigma-$) and reservoir- ($R-$) particles respectively.

2.1 Equation of motion

The equations of motion for the t.p. read:

$$\dot{\mathbf{x}} = \mathbf{v} \, ; \qquad \dot{\mathbf{v}} = \mathbf{F_0}(\mathbf{x}, \mathbf{v}) \, + \, \lambda \, \mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t) \tag{1}$$

The force $\mathbf{F_0}(\mathbf{x}, \mathbf{v})$ is due to the external field; in the case of an electrostatic plasma, i.e. $N+1$ particles of species $\alpha' \in \{\alpha_j\} = \{e, i, ...\}$ ($\equiv$ *electrons, ions, ...*) (charge e_j^α, mass m_j^α, $j = 1, 2, ..., N, \Sigma$) in a uniform magnetic field, it represents the Lorentz force

$$\mathbf{F_L}(\mathbf{v}) = \frac{e}{c}(\mathbf{v} \times \mathbf{B}) \tag{2}$$

The *interaction* force

$$\mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}; t) = -\frac{\partial}{\partial \mathbf{x}} \sum V(|\mathbf{x} - \mathbf{x_j}|)$$

problem of motion is known (in the form of (4) in the text); for the sake of reference, such cases include motion in (i) a magnetic field (ii) a harmonic potential field (oscillator models) and the limit case of (iii) no field (free motion).

represents the sum of random interactions between Σ and the heat bath (assumed to be in equilibrium); it can be proved to be a stationary Gaussian process with zero mean-value.

2.2 Single-particle dynamics

In the following we shall assume that the zeroth-order problem of motion:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \frac{1}{m}\mathbf{F_0}(\mathbf{x},\mathbf{v})\end{pmatrix} \tag{3}$$

yields a known analytic solution in the form:

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{M}'(t)\,\mathbf{x} + \mathbf{N}'(t)\,\mathbf{v}\\ \mathbf{x}(t) &= \mathbf{x} + \int_0^t dt'\;\mathbf{v}(t') = \mathbf{M}(t)\mathbf{x} + \mathbf{N}(t)\mathbf{v}\end{aligned}$$

i.e.

$$\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{M}(t) & \mathbf{N}(t)\\ \mathbf{M}'(t) & \mathbf{N}'(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} = \mathbf{E}(t)\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} \tag{4}$$

[2] with the initial conditions: $\{\mathbf{x}(0), \mathbf{v}(0)\} = \{\mathbf{x}, \mathbf{v}\}$ implying $\mathbf{E}(t=0) = \mathbf{I}$.

As obvious, the following group properties have to be respected:

$$\begin{aligned}&(i) \qquad \mathbf{E}(t+t') = \mathbf{E}(t)\,\mathbf{E}(t')\\ &(ii) \qquad \mathbf{E}^{-1}(t) = \mathbf{E}(-t) \qquad\qquad \forall t \in \Re\end{aligned} \tag{5}$$

implying a set of relations to be satisfied by the $d \times d$ matrices in (4):

$$\begin{aligned}\mathbf{N}'(t)\mathbf{N}'(t') &= \mathbf{N}'(t+t') \qquad \forall t,t' \in \Re\\ \mathbf{N}(t') + \mathbf{N}(t)\mathbf{N}'(t') &= \mathbf{N}(t+t') \qquad \forall t,t' \in \Re\end{aligned} \tag{6}$$

Note that, in general:

$$\mathbf{N}(t)\mathbf{N}(t') \neq \mathbf{N}(t+t') \qquad \forall t,t' \in \Re$$

Also:

$$\mathbf{N'}^{-1}(t) = \mathbf{N}'(-t) \qquad \forall t \in \Re$$

(yet $\mathbf{N}^{-1}(t) \neq \mathbf{N}(-t)$).

[2] In a $d-$dimensional problem, $\{\mathbf{M}(t), \mathbf{N}(t)\}$ are $d \times d$ matrices whose form depends on the particular aspects of the dynamical problem taken into consideration; properly speaking, one has

$$\begin{pmatrix} M_{ij}(t) & N_{ij}(t)\\ M'_{ij}(t) & N'_{ij}(t)\end{pmatrix} = \begin{pmatrix}\frac{\partial x_i^{(0)}(t)}{\partial x_j} & \frac{\partial x_i^{(0)}(t)}{\partial v_j}\\ \frac{\partial v_i^{(0)}(t)}{\partial x_j} & \frac{\partial v_i^{(0)}(t)}{\partial v_j}\end{pmatrix}$$

thus (4) may be viewed as a linearized (in x_j, v_j) solution of the - possibly nonlinear - 'free' (i.e. collisionless) motion problem.

2.3 "Free" motion in a magnetic field

Let us consider the simple case of a uniform external magnetic field in the $\hat{z}$ direction. The zeroth-order ($\sim \lambda^0$) problem of motion yields the well-known (helicoidal) solution:

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{N}(t)\,\mathbf{v}(0) \qquad\qquad \mathbf{v}(t) = \mathbf{R}(t)\,\mathbf{v}(0)$$

i.e.

$$\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{N}(t) \\ \mathbf{0} & \mathbf{R}(t) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} \tag{7}$$

where

$$\mathbf{R}^\alpha(t) = \mathbf{N}'(t) = \begin{pmatrix} \cos\Omega t & s\,\sin\Omega t & 0 \\ -s\,\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{N}(t) = \int_0^t dt'\,\mathbf{R}^\alpha(t) = \Omega^{-1} \begin{pmatrix} \sin\Omega t & s\,(1-\cos\Omega t) & 0 \\ s\,(\cos\Omega t - 1) & \sin\Omega t & 0 \\ 0 & 0 & \Omega t \end{pmatrix} \tag{8}$$

(i.e. $\mathbf{M} = \mathbf{I}$, $\mathbf{M}' = \mathbf{0}$, $\mathbf{N}' = \mathbf{R}(t)$, cf. (4)); Ω is the gyro-frequency of particle j (of species $\alpha \in \{e, i, ...\}$, mass m_α, charge e_α):

$$\Omega = \Omega_j \equiv \frac{|e_\alpha| B}{m_\alpha c} \tag{9}$$

and $s^j = sgn(e_j) = \pm 1$. Relations (6) are satisfied, as may easily be checked; furthermore, relations (8) satisfy:

$$\mathbf{R}(-t) = \mathbf{R}^T(t), \qquad \mathbf{N}(-t) = -\mathbf{N}^T(t)$$

Remember that $\mathbf{N}(t=0) = \mathbf{0}$ and $\mathbf{N}'(t=0) = \mathbf{I}$. Also note that, 'switching off' the field (i.e. for $\Omega = 0$) we obtain $\mathbf{N} = t\,\mathbf{I}$ and $\mathbf{N}' = \mathbf{I}$ and thus recover the free motion (Landau) limit.

3 Statistical formulation - the kinetic equation

Let $\rho = \rho(\{\mathbf{X}, \mathbf{X_R}\}; t)$ ($F = F(\mathbf{X_R})$) be the total (reservoir) phase-space distribution function (df), normalized to unity: $\int d\mathbf{X}\,\rho = 1$ ($\int d\mathbf{X_R}\,F = 1$); the subscript will be omitted in the following where Σ is understood.

The equation of continuity in phase space reads:

$$\frac{\partial \rho}{\partial t} + \mathbf{v_j}\frac{\partial \rho}{\partial \mathbf{x_j}} + \frac{\partial}{\partial \mathbf{v_j}}\left(\frac{1}{m}\,\mathbf{F_j}\,\rho\right) = 0 \tag{10}$$

where a summation over j $(= 1, 2, ..., N, \Sigma)$ is understood.

3.1 Reduction of the Liouville equation - BBGKY hierarchy

The standard procedure consists in defining appropriate 's-body' reduced distribution functions (rdf), among which the (1−body-) test-particle rdf:

$$f(\mathbf{x}, \mathbf{v}; t) = (I,\, \rho)_R \equiv \int_{\Gamma_R} d\mathbf{X_R}\, \rho$$

and then appropriately integrating the $N-$particle Liouville equation in order to obtain a hierarchy of coupled equations of evolution of the rdf's. Since this is more or less a standard procedure [4], the details will be omitted here [3]. In order to obtain an equation of evolution for $f(t)$, the *BBGKY hierarchy* of equations thus obtained can be truncated to 2nd order in λ by assuming interactions to be weak (i.e. $\lambda \ll 1$). One thus obtains the system:

$$\begin{aligned}(\partial_t - L_0^{\Sigma})\, f(\mathbf{X}; t) &= \lambda^2 \int d\mathbf{X_1}\, L_I\, g(\mathbf{X}, \mathbf{X_1}; t) \, + \mathcal{O}(\lambda^3) \\ (\partial_t - L_0^{\Sigma} - L_0^{1})\, g(\mathbf{X}, \mathbf{X_1}; t) &= \lambda\, L_I\, F_1(\mathbf{X_1})\, f(\mathbf{X}) + \mathcal{O}(\lambda^2)\end{aligned} \tag{11}$$

where L_0^j is the "free" Liouvillian *in* the field:

$$L_0^j \cdot = -\mathbf{v_j}\, \frac{\partial \cdot}{\partial \mathbf{x_j}} - \frac{1}{m_j}\, \frac{\partial}{\partial \mathbf{v_j}}\, (\mathbf{F_0}\, \cdot) \tag{12}$$

and $L_I \equiv L_{\Sigma 1}$ is the binary interaction operator $L_I \equiv L_{\Sigma 1}$ where:

$$L_{ij} = -\mathbf{F_{int}}(|\mathbf{x_i} - \mathbf{x_j}|) \left(\frac{1}{m_i} \frac{\partial}{\partial \mathbf{v_i}} - \frac{1}{m_j} \frac{\partial}{\partial \mathbf{v_j}} \right) \tag{13}$$

$(i, j \in \{\Sigma, 1_R^{\alpha}\})$. As obvious, $f = f(\mathbf{X}^{\alpha}; t)$, $F_1(\mathbf{X}_{1_R^{\alpha'}})$ and $f_2(\mathbf{X}^{\alpha}, \mathbf{X}_{\mathbf{1}}^{\alpha'}; t)$ denote the $\Sigma-$1-body, $R-$1-body and $(1_R^{\alpha'} + \Sigma^{\alpha})-$2-body rdf's respectively and $g = g(\mathbf{X}^{\alpha}, \mathbf{X}_{\mathbf{1}}^{\alpha'}; t)$ is the 'two-body' $(1_R^{\alpha'} + \Sigma^{\alpha})$ correlation function:

$$g(\mathbf{X}, \mathbf{X_1}; t) = f_2(\mathbf{X}^{\alpha}, \mathbf{X}_{\mathbf{1}}^{\alpha'}; t) - F(\mathbf{X}_{\mathbf{1}}^{\alpha'}) f(\mathbf{X}^{\alpha}; t)$$

(in a multi-component plasma, a summation over particle species α' is to be understood in the rhs of (11)). Note that the mean-field *(Vlasov)* term, in order λ^1, disappears since we have assumed the reservoir to be in a homogeneous equilibrium state $F = n_{\alpha'} \phi_{eq}^{\alpha'}(\mathbf{v_1})$ (essentially a Maxwellian state $\phi_{Max}(v_1)$).

