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Kinetic Theory for a Test-Particle Weakly Coupled to a Heat Bath. Application to Magnetized Plasmas

by

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Στους γονείς μου, Μαρία και Μιχάλη
με ευγνωμοσύνη

Sketch of a foreword…

Caminante, no hay camino.
Se hace camino al handar…
Antonio Machado*

This text is the result of personal study which lasted for a few years and involved a good deal of patient work and often tedious analytical manipulations. The process was not always easy. It was personally conceived as a long venture through the forest of scientific thought, tasting the fruit of knowledge trees at random and trying to find a way through by trying one direction or another. Definitely, the (vague) knowledge of the destination did not always imply knowledge of the exact path to follow, so finding my way around would have never been possible without the support of people met on the way, providing support, guidance, ideas or plainly inspiration (unconsciously sometimes). A few words are in row, as a least expression of gratitude to those who helped me so far.

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* *Wanderer, there’s no path. The path is made by walking...*
Antonio Machado (Spanish poet).

teaching staff of the team have proved excellent colleagues and friends and have contributed to a sane working atmosphere and teaching framework.

This effort would have never given its fruit without the continuous support from friends and colleagues, throughout these years. I am enormously thankful to all the colleagues at the ULB for numerous discussions, of either scientific or philosophical, literary, political, cultural content. I grasp the opportunity to mention Pavlos Akritas, Glenn Barnich, Vasilis Basios, Olivier Debliquy, Laurent Houart, Giorgos Kalogiannakis, Costas Karamanos, Stéphane Louies, omitting (but *not* forgetting) quite a few. Mustapha. Tlidi was a tireless source of technical and friendly advice. Kallia Petraki conveyed technical skills on computing and lots more. From the outside world, Ingmar Sandberg has always been a dear friend and a good colleague; Olivier Agullo, Thierry Dauxois and Donal McKernan have helped a lot with advice and hospitality.

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The urge is great to end this note by dedicating the thesis to the memory of Stephanos Pnevmatikos, who tragically passed away a few years ago. His charismatic qualities as a science teacher and personal tutor are unforgettable. He has succeeded in convincing me to go on, at some critical point of my "trajectory" when I was admittedly not quite sure I wanted to. I feel this work is somehow *his*, as well…

I. K.

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Figures 2.1 was adapted from reference [22]; figures 2.2-3 were adapted from [50].

Contents

Chapter 1

Introduction

Summary

We present a brief overview of kinetic theory for interacting particles, giving emphasis on