3.2 The generalized master equation

In order to obtain a closed equation for f, the system of equations (11) can be decoupled by solving the second equation and then substituting into the first. Neglecting correlations

[3] In fact, the precise formulation in a test-particle problem is quite similar but *not* identical to the one found in [4], since one must distinguish $s-$body rdf's where the t.p. is, or is not, included. The details will be provided in [3]

at $t = 0$, f is found to obey a (non-markovian) *generalized master equation (GME)* :

$$\partial_t f(\mathbf{x}, \mathbf{v}; t) = L_0 \, f(\mathbf{x}, \mathbf{v}; t) + \lambda^2 \, n \int_0^t d\tau \int d\mathbf{x_1} \, d\mathbf{v_1} \, L_I \, U^{(0)}(\tau) \, L_I \, \phi_{eq}(\mathbf{v_1}) \, f(\mathbf{x}, \mathbf{v}; t - \tau) \quad (14)$$

($n = n_{\alpha'} = \frac{N_{\alpha'}}{V}$ is the particle density; a summation over species α' is understood) All operators were defined in the previous paragraph. By

$$U^{(0)}(t) \equiv Exp(L_0 \, t)$$

we denote the 'free' (collisionless) Liouville time-evolution operator (*'propagator'*).

3.3 Markovian approximation

Adopting a standard 'markovian' approximation, which consists in substituting with the zeroth-order solution - assuming, that is, that

$$f(t - \tau) \approx e^{-L_0 \tau} \, f(t) \; \equiv \; U^{(0)}(-\tau) \, f(t)$$

is sufficient to this order - we obtain the *markovian* master equation:

$$\partial_t f - L_0 \, f \, = \, n \int_0^t d\tau \int d\mathbf{x_1} \, d\mathbf{v_1} \, L_I \, U^{(0)}(\tau) \, L_I \, \phi_{eq}(\mathbf{v_1}) \, U^{(0)}(-\tau) \, f \quad (15)$$

($f = f(\mathbf{v}; t)$); the 'tag' λ^2 in the rhs will be omitted for simplicity. The asymptotic limit, $t \to \infty$, is most often considered in literature, essentially yielding time-independent coefficients in the kinetic equation.

4 Kinetic equation

One is now left with the task of evaluating the kernel in the collision integral (rhs) of the master equation, taking into account the specific features of the particular physical problem considered. This is done by explicitly substituting from (13) into (15) and then evaluating the action of the propagator $U^{(0)}(t)$ on functions of the phase-space variables $\mathbf{X} \equiv \{\mathbf{x}, \mathbf{v}\}$ e.g.

$$U^{(0)}(t) f(\mathbf{X}) \equiv U^{(0)}(t) f(\mathbf{X}; 0) = f(\mathbf{X}; t) = f(\mathbf{X}(\mathbf{0}); t) = f(\mathbf{X}(-\mathbf{t}); 0) \equiv f(\mathbf{X}(-\mathbf{t}))$$

(this is actually a consequence of the Liouville theorem [4], [5]) [4] and so forth. A key element often neglected in the past is the fact that $U^{(0)}(t)$ does *not* commute with phase-space gradients $\frac{\partial}{\partial \mathbf{v}}$, ...; more precisely, when dealing with the exact form of the kernel, one encounters the expression:

$$\mathbf{D}_{\mathbf{V_i}}(t) \equiv U(t) \frac{\partial}{\partial \mathbf{v_i}} U(-t) \qquad i = \Sigma, \; 1^R$$

[4]The propagator formalism is exhaustively studied in [ref. Misguich-Balescu...]. A brief but concise discussion of the theory can be found in [1], [4].

which can be evaluated in terms of the solution (4) of the dynamical problem; we find:

$$\mathbf{D}_{V_i}(t) = \mathbf{N_i}^T(t)\ \frac{\partial}{\partial \mathbf{x_i}} + \mathbf{N_i'}^T(t)\ \frac{\partial}{\partial \mathbf{v_i}} \tag{16}$$

in full agreement with the results in [6]; a similar expression can be obtained for the space-gradient $\frac{\partial}{\partial \mathbf{x}}$.

The matrices in (16) ($\mathbf{N_i}(t), ...$) contain the signature of the external field, since they have been defined through the solution of the specific dynamical problem (see in Section 2). The force field considered is thus seen to enter, just as expected, the collision term of the kinetic equation in an explicit manner, and this is true for any specific problem considered.

4.1 Fokker-Planck-equation

By explicitly evaluating the kernel in (15) we find the 2nd order PDE:

$$\begin{aligned}\frac{\partial f}{\partial t} \ + \ \frac{1}{m}\mathbf{F}\frac{\partial f}{\partial \mathbf{v}} \quad &= \quad \frac{\partial}{\partial \mathbf{v}}[\mathbf{A}\frac{\partial}{\partial \mathbf{v}} \ + \ \mu\, \mathbf{a}]\ f \\ &\equiv \quad \mathcal{C}\{f(\mathbf{v};t)\} \end{aligned} \tag{17}$$

($\mu \equiv m/m_1^\alpha$). Remember that $\mathbf{F}$ is the force which is due to the external force field ($\mathbf{F_0}$ in §2.1); once more, don't forget that the mean-field force (Vlasov term) that one would expect to see in the lhs of such a kinetic equation cancels once we took the background to be in homogeneous equilibrium.

The above equation can be cast into the form of a *Fokker-Planck-type equation* :

$$\frac{\partial f}{\partial t} \ + \ \frac{1}{m}\mathbf{F}\frac{\partial f}{\partial \mathbf{v}} \ = \ \frac{\partial^2}{\partial v_i \partial v_j}(A_{ij} f) \ - \ \frac{\partial}{\partial v_i}(\mathcal{F}_i f) \tag{17-bis}$$

where $\mathbf{A} = \mathbf{A}(\mathbf{x}, \mathbf{v}; field)$ is a 3×3 *diffusion matrix* and $\mathcal{F} = \mathcal{F}(\mathbf{x}, \mathbf{v}; field)$ is a $3d$ *dynamical friction* vector defined as:

$$\mathcal{F}_i = -\mu a i + \partial A_{ij}/\partial v_j \tag{18}$$

The *rhs* of this equation is reminiscent of the form of a Fokker-Planck equation; however, the coefficients in it are not constant, but depend on phase-space variables *and* on the external force field.

4.2 Coefficients

Let us define the two-time interaction-force correlation matrix:

$$\begin{aligned}\mathbf{C} \ &= \ \langle \mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2)\rangle_R \\ &\equiv \ \int_{\Gamma_1} d\mathbf{x_1} d\mathbf{v_1}\ \phi_{eq}^{\alpha'}(\mathbf{v_1})\ \mathbf{F_{int}}(|\mathbf{x}(t_1) - \mathbf{x_1}(t_1)|)\ \mathbf{F_{int}}(|\mathbf{x}(t_2) - \mathbf{x_1}(t_2)|) \\ &= \ \mathbf{C}(\mathbf{x}, \mathbf{v}; t_1, t_2)\end{aligned}$$

and

$$\begin{aligned}
\mathbf{d} &= \int_{\Gamma_1} d\mathbf{x_1} d\mathbf{v_1}\, \mathbf{F_{int}}(|\mathbf{x}(t_1) - \mathbf{x_1}(t_1)|)\, \mathbf{F_{int}}(|\mathbf{x}(t_2) - \mathbf{x_1}(t_2)|) \\
&\qquad \underline{\underline{\mathbf{R}}}_{\mathbf{1}}{}^{T}(\tau)\, \frac{\partial \phi_{eq}^{\alpha'}(\mathbf{v_1})}{\partial \mathbf{v_1}} \\
&= \mathbf{d}(\mathbf{x}, \mathbf{v}; t_1, t_2) \qquad (19)
\end{aligned}$$

The coefficients in eq. (17) are conveniently expressed in terms of the above quantities:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}, \mathbf{v}) &= \frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \\
&\qquad \mathbf{F_{int}}(|\mathbf{x} - \mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau) - \mathbf{x_1}(-\tau)|) \mathbf{N}'^{T}(\tau) \\
&= \frac{n}{m^2} \int_0^\infty d\tau\, \underline{\underline{\mathbf{C}}}(\mathbf{x}, \mathbf{v}; t, t-\tau) \underline{\underline{\mathbf{N}}}'^{T}(\tau) \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(\mathbf{x}, \mathbf{v}) &= -\frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \\
&\qquad \mathbf{F_{int}}(|\mathbf{x} - \mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau) - \mathbf{x_1}(-\tau)|)\, \underline{\underline{\mathbf{R}}}_{\mathbf{1}}^{\mathbf{T}}(\tau)\, \frac{\partial \phi(\mathbf{v_1})}{\partial \mathbf{v_1}} \\
&= -\frac{n}{m^2} \int_0^\infty d\tau\, \mathbf{d}(\mathbf{x}, \mathbf{v}; t, t-\tau) \qquad (21)
\end{aligned}$$

5 Case of interest: electrostatic plasma in a magnetic field

The results of the previous section are valid, just as such (i.e. precisely eqs. (17), (17-bis), along with definitions (18) - (21) for the coefficients) upon substitution with the exact form (8) for the dynamic matrices.

5.1 Plasma kinetic equation

By applying the above results in the magnetized plasma case (considering a uniform magnetic field along $\hat{z}$), the diffusion matrix is found to be of the form:

$$\mathbf{A}(t) = \begin{pmatrix} D_\perp & D_\angle & 0 \\ D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \qquad (22)$$

The test-particle df $f(\mathbf{v}; t)$ is thus found to obey the **plasma kinetic equation**:

$$\begin{aligned}
\frac{\partial f}{\partial t} + \frac{e}{mc}(\mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}} &= \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \Big) \Big[D_\perp(\mathbf{v}) f \Big] + \frac{\partial^2}{\partial v_z^2} [D_\parallel(\mathbf{v}) f] \\
&\quad - \frac{\partial}{\partial v_x} \Big[\mathcal{F}_x(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_y} \Big[\mathcal{F}_y(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_z} \Big[\mathcal{F}_z(\mathbf{v})\, f \Big] \qquad (23)
\end{aligned}$$

This equation (presented in [2]) is formally similar to a linearized form of a kinetic equation derived in the past [7].

All the coefficients are defined in the following paragraph.

5.2 Plasma coefficients

Notice the particular form of the diffusion matrix $\mathbf{D}$ in (22) which reflects the cylindrical-symmetry of the problem; remember that the celebrated Landau collision term for electrostatic plasma (in *no* external field) presents a *spherical* type of symmetry [1]. This is a manifestation of the Onsager symmetry principle [8], as extensively argued in [1]. Furthermore, as suggested in §5.5-B therein, one would expect $\mathbf{D}$ to become diagonal (in fact proportional to the unit matrix) should the field be "switched off". One may straightforward check that our matrix elements defined in the next paragraph *do* satisfy this criterion.

The diffusion coefficients are actually all functions of $\{v_\perp, v_\|\}$ only (remember that the components $v_\perp \equiv (v_x^2+v_y^2)^{1/2}$ and $v_\| \equiv v_z$ of the velocity are conserved under the action of a constant Lorentz force; cf. (2), (7), (8)). They are defined by:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} \\ D_\| \end{array} \right\} = \sum_{\alpha'} \frac{1}{m_\alpha^2} \int_0^t d\tau \left\{ \begin{array}{c} \left\{ C_\perp^{\alpha,\alpha'} \right\} \\ C_\|^{\alpha,\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\Omega^\alpha\tau \\ (-s^\alpha)\,\frac{1}{2}\sin\Omega^\alpha\tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{24}$$

where $C_{\{\perp,\|\}}^{\alpha,\alpha'}(v_\perp, v_\perp; \Omega)$ are (diagonal) elements of the force-correlation matrix $\mathbf{C}(\tau) = \langle \mathbf{F_{int}}(t)\ \mathbf{F_{int}}(t-\tau)\rangle_R$; they come out to be:

$$C_*^{\alpha,\alpha'} = n_{\alpha'}(2\pi)^3 \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\ e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\ k_*^2 \tag{25}$$

($* \in \{\perp, \|\}$; a summation over n, m is understood) where v_i ($v_{1,i}$), $i = 1,2,3$ denote the velocity coordinates of the test- (R-) particle of species $\alpha_\Sigma \equiv \alpha$ ($\alpha_1 = \alpha' \in \{e, i, ...\}$) respectively (the α index will be dropped where obvious). Finally, $\tilde{V}_k$ stands for the Fourier transform of $V(r)$:

$$\tilde{V}_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int d\mathbf{r}\ V(\mathbf{r}(t))\ e^{i\mathbf{kr}(t)}\ , \qquad V(\mathbf{r}(t)) = \int d\mathbf{k}\ \tilde{V}_{\mathbf{k}}\ e^{-i\mathbf{kr}(t)} \tag{26}$$

Remember that $V = V(|\mathbf{r}|) = V(r)$ implies $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$ (so $V_{\mathbf{k}} = V_{-\mathbf{k}} = V_k$)

Notice the explicit dependence on the magnetic field through $\Omega = \Omega_i^{\alpha_i}$ ($i = \Sigma^\alpha, 1^{\alpha'}$) and also on the form of the reservoir equilibrium d.f. $\phi_{eq} = \phi_{eq}(v_\perp, v_\|)$ and the interaction potential $V(r)$.

In a single species system all coefficients are functions of $\mathbf{v} - \mathbf{v_1} \equiv \mathbf{g}$; one may prove that

$$a_i = -\partial A_{ij}/\partial v_j \tag{27}$$

The dynamical friction terms $\mathcal{F}_i$ are then given by:

$$\mathcal{F}_x = (1+\mu)\,(\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y}) \qquad \mathcal{F}_y = (1+\mu)\,(-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y})$$
$$\mathcal{F}_z = (1+\mu)\,\frac{\partial D_\parallel}{\partial v_z} \qquad \mu = \frac{m_\alpha}{m_{\alpha'}} \tag{28}$$

6 Explicit construction of the coefficients for a single-species plasma

The exact form of the coefficients presented above can be computed once one has chosen a specific form of interaction potential $V(r)$ and equilibrium reservoir distribution function $\phi_{eq}(v_1)$. Once an appropriate reference frame is chosen (see figure 1), the integrals can be carried out in convenient polar coordinates $\{k_\perp, \alpha, k_\parallel\}$ [5], $\{v_{1,\perp}, \beta, v_{1,\parallel}\}$, so that

$$\int_0^{2\pi} d\alpha \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^\infty dk_\parallel \cdots = (2\pi) \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^\infty dk_\parallel \cdots = ... \tag{29}$$

and so forth (note that neither ϕ_{Max} nor $\tilde{V}_k$ depend on the angle variable).

The calculation is tedious but straightforward. For the sake of clarity in presentation, only the final results will be presented in the next two paragraphs. The detailed calculation is provided in the Appendix [9].

6.1 Maxwellian distribution function

Let us carry out the v_1- integration in (25) by assuming ϕ_{eq} to be a Maxwellian of the form:

$$\phi_{Max}^{\alpha'}(v_1) = \prod_{i=1,2,3} \phi_0^{(i,\alpha')}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \tag{30}$$

$(\phi_0^{(i)} = (\frac{m_{\alpha'}}{2\pi T_{\alpha'}^{(i)}})^{1/2} \equiv \frac{1}{\sqrt{\pi\sigma_i^{\alpha'}}}$; $\quad \sigma_i^{\alpha'} \equiv 2\, {v_{i,th}^{\alpha'}}^2 \equiv \frac{2T_i^{\alpha'}}{m_{\alpha'}} \quad \forall i \in \{1,2,3\} \equiv \{x,y,z\}$ [6].

For a two-temperature plasma (i.e. $\sigma_1 = \sigma_2 = \sigma_\perp$, $\sigma_3 = \sigma_\parallel$) we obtain (see in the Appendix for details):

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{n_{\alpha'}}{m_\alpha^2} (2\pi)^4 e^{-v_\parallel^2/\sigma_\parallel^{\alpha'}} \int_0^t d\tau \int_0^\infty dk_\perp \Big[\int_{-\infty}^\infty dk_\parallel\, k_\parallel^{\{0,2\}} e^{-\sigma_\parallel^{\alpha'} \left(k_\parallel \tau - i\frac{2v_\parallel}{\sigma_\parallel^{\alpha'}}\right)^2/4} \tilde{V}_k^2 \Big]$$

[5] The angle variable α should not be mistaken for the species 'tag' used previously.

[6] We shall later set $T_\perp = T_\parallel = T^\alpha$ for simplicity, i.e.

$$\phi_{eq}^\alpha(v_1) = \big(\frac{m_\alpha}{2\pi T_\alpha}\big)^{\frac{3}{2}}\, e^{-\frac{m_\alpha (\mathbf{v}_1 - \mathbf{u})^2}{2T_\alpha}} \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{v_\alpha^3}\, e^{-\frac{(\mathbf{v}_1 - \mathbf{u})^2}{2v_{th,\alpha}^2}}$$

($\mathbf{u} = 0$ in this text).

$$k_\perp^{\{3,1\}}\, e^{-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'} \tau}{2}} J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2}) \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_\alpha \tau \\ (-s)\, \frac{1}{2} \sin \Omega_\alpha \tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{31}$$

(a summation over particle species α' will be understood wherever α' appears). Obviously, m (n) in $\{m, n\}$ correspond to the upper (lower) i.e. $\perp$ ($\|$) parts respectively.

6.2 Debye interactions

In fact, relation (31) holds as it stands for *any* particular form of (long-range) central interaction potential $V(r)$. Let us now explicitly consider a Debye potential:

$$V(r) = e^2 \frac{e^{-k_D r}}{r}, \qquad \tilde{V}_k = \frac{e^2}{2\pi^2} \frac{1}{k^2 + k_D^2} = \frac{\tilde{V}_0}{k_\perp^2 + k_\|^2 + k_D^2}$$

($\lambda_D = k_D^{-1} = (\frac{4\pi e^2 n}{k_B T})^{-1}$ is the Debye length [1]).

The integral(s) in $k_\|$, say $I_{k_\|}^{\{\perp,\|\}}$ (within brackets in (31)), can now be explicitly evaluated (once more, see in the Appendix for details); the calculation yields:

$$I_{k_\|}^{\{\perp,\|\}} = e^{v_\|^2/\sigma_\|^{\alpha'}} \frac{1}{\left\{ \begin{array}{c} \hat{k}_\perp^3 \\ \hat{k}_\perp \end{array} \right\}} F_{\{\perp,\|\}}^{\alpha'}$$

where the functions $F = F_{\{\perp,\|\}}^{\alpha'}(k_\perp, v_\|, \tau;\, \sigma_\|^{\alpha'})$ are given by:

$$F_{\{\perp,\|\}}^{\alpha'} = \pm \frac{\sqrt{\pi}}{2} \sqrt{\sigma_\|^{\alpha'}}\, \hat{k}_\perp \tau\, e^{-v_\|^2/\sigma_\|^{\alpha'}}$$
$$+ \frac{\pi}{4} e^{\sigma_\|^{\alpha'} \hat{k}_\perp^2 \tau^2/4} \sum_{s=+1,-1} \Big[e^{s \hat{k}_\perp v_\| \tau} \,(1 \mp \sigma_\|^{\alpha'} \hat{k}_\perp^2 \tau^2/2 \mp s \hat{k}_\perp v_\| \tau)\, Erfc(\frac{1}{2}\sqrt{\sigma_\|^{\alpha'}}\hat{k}_\perp \tau + s \frac{v_\|}{\sqrt{\sigma_\|^{\alpha'}}}) \Big] \tag{32}$$

and

$$\hat{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$$

- the upper (lower) signs corresponding to the $\perp$ ($\|$)- parts respectively. $Erfc(x)$ is the *complementary* error function:

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The coefficients in (31) i.e. (24) (actually functions of $\{v_\perp, v_\|, t;\, \sigma_\perp^{\alpha'}, \sigma_\|^{\alpha'}, \Omega^{\{\alpha',\alpha\}}\}$) now become:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} \\ D_\| \end{array} \right\} = \frac{n_{\alpha'}}{m_\alpha^2} 4 e_\alpha^2 e_{\alpha'}^2 \int_0^t d\tau \int_0^\infty dk_\perp\, e^{-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'} \tau}{2}} J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2})$$
$$\Big(1 - \frac{k_D^2}{k_D^2 + k_\perp^2}\Big)^{\{3/2,1/2\}} \left\{ \begin{array}{c} \left\{ \begin{array}{c} F_\perp^{\alpha'} \end{array} \right\} \\ F_\|^{\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_\alpha \tau \\ \frac{-s}{2} \sin \Omega_\alpha \tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{33}$$

where the functions $F = F^{\alpha'}_{\{\perp,||\}}(k_\perp, v_{||}, \tau;\, \sigma^{\alpha'}_{||})$ were defined above. Note that the integrand vanishes at infinity i.e. at $k_\perp \to \infty$ (and also at $\tau \to \infty$). Futhermore, the limit of the integrands at $k_\perp \to 0$ is finite (the same holds for $\tau \to 0$).

6.3 Force correlations

Remember that the diffusion coefficients $D_{ij}(t)$ are related to the force correlation function(s) $C_*(\tau)$ ($* \equiv \perp, ||$) through expression (24) above. The correlations $C_*(\tau)$ therefore read:

$$\begin{aligned} C^{\alpha}_{\{\perp,||\}} &= \sum_{\alpha'} C^{\alpha\,\alpha'}_{\{\perp,||\}} \\ &= 4e_\alpha^2 \sum_{\alpha'} n_{\alpha'} e_{\alpha'}^2 \int_0^\infty dk_\perp\, e^{-\sigma^{\alpha'}_\perp \frac{k_\perp^2}{\Omega^2_{\alpha'}} \sin^2 \frac{\Omega_{\alpha'}\tau}{2}}\, J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin\frac{\Omega_\alpha \tau}{2})\; F^{\alpha'}_{\{\perp,||\}} \\ &\qquad \left(1 - \frac{k_D^2}{k_D^2 + k_\perp^2}\right)^{\{3/2,1/2\}} \end{aligned} \tag{34}$$

(compare (33) to (24) where $F^{\alpha'}$ is now defined by:

$$F^{\alpha'}_{\{\perp,||\}} = \pm\sqrt{\pi}\,\phi\, e^{-\hat{v}^2_{||}} + \frac{\pi}{4} e^{\phi^2} \sum_{s=+1,-1} \Big[e^{s2\,\phi\,\hat{v}_{||}} \,(1 \mp 2\,\phi^2 \mp s2\,\phi\,\hat{v}_{||})\; Erfc(\phi + s\,\hat{v}_{||}) \Big] \tag{35}$$

and:

$$\phi = \frac{1}{2}\sqrt{\sigma^{\alpha'}_{||}}\,\hat{k}_\perp \tau\,, \qquad \tilde{v}_{||} = v_{||}/\sqrt{\sigma^{\alpha'}_{||}}$$

(this is essentially a re-shaped form of (32)). Notice that the correlation function is an *even* function with respect to time τ (as expected!).

One may therefore study the correlations $C_*(\tau)$ in terms of τ, $v_\perp, v_{||}$ *and* the field (i.e. Ω) and then integrate in τ to obtain the corresponding expressions for the diffusion coefficients (as functions of t, $v_\perp, v_{||}$, Ω).

Let us point out that all the above expressions are valid for a *multi-component* (α, $\alpha' \in \{e, i, ...\}$), *two-temperature* ($T_\perp$, $T_{||}$) plasma, just as they are. In the following, however, we shall assume that $\alpha = \alpha'$ and $T_\perp = T_{||}$, for the sake of simplicity.

7 Reduced form of the coefficients - variable scaling (one-component plasma)

Let us try to derive a non-dimensional form of the above coefficients.

In the following, we shall set $\alpha = \alpha'$ and $\sigma_\perp = \sigma_{||} = \sigma$ in the above formula.

7.1 Correlations

The integration variable $k_\perp$ in the previous paragraph can be rescaled to the non-dimensional variable:

$$x = \frac{\hat{k}_\perp}{k_D} = (1 + \frac{k_\perp^2}{k_D^2})^{1/2}$$

$(k_D \neq 0)$. Relation (34) now becomes:

$$C^{\alpha}_{\{\perp,\|\}}(\tau) \;=\; 4\,n\,e^4\,k_D \int_1^{x_{max}} dx\; e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\Omega\tau}{2}} \left(1-\frac{1}{x^2}\right)^{\{1,0\}} J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\Omega\tau}{2})\; F_{\{\perp,\|\}} \quad (36)$$

where $F = F(\phi(x,\tau), \tilde{v}_\|)$ is given by:

$$F^{\alpha'}_{\{\perp,\|\}} = \pm\sqrt{\pi}\,\phi\,e^{-\hat{v}_\|^2} + \frac{\pi}{4}\,e^{\phi^2} \sum_{s=+1,-1} \left[e^{s2\,\phi\,\hat{v}_\|}\,(1 \mp 2\,\phi^2 \mp s2\,\phi\,\hat{v}_\|)\,Erfc(\phi + s\,\hat{v}_\|) \right] \quad (37)$$

Its arguments are:

$$\phi = \frac{1}{\sqrt{2}}\,\omega_{p,\alpha}\,\tau\,x\,, \qquad \hat{v}_\| = \tilde{v}_\| \equiv v_\|/\sqrt{\sigma_\|} = v_\|/\sqrt{\sigma}$$

Also,

$$\tilde{v}_\perp \equiv v_\perp/\sqrt{\sigma_\perp} = v_\perp/\sqrt{\sigma}\,, \qquad \lambda = \sqrt{\sigma_\perp}\,\frac{k_D}{\Omega} = \cdots = \sqrt{2}\,\frac{\omega_p}{\Omega}$$

Note that

$$\phi = \frac{\lambda}{2}\,(\Omega\,\tau)\,x$$

Remember that $\sigma_\alpha = 2\,k_B\,T_\alpha/m_\alpha = 2\,v_{th,\alpha}^2$ is related to the thermal velocity (i.e. the temperature), $\Omega_\alpha = e_\alpha\,B/m_\alpha\,c$ is the cyclotron (gyroscopic) frequency, $k_D = \frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha}$ is the Debye wave-number and $\omega_{p,\alpha} = (\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha})^{1/2}$ is the plasma (Langmuir) frequency (so $\omega_p = \sqrt{\sigma\,k_D/2}$). Notice the interplay of collision and magnetic field scales through $\lambda \approx \frac{T_{gyro}}{T_{coll}} \equiv \frac{v_{thermal}}{v_{Alfven}}$

The correlations $C(\tau)$ are now expressed as a single definite integral in x from 1 to x_{max}. As a matter of fact, the integral diverges at an initial stage in $\tau \approx 0$, as one may readily check by setting $\tau = 0$ in the above formulae. Nevertheless, the integral converges everywhere outside an infinitesimal region close to $\tau = 0$. In fact, a thorough numerical study of the integrand actually reveals a rapidly convergent character; indeed, un upper cutoff of k_{max} equal to 5, has been checked and came out to be sufficient above, say typically, a time threshold τ_{min} equal to 0.01 gyration periods, as the value of the integral is thus preserved up to a precision of 10^{-4} at least. This divergence is actually due to the short distance (long wave-number) divergence of the interaction potential $V(r)$ (see that the divergence is already present if one sets $\tau = 0$ in the initial formula (25)), and therefore reflects the limitations imposed by the weak-coupling ('no-close-encounter') approximation.

In order to avoid this mathematical divergence, the upper integration limit k_{max} (cf. old formulae) can be taken, instead of infinity (i.e. $x_{max} = \infty$) to be equal, say, to the inverse collision parameter for a 90-degree deflection: $k_{max} = 3k_B T/e^2$ (cf. [5], p. 130); the latter leads to an upper value of the order of $x_{max} \approx \rho^{-3/2} \gg 1$:

$$x_{max} = (1 + k_{max}^2/k_D^2)^{1/2} = \ldots = (1 + \frac{9}{4\pi\rho^3})^{1/2} \approx 3/2\sqrt{\pi}\rho^{-3/2}$$

where $\rho = \frac{e^2 n^{1/3}}{k_B T} \ll 1$ is the plasma parameter.

Essentially, for a given set of parameter values (T, n, m_α, e_α and B), one has to determine the values of ω_p, Ω and then λ; the above formulae can then be studied as functions of τ (once the integration in x is carried out numerically), where all external parameters enter through ω_p and Ω.

7.2 Diffusion coefficients

Once the correlations $C_{\perp,\|}(\tau)$ are evaluated as a function of τ (or, rather, $\Omega\tau$), the final coefficients $D_{\perp,\angle,\|,\ldots}(t)$ are then defined as a definite integral in τ (from 0 to t [7]): see relation (24).

8 Numerical study of the coefficients

8.1 Typical parameter values

Choosing a set of typical values, i.e. a temperature of $T = 10\,KeV$ and a particle density of $n = 10^{14}\,cm^{-3} = 10^{20}\,m^{-3}$, we obtain a plasma frequency $\omega_{p,e} = 5.64 \cdot 10^{11}\,s^{-1}$ and a cyclotron (gyro-) frequency of:

$$\Omega_e = 1.76\,10^{11} \times B \qquad s^{-1}$$

(B is expressed in Tesla) implying a plasma parameter of $\rho = 2.5 \cdot 10^{-8} \ll 1$ as well as $x_{max} \approx 2 \cdot 10^{11} \gg 1$. The parameters in the previous paragraphs are thus given by:

$$\lambda = 4.5 \times B^{-1}$$

and

$$\phi = (2.25 \times B^{-1}) \times (\Omega\tau) \times x$$

(B is expressed in Tesla).

[7] All quantities are found to converge in the asymptotic limit $t \to \infty$.

8.2 Evolution in time

Let us study the behaviour in time of the force correlations and the diffusion coefficients, for a fixed set of parameter values (see above). Unless otherwise stated, we shall choose the magnetic field to be $B = 3\,T$ (so that $\lambda = 1.5$).

8.2.1 Force correlations $C_{\{\perp,\|\}}(\tau)$ vs. τ

The force correlations $C_\perp(\tau)$, $C_\|(\tau)$ given by (36), (37) are represented in figures 2a,b, 3a,b respectively, as functions of the *time*-integration variable τ (cf. (24)) (measured in gyration periods $\Omega\tau$), for different values of the magnitude of the magnetic field B ($\sim \Omega$). Correlations are found to decrease fast in time. In both cases, the magnetic field seems to enhance correlation, since the higher its magnitude B ($\sim \Omega$; cf (9)), the higher the value of both $C_{\{\perp,\|\}}(\tau)$; see figures 2a, 3a. Physically speaking, this fact reflects particle confinement by the magnetic field, since particles 'stick' to their helicoidal trajectory around the magnetic field lines and thus 'feel' each other for longer periods of time. Note that $C_\|(\tau)$ decreases *faster* than $C_\perp(\tau)$. Also notice the peaks appearing every gyration period in the latter (actually the signature of the magnetic field) which are absent from the former: they are smoothed out in a few gyration periods; compare figures 2b, 3b. Nevertheless, in both cases, particle interactions seem to be completely decorrelated after a few gyration periods.

Furthermore, by increasing particle velocity $v_\perp, v_\|$ we reduce force correlations *faster* : see figure 3a, b. Once again, this is quite expected, since the higher the velocity the *less* particles see each other while moving.

8.2.2 Diffusion coefficients $D_{\{\perp,\|,\cdot\}}(t)$ vs. t

We saw that the diffusion coefficients $D_\perp(t)$, $D_\angle(t)$, $D_\|(t)$ actually correspond to the surface under the curves $C_{\perp,\|}(\tau)$ vs. τ multiplying an oscillating function of time (e.g. definite integrals of $C_{\perp,\angle,\|}(\tau) \times \cos\Omega\tau$ in τ and so forth; cf. (24)). Therefore, their evolution in time, depicted in figures 5a, b, c was quite expected: they start from an initial zero-value and soon evolve towards a final asymptotic value which remains practically constant after a fews gyration periods. Notice the local "jumps" every cyclotron period - absent from $D_\|(t)$, since the $z-$direction ($\| \mathbf{B}$) yields no memory of the field - which are the consequence of the small peaks in the $\perp -$correlations; their influence decreases to null after a few gyration periods.

Remember that our diffusion coefficients $D_*(t)$ are related to the *inverse* of the time needed for relaxation towards equilibrium (see, for instance, chapter 3 in [10]). Therefore, the difference in magnitude between the coefficients (for the same set of physical parameters, see e.g. fig. 5) seems to point out that collisions along the $z-(\| -)$direction are more efficient, since they lead to relaxation faster than the ones on the $xy-(\perp -)$plane.

Figure 6 shows $D_{\perp}(t)$ for different values of the field (i.e. the cyclotron frequency Ω). We can see that the asymptotic value of the diffusion coefficients $D(t)$ (as $t \to \infty$) depends rather dramatically on the magnetic field: in fact, the higher the field, the higher the final value $D_{\perp}(\infty)$. [8]. We see that the magnetic field *favours* thermalization (i.e. relaxation of the distribution function towards a maxwellian state), since it increases the value of D_*, thus reducing relaxation times. Once more, this seems to agree with physical intuition (the more 'confined' the particles, the more they influence each other and the more efficient collisions are for relaxation); this is nevertheless in contradiction with what has often been suggested (yet never rigorously studied) in the past (the influence of the magnetic field on the collision term has always either been under-estimated [11] or deliberately neglected [12] when discussing the mathematical properties of collision terms as related to the physical - transport - behaviour of plasma).

9 Conclusion

In conclusion, we have obtained a kinetic equation for a test-particle weakly interacting with a large heat-bath in thermal equilibrium, focusing on the explicit dependence of the collision term on the external force field, as well as on the phase-space parameters $\mathbf{x}, \mathbf{v}$, in general.

Furthermore, considering magnetized plasma, we have obtained new expressions for transport coefficients in terms of physical parameters (density, temperature, magnetic field magnitude) and velocity components $v_{\perp}, v_{\parallel}$. We have pointed out that the magnetic field plays a significant role in the value of the diffusion coefficients (and the resulting transport properties) of plasma.

[8]The same feature is present in $D_{\angle}(t)$, and little less in $D_{\parallel}(t)$

Appendix

We saw that the $D_\perp$, $D_\angle$ and $D_\parallel$ coefficients appearing in the text are given by the expressions (24), (25) i.e.

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} \frac{n_{\alpha'}}{m^2} (2\pi)^3 \int_0^t d\tau \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2 + k_y^2)\, \cos \Omega^\alpha \tau \\ (-s^\alpha)\, \frac{1}{2}\,(k_x^2 + k_y^2)\, \sin \Omega^\alpha \tau \\ k_z^2 \end{array} \right\} \tag{38}$$

where, once again, v_i ($v_{1,i}$), $i = 1, 2, 3$ denote the velocity coordinates of the test-particle (reservoir-particle) of species $\alpha_\Sigma \equiv \alpha$ ($\alpha_1 = \alpha' \in \{e, i, ...\}$) respectively (the α index will be dropped where obvious) and $\tilde{V}_k$ stands for the Fourier transform of the (long-range) interaction potential, defined in (26).

Remember that the dynamical friction vector $\mathcal{F}$ is also defined through the above coefficients (see in §5.2).

A Eliminating integrals... - from (24) to (31)

A.1 The velocity integration $\int d^3\mathbf{v_1}$

The v_1- integration in (38) can be carried out at this stage, once one assumes an analytic form for the equilibrium reservoir distribution function (df) $\phi_{eq}(\mathbf{v_1})$ [9]. Here, it will be explicitly taken to be a Maxwellian of the form (30) i.e.:

$$\phi_{Max}^{\alpha'}(v_1) = \prod_{i=1,2,3} \phi_0^{(i,\alpha')}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \tag{39}$$

($\phi_0^{(i)} = (\frac{m_{\alpha'}}{2\pi T_{\alpha'}^{(i)}})^{1/2} \equiv \frac{1}{(2\pi)^{1/2}\, v_{i,th}^{\alpha'}} \equiv \frac{1}{\sqrt{\pi \sigma_i^{\alpha'}}}$; $\quad \sigma_i^{\alpha'} \equiv 2\, {v_{i,th}^{\alpha'}}^2 \equiv \frac{2T_i^{\alpha'}}{m_{\alpha'}} \quad \forall i \in \{1, 2, 3\} \equiv \{x, y, z\}$; let us assume here that $\sigma_1^{\alpha'} = \sigma_2^{\alpha'} = \sigma_\perp$, $\sigma_3^{\alpha'} = \sigma_\parallel$; the summation over particle species α' is omitted in the following).

By substituting from (39) into (38) we obtain:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\parallel \end{array} \right\} =$$

$$= \frac{n_{\alpha'}}{m^2} (2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \int d\mathbf{v_1} \prod_i \left[\phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \right] e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\, \tilde{V}_k^2\, e^{ik_n N_{nm}(\tau) v_m}$$

[9] Remember that, actually the homogeneous equilibrium df can be *any* function of $\{v_\perp, v_\parallel\}$.

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array} \right\}$$

$$= \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \prod_i \Big[\int dv_{1,i}\, \phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}}\, e^{-ik_n N_{ni}^{\alpha'}(\tau) v_{1,i}} \Big]\, \tilde{V}_k^2\, e^{ik_n N_{nm}(\tau) v_m}$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array} \right\}$$

$$= \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \prod_i \Big[\int dv_{1,i}\, \phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}}\, e^{-ip_i^{\alpha'} v_{1,i}} \Big]\, \tilde{V}_k^2\, e^{ip_m v_m}$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array} \right\} \qquad (40)$$

where we defined: $k_n N_{nm}^{\beta} \equiv p_m^{\beta}$ (β is either α or α'; a summation over m is understood where appropriate) [10]. Note that:

$$p_m^{\alpha'}(\mathbf{k};\tau) = \sum_{n=1}^{3} k_n N_{nm}^{\alpha'}(\tau) = \sum_{n=1}^{3} k_n \int_0^{\tau} R_{nm}^{\alpha'}(t')dt' \qquad (41)$$

The definite integrals in brackets are of a well-known general form, which can be shown to yield:

$$\int_{-\infty}^{\infty} e^{-iAx}\, e^{-Bx^2}\, dx \;=\; e^{-\frac{A^2}{4B}} \int_{-\infty}^{\infty} e^{-B(x+i\frac{A}{2B})^2}\, dx \;=\; e^{-A^2/4B}\, \frac{\sqrt{\pi}}{\sqrt{B}}$$

i.e.

$$\int_{-\infty}^{\infty} e^{-ip_i^{\alpha'} v_{1,i}}\, e^{-\frac{v_{1,i}^2}{\sigma_i^{\alpha'}}}\, dv_{1,i} \;=\; \sqrt{\pi\sigma_i^{\alpha'}}\; e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4}$$

Therefore, the above expression (40) becomes:

$$\left\{ \begin{array}{c} D_{\perp} \\ D_{\angle} \\ D_{\parallel} \end{array} \right\} \;=\; \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^{t\to\infty} d\tau \int d\mathbf{k} \prod_i \Big[\phi_0^{(i)}\, \sqrt{\pi\sigma_i^{\alpha'}}\, e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4} \Big]\, \tilde{V}_k^2\, e^{ip_m^{\alpha} v_m}$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array} \right\}$$

$$= \;\frac{n_{\alpha}}{m^2}(2\pi)^3 \int_0^{t\to\infty} d\tau \int d\mathbf{k} \prod_i \Big[e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4}\, e^{ip_i^{\alpha} v_i} \Big]\, \tilde{V}_k^2$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array} \right\} \qquad (42)$$

[10] In the limit $\Omega \to 0$ we have: $p_m \to k_m \tau$.

(remember the definition of $\phi_0^{(i)}$). A summation over α' is understood; once more, let us remind that α (α') denotes the test- (1^R-) particle species respectively.

From now on, we will restrict ourselves to the *single-species* (electron) plasma case (species indexes are now dropped for simplicity). Therefore, if (and *only if*) $\alpha_\Sigma = \alpha_1$ ($\alpha = \alpha'$), one may write:

$$-\sigma_j\, p_j^2/4 + i p_j v_j \;=\; -\frac{1}{4}\,\sigma_j \left[(p_j^2 - i\frac{2v_j}{\sigma_j})^2 + \frac{4v_j^2}{\sigma_j^2}\right] \;=\; -\frac{1}{4}\,\sigma_j\,(p_j^2 - i\frac{2v_j}{\sigma_j})^2 - \frac{v_j^2}{\sigma_j}$$

Thus, setting:

$$q_m(\mathbf{k};\tau;v) \;=\; p_m - i\frac{2v_m}{\sigma_m} \;\equiv\; \sum_{n=1}^{3} k_n N_{nm}(\tau) - i\frac{2v_m}{\sigma_m} \tag{43}$$

(cf. (41)[11]) we have:

$$\begin{aligned}
\left\{\begin{array}{c} D_\perp \\ D_\angle \\ D_\parallel \end{array}\right\} \;&=\; \frac{n_\alpha}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k}\, \prod_i \Big(e^{-\sigma_i\, q_i^2/4}\, e^{-v_i^2/\sigma_i} \Big)\, \tilde{V}_k^2 \\
&\qquad\qquad \left\{\begin{array}{c} \frac{1}{2}\,(k_x^2 + k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2 + k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array}\right\} \\
&=\; \frac{n_\alpha}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k}\; e^{-\sum_i \sigma_i\, q_i^2/4}\, e^{-\sum_i v_i^2/\sigma_i}\, \tilde{V}_k^2 \\
&\qquad\qquad \left\{\begin{array}{c} \frac{1}{2}\,(k_x^2 + k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2 + k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array}\right\} \\
&=\; \frac{n_\alpha}{m^2}(2\pi)^3\; e^{-v_i^2/\sigma_i} \int_0^t d\tau \int d\mathbf{k}\; e^{-\sigma_i\, q_i^2/4} \tilde{V}_k^2 \\
&\qquad\qquad \left\{\begin{array}{c} \frac{1}{2}\,(k_x^2 + k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2 + k_y^2)\,\sin\Omega\tau \\ k_z^2 \end{array}\right\}
\end{aligned}$$

i.e.

$$\begin{aligned}
\left\{\begin{array}{c} D_\perp \\ D_\angle \end{array}\right\} \;&=\; \frac{n_\alpha}{m^2}(2\pi)^3\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\; e^{-\sigma_m\, q_m^2/4}\, k_\perp^2\, \tilde{V}_k^2 \right) \left\{\begin{array}{c} \frac{1}{2}\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,\sin\Omega\tau \end{array}\right\} \\
D_\parallel \;&=\; \frac{n_\alpha}{m^2}(2\pi)^3\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\; e^{-\sigma_m\, q_m^2/4}\, k_\parallel^2\, \tilde{V}_k^2 \right)
\end{aligned} \tag{44}$$

(once more, a summation over m is understood where appropriate; thus $e^{-v_m^2/\sigma_m} = e^{-v_\perp^2/\sigma_\perp}\, e^{-v_\parallel^2/\sigma_\parallel}$). Note that the integrals within parenthesis in the last relations present

[11] $[q_m] = [p_m] = 1/[v] = L^{-1}T^1$; in the limit $\Omega \to 0$ we have: $q_m \to k_m\tau - i\frac{2u_m}{\sigma}$.

a cylindrical symmetry due to the existence of the $\underline{\underline{N}}$ matrix in it (cf. (43)); in the absence of an external magnetic (or *any*) field, it reduces to a *spherically* symmetric form, as $\mathbf{N}(\tau) \to \tau \mathbf{I}$ [12].

For convenience we shall define the quantities in parenthesis in the latter relation as [13]:

$$I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}} \equiv \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\; \tilde{V}_k^2 \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\} \tag{45}$$

As a matter of fact, relation (31) holds as it stands for *any* particular form of $V(r)$. However, this is not the most general form in terms of $V(r)$ (still not specified, that is). Remember that, in principle, $V = V(|\mathbf{r}|) = V(r)$ implies that $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$ ($= V(k_\perp^2 + k_\|^2)$ in a cylindrical-symmetric problem). Therefore the Fourier transform of the interaction potential does *not* depend on the angle α ($\equiv (\hat{x}, \mathbf{k}_\perp)$) and the corresponding angle integration inside the triple integral $I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}}$ can be performed straightaway, giving a final double integration in $k_\perp$, $k_\|$. This is precisely what we will do in the next paragraph.

A.2 The Fourier-integral(s) $\int d^3\mathbf{k}... = I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}}$

Let us remark that q^2 can be expressed in an elegant manner as [14]:

$$q^2 = \sum_m q_m^2 = \sum_m (p_m - i\frac{2v_m}{\sigma_m})^2 = \sum_m (p_m^2 - 4\frac{v_m^2}{\sigma_m^2} - 4i\frac{p_m v_m}{\sigma_m}) \equiv p^2 - 4\frac{v^2}{\sigma^2} - 4i\frac{\mathbf{p}\cdot\mathbf{v}}{\sigma}$$

and can be decomposed into $\{x, y\} \equiv \perp$ and $\{z\} \equiv \|$- parts:

$$q^2 = (p_x - i\frac{2v_x}{\sigma_x})^2 + (p_y - i\frac{2v_y}{\sigma_y})^2 + (p_z - i\frac{2v_z}{\sigma_z})^2 \equiv q_\perp^2 + q_\|^2$$

---

[12] If $\Omega \to 0$, the $\mathbf{k}-$ integral (cf. (31)) in $D_{11} = D_{22} = D_{33}$ will be:

$$D_{jj} = \cdots \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\, \tilde{V}_k^2\, k_j^2 = \cdots \int_{-\infty}^{\infty} dk_i\; e^{-\sigma\, q_i^2/4} \int_{-\infty}^{\infty} dk_l\; e^{-\sigma\, q_l^2/4} \int_{-\infty}^{\infty} dk_j\; e^{-\sigma\, q_j^2/4}\, \tilde{V}_k^2\, k_j^2$$

$(i \neq l \neq j = 1, 2, 3)$

[13] e.g. for a Coulomb potential $V(r) = e^2/r$, (45) reduces to:

$$I_{Coulomb}{}_{\mathbf{k}^{(3)}}^{\{\perp,\|\}} \equiv \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\, (\frac{e^2}{2\pi^2})^2\, \frac{1}{(\sum_{i=1}^3 k_i^2)^2} \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\}$$

[14] In the limit $\Omega \to 0$ we have:

$$q^2 \to \sum_m q_m^2 = \sum_m (k_m\tau - i\frac{2v_m}{\sigma_m})^2 = \sum_m (k_m^2\tau^2 - 4\frac{v_m^2}{\sigma_m^2} - 4i\frac{k_m v_m \tau}{\sigma_m}) \equiv k^2\tau^2 - 4\frac{v^2}{\sigma_m^2} - 4i\frac{\tau(\mathbf{k}\cdot\mathbf{v})}{\sigma_m}$$

By making use of the explicit definition of the $\mathbf{p}$ vector above, as well as that of the $\underline{\underline{N}}$ matrix in it (cf. paper (4)), we find

$$q_{\|}^2 = (k_{\|}\tau - i\frac{2v_{\|}}{\sigma_{\|}})^2 \tag{46}$$

$$\begin{aligned} q_{\perp}^2 &= p_{\perp}^2 - 4\frac{v_{\perp}^2}{\sigma_{\perp}^2} - 4i\frac{\mathbf{p}_{\perp}\cdot\mathbf{v}_{\perp}}{\sigma_{\perp}} \\ &= \Omega^{-2}\left(k_x^2 + k_y^2\right) 2\,(1-\cos\Omega\tau) - 4\frac{v_x^2+v_y^2}{\sigma_{\perp}^2} \\ &\quad - i\frac{2}{\sigma_{\perp}}\Omega^{-1}\Big[(k_x v_x + k_y v_y)\sin\Omega\tau - s(1-\cos\Omega\tau)(k_y v_x - k_x v_y)\Big] \end{aligned} \tag{47}$$

in cartesian coordinates. In cylindrical coordinates, one may set:

$$\mathbf{k} = \begin{pmatrix} k_{\perp}\cos\alpha \\ k_{\perp}\sin\alpha \\ k_{\|} \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_{\perp}\cos\theta \\ v_{\perp}\sin\theta \\ v_{\|} \end{pmatrix}$$

so relation (47) takes the form:

$$\begin{aligned} q_{\perp}^2 &= 2\frac{k_{\perp}^2}{\Omega^2}(1-\cos\Omega\tau) - 4\frac{v_{\perp}^2}{\sigma_{\perp}^2} - i\frac{4}{\sigma_{\perp}}\frac{k_{\perp}v_{\perp}}{\Omega}\Big[\sin(\theta-\alpha) - \sin(\theta-\alpha-s\Omega\tau)\Big] \\ &= 4\frac{k_{\perp}^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2} - 4\frac{v_{\perp}^2}{\sigma_{\perp}^2} - i\frac{4}{\sigma_{\perp}}\frac{k_{\perp}v_{\perp}}{\Omega}\,2\,\sin(s\frac{\Omega\tau}{2})\cos(\theta-\alpha-s\frac{\Omega\tau}{2}) \end{aligned} \tag{48}$$

and the $\mathbf{k}$- integral(s) in parenthesis in eq. (31) become:

$$\int d\mathbf{k}\; e^{-\sigma\, q^2/4}\,\tilde{V}_k^2\, k_i\, k_l =$$

$$\int_0^{\infty} dk_{\perp}\, k_{\perp} \int_{-\infty}^{\infty} dk_{\|} \int_0^{2\pi} d\alpha\; \tilde{V}_k^2 \begin{pmatrix} k_{\perp}\cos\alpha \\ k_{\perp}\sin\alpha \\ k_{\|} \end{pmatrix}_i \begin{pmatrix} k_{\perp}\cos\alpha \\ k_{\perp}\sin\alpha \\ k_{\|} \end{pmatrix}_j e^{-\sigma_{\perp}\, q_{\perp}^2/4}\, e^{-\sigma_{\|}\, q_{\|}^2/4}$$

that is

$$\begin{aligned} I_{\mathbf{k}^{(3)}}^{(\perp,\|)} &= \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\,\tilde{V}_k^2 \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\} = \\ &\qquad \int_0^{\infty} dk_{\perp}\, k_{\perp} \int_{-\infty}^{\infty} dk_{\|} \int_0^{2\pi} d\alpha\, e^{-\sigma_{\perp}\, q_{\perp}^2/4}\, e^{-\sigma_{\|}\, q_{\|}^2/4}\,\tilde{V}_k^2 \left\{ \begin{array}{c} k_{\perp}^2 \\ k_{\|}^2 \end{array} \right\} \\ &= \int_0^{\infty} dk_{\perp} \left\{ \begin{array}{c} k_{\perp}^3 \\ k_{\perp} \end{array} \right\} \left(\int_0^{2\pi} d\alpha\; e^{-\sigma_{\perp}\, q_{\perp}^2/4}\right) \left[\int_{-\infty}^{\infty} dk_{\|}\; e^{-\sigma_{\|}\, q_{\|}^2/4}\,\tilde{V}^2_{k\equiv(k_{\perp}^2+k_{\|}^2)^{1/2}} \left\{ \begin{array}{c} 1 \\ k_{\|}^2 \end{array} \right\}\right] \\ &\equiv \int_0^{\infty} dk_{\perp} \left\{ \begin{array}{c} k_{\perp}^3 \\ k_{\perp} \end{array} \right\} I_{\alpha}\; I_{k_{\|}}^{\{\perp,\|\}} \end{aligned} \tag{49}$$

IMPORTANT: Note the appearance on the magnetic field *only* in the first (angle-) integration I_{α}. Furthermore, the exact form of the interaction potential *only* enters the (rest of the) $k-$ integral.

The integral I_α will be analytically evaluated in the next paragraph. The remaining part of the $\mathbf{k}-$ integral has to be evaluated once a specific form of interaction potential is chosen [15].

A.2.1 The α- integration

The α- integral in parenthesis in (49), say I_α, can now be evaluated analytically. For convenience, we can use expression (48) which, once substituted into I_α, yields:

$$
\begin{aligned}
I_\alpha &= \int_0^{2\pi} d\alpha \, e^{-\sigma_\perp q_\perp^2/4} \\
&= \int_0^{2\pi} d\alpha \, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \, e^{i \frac{k_\perp v_\perp}{\Omega} 2 \sin(s \frac{\Omega\tau}{2}) \cos(\theta - \alpha - s\frac{\Omega\tau}{2})} \\
&= e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \int_0^{2\pi} d\alpha \, e^{i 2 \frac{k_\perp v_\perp}{\Omega} \sin(s\frac{\Omega\tau}{2}) \sin(\frac{\pi}{2} - \theta + \alpha + s\frac{\Omega\tau}{2})}
\end{aligned}
$$

We may now use the Bessel function identity:

$$
e^{i\, x \sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\,\phi} \qquad \forall x, \phi \in \Re \tag{50}
$$

to obtain [16]:

$$
I_\alpha = (2\,\pi)\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \, J_O(2 \frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}) \tag{51}
$$

where $J_O(x)$ is the zeroth-order Bessel function of the first kind (actually an *even* function of x).

Remark: In the limit $\Omega \to \infty$: $I_\alpha \to 2\,\pi\, e^{-v_\perp^2/\sigma_\perp}$ [17].

A.2.2 Final form of the coefficients for an arbitrary potential V(r)

By substituting from (51) into (49) and then back into (31) we obtain:

$$
\begin{aligned}
\left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} &= \frac{n_\alpha}{m^2} (2\pi)^4 \, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp \, k_\perp^3 \Big[\int_{-\infty}^{\infty} dk_\parallel \; e^{-\sigma_\parallel q_\parallel^2/4} \, \tilde{V}_k^2 \Big] \\
& \quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, J_O(2 \frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}) \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega\tau \\ (-s)\, \frac{1}{2} \sin \Omega\tau \end{array} \right\} \\
D_\parallel &= \frac{n_\alpha}{m^2} (2\pi)^4 \, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp \, k_\perp \Big[\int_{-\infty}^{\infty} dk_\parallel \, k_\parallel^2 \; e^{-\sigma_\parallel q_\parallel^2/4} \, \tilde{V}_k^2 \Big] \\
& \quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, J_O(2 \frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2})
\end{aligned} \tag{52}
$$

---

[15] e.g. $I_{k_\parallel}^{\{\perp,\parallel\}}$ first and then $k_\perp-$ or vice versa.

[16] Note that the integration has invoked a $\delta_{n,0}^{Kr}$ Kronecker symbol, thus setting n to zero i.e. $\int_0^{2\pi} e^{in\alpha} d\alpha = 2\pi\, \delta_{n,0}^{Kr}$.

[17] In the limit $\Omega \to 0$ we have:

$$
I_\alpha \to (2\pi)\, e^{-\sigma_\perp k_\parallel^2 \tau^2/2} \, e^{v_\perp^2/\sigma_\perp} \, J_0(k_\perp v_\perp \tau)
$$

Let us point out that $k_{||}$ only appears inside the brackets (so the $k_{||}-$ integration may be easier to carry out first).

Relation (52) is exactly (31) in the text (cf. (29)); it is the most general form of the diffucion coefficients for a test-particle in a maxwellian background, interacting through a central potential $V(r)$ (which remains to be specified).

B Explicit calculation for a Debye potential - from (31) to (33)

Let us now explicitly assume that the (long-range) interaction potential $V(r)$ is a Debye potential:

$$V(r) = e^2 \frac{e^{-k_D\, r}}{r}\,, \qquad \tilde{V}_k = \frac{e^2}{2\pi^2}\frac{1}{k^2+k_D^2} = \frac{\tilde{V}_0}{k_\perp^2 + k_{||}^2 + k_D^2}$$

($\lambda_D = k_D^{-1} = (\frac{4\pi e^2 n}{k_B T})^{-1}$ is the Debye length [1]). ($V_0 = e_\alpha\, e_{\alpha'}$, $\tilde{V}_0 = \frac{e_\alpha\, e_{\alpha'}}{2\pi^2}$ are appropriate constant quantities

Eq. (52) directly becomes:

$$\begin{aligned}
\left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} &= \frac{4\, n_\alpha\, e_\alpha^2}{m_\alpha^2}\, e^{-v_{||}^2/\sigma_{||}} \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp^3 \Big[\int_{-\infty}^\infty dk_{||}\; e^{-\sigma_{||}\, q_{||}^2/4}\, \frac{1}{k^4} \Big] \\
&\quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O\big(2\frac{k_\perp v_\perp}{\Omega}\, \sin\frac{\Omega\tau}{2}\big) \left\{ \begin{array}{c} \frac{1}{2}\, \cos\Omega\tau \\ (-s)\, \frac{1}{2}\, \sin\Omega\tau \end{array} \right\} \\
D_{||} &= \frac{4\, n_\alpha\, e_\alpha^2}{m_\alpha^2}\, e^{-v_{||}^2/\sigma_{||}} \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp \Big[\int_{-\infty}^\infty dk_{||}\, k_{||}^2\; e^{-\sigma_{||}\, q_{||}^2/4}\, \frac{1}{k^4} \Big] \\
&\quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O\big(2\frac{k_\perp v_\perp}{\Omega}\, \sin\frac{\Omega\tau}{2}\big) \qquad (53)
\end{aligned}$$

(obviously $k^2 = \sum k_i^2 = k_x^2 + k_y^2 + k_z^2 \equiv k_\perp^2 + k_{||}^2$).

B.1 The $k_{||}$- integration

We may now attempt to evaluate the $k_{||}$- integral (in brackets in (53)), say $I_{k_{||}}^{(\perp,||)}$ [18]. Substituting from expression (46) we obtain:

$$\begin{aligned}
I_{k_{||}}^{\{\perp,||\}} &= \int_{-\infty}^\infty dk_{||}\; e^{-\sigma_{||}\, q_{||}^2/4}\, \frac{1}{(k_\perp^2+k_{||}^2)^2} \left\{ \begin{array}{c} 1 \\ k_{||}^2 \end{array} \right\} \\
&= \int_{-\infty}^\infty dk_{||}\; e^{-\sigma_{||}\left(k_{||}\tau - i\frac{2v_{||}}{\sigma_{||}}\right)^2/4}\, \frac{1}{(k_\perp^2+k_{||}^2)^2} \left\{ \begin{array}{c} 1 \\ k_{||}^2 \end{array} \right\}
\end{aligned}$$

[18] Note the absence of the field i.e. Ω in the $||$ $-$ part of the formulae; thus, the results of this paragraph are valid as they stand in the free-of-field (free motion) case, as well.

$$= \quad e^{v_{\|}^2/\sigma_{\|}} \int_{-\infty}^{\infty} dk_{\|}\, e^{-\sigma_{\|}\, k_{\|}^2 \tau^2/4}\, \frac{\cos(k_{\|}\, v_{\|}\, \tau)}{(k_\perp^2 + k_{\|}^2)^2} \left\{ \begin{array}{c} 1 \\ k_{\|}^2 \end{array} \right\} \tag{54}$$

(note that the imaginary part of the integral cancels for reasons of symmetry, as the integrand in it is an *odd* function of $k_{\|}$). The integrals in (54) can be found to yield [19]:

$$I_{k_{\|}}^{\{\perp,\|\}} = \frac{1}{\left\{ \begin{array}{c} \hat{k}_\perp^3 \\ \hat{k}_\perp \end{array} \right\}} \Big\{ \pm \frac{\sqrt{\pi}}{2}\, \sqrt{\sigma_{\|}}\, \hat{k}_\perp \tau + \frac{\pi}{4} e^{v_{\|}^2/\sigma_{\|}}\, e^{\sigma_{\|} \hat{k}_\perp^2 \tau^2/4}$$

$$\sum_{s=+1,-1} \Big[e^{s \hat{k}_\perp v_{\|} \tau}\, (1 \mp \sigma_{\|} \hat{k}_\perp^2 \tau^2/2 \mp s \hat{k}_\perp v_{\|} \tau)\, Erfc(\frac{1}{2} \sqrt{\sigma_{\|}} \hat{k}_\perp \tau + s \frac{v_{\|}}{\sqrt{\sigma_{\|}}}) \Big] \Big\} \tag{55}$$

where the upper (lower) sign holds for $I^{(\perp)}$ ($I^{(\|)}$) and

$$\hat{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$$

. The complementary error function $Erfc(x)$ is defined as:

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Note that the integrals $I_{k_{\|}}^{(\perp,\|)}$:
(i) converge to zero at both $k_\perp \to \infty$ and $\tau \to \infty$, as - more or less - expected.
(ii) give a *finite* limit at $\tau \to 0$ [20] :

$$\lim_{\tau \to 0} I_{k_{\|}}^{\{\perp,\|\}} = \frac{\pi}{2} e^{v_{\|}^2/\sigma_{\|}}$$

and diverge precisely as $k_\perp^{\{3,1\}}$, respectively, at $k_\perp \to 0$:

$$\lim_{k_\perp \to 0} (k_\perp^{\{3,1\}} I_{k_{\|}}^{\{\perp,\|\}}) = \frac{\pi}{2} e^{v_{\|}^2/\sigma_{\|}}$$

B.2 Coefficients (final form of ...)

Expressions (31) for the coefficients (actually functions of $\{v_\perp, v_{\|}, t;\, \sigma_\perp, \sigma_{\|}, \Omega\}$) thus finally yield:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \end{array} \right\} \\ D_{\|} \end{array} \right\} = \frac{n_{\alpha'}}{m_\alpha^2} 4 e_\alpha^2 e_{\alpha'}^2 \int_0^t d\tau \int_0^\infty dk_\perp\, e^{-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'} \tau}{2}}\, J_O(2 \frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2})$$

$$\Big(1 - \frac{k_D^2}{k_D^2 + k_\perp^2}\Big)^{\{3/2,1/2\}} \left\{ \begin{array}{c} \{\, F_\perp^{\alpha'} \,\} \\ F_{\|}^{\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_\alpha \tau \\ \frac{-s}{2} \sin \Omega_\alpha \tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{56}$$

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[19]Details will be available in [3]; they were omitted here, for the sake of brevity in presentation.
[20]Note that $Erfc(x) + Erfc(-x) = 2$

where the functions $F = F^{\alpha'}_{\{\perp,\|\}}(k_\perp, v_\|, \tau;\, \sigma^{\alpha'}_\|)$ are given by:

$$F^{\alpha'}_{\{\perp,\|\}} = \pm\frac{\sqrt{\pi}}{2}\,\sqrt{\sigma^{\alpha'}_\|}\,\hat{k}_\perp\,\tau\,e^{-v_\|^2/\sigma^{\alpha'}_\|}$$
$$+\frac{\pi}{4}\,e^{\sigma^{\alpha'}_\|\hat{k}_\perp^2\tau^2/4}\sum_{s=+1,-1}\Big[e^{s\hat{k}_\perp v_\|\tau}\,(1\mp\sigma^{\alpha'}_\|\hat{k}_\perp^2\tau^2/2\mp s\hat{k}_\perp v_\|\tau)\,Erfc(\frac{1}{2}\sqrt{\sigma^{\alpha'}_\|}\hat{k}_\perp\tau+s\frac{v_\|}{\sqrt{\sigma^{\alpha'}_\|}})\Big] \quad (57)$$

and

$$\hat{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$$

- the upper (lower) signs corresponding to the $\perp$ ($\|$)- parts respectively. $Erfc(x)$ is the *complementary* error function:

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

(the subscript α, denoting the t.p. species in Ω, m, σ was dropped).

These are precisely relations (33) and (32) in the text.

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(a) **(b)**

Figure 2

The *perpendicular* (interaction) force-correlation function $C_{\perp}\left(\tau; v_{\perp}, v_{//}; B\right)$ as a function of time (scaled over a cyclotron period T). In ascending order, the magnitude of the magnetic field is set to B = *1, 2, 3* Tesla respectively. Both velocity components $v_{\perp}$ and $v_{//}$ are set to be equal to $v_{th} = \left(T / m\right)^{1/2}$. In (a) $C_{\perp}$ can be seen to decrease fast in time, still bearing a "tail" of gradually smoothed out peaks every gyration period (a signature of the magnetic field); see (b).

(a) **(b)**

Figure 3

The *parallel* (interaction) force-correlation function $C_{//}\left(\tau; v_{\perp}, v_{//}; B\right)$ as a function of time (scaled over a cyclotron period T). In ascending order, the magnitude of the magnetic field is set to B = *1, 2, 3* Tesla respectively. Both velocity components $v_{\perp}$ and $v_{//}$ are set to be equal to $v_{th} = \left(T / m\right)^{1/2}$. In (a) $C_{//}$ can be seen to decrease in time very fast (faster than $C_{\perp}$); note the absence of "tail" as in fig. 2b (since motion along the magnetic field is essentially "free"); see (b).

Figure 4

$C_{\perp}\left(\tau; v_{\perp}, v_{//}; B\right)$ as a function of time (scaled over a cyclotron period T). In descending order, the magnitude of the perpendicular velocity component $v_{\perp}$ is set to $v_{th} = (T/m)^{1/2}$ and $1.5\, v_{th}$ respectively. $v_{//}$ is set to be equal to v_{th} . and the magnetic field is set to $B = 3$ Tesla $C_{\perp}$ can be seen to decrease faster in time as $v_{\perp}$ increases.

Figure 5

In ascending order: diffusion coefficients $D_{\angle}$, $D_{\perp}$, $D_{//}$ (functions of $\{t; v_{\perp}, v_{//}; B\}$) represented against time t (scaled over a cyclotron period T). Both velocity components $v_{\perp}$ and $v_{//}$ are set to be equal to $v_{th} = (T/m)^{1/2}$. and $B = 3$ T.

Figure 6

In ascending order: diffusion coefficient $D_\perp$ against time t (scaled over T) for values of B = *1, 2, 3* T respectively. Both velocity components $v_\perp$ and $v_{//}$ are set to be equal to $v_{th} = (T/m)^{1/2}$. The upper curve (B = *3* T) corresponds to the middle curve in fig. 5.

Figure 7

(Long time) behaviour of the diffusion coefficient $D_\perp$ against t (scaled over *T*) for B = *3* T (cf. fig 6). Both $v_\perp$ and $v_{//}$ are set to be equal to v_{th}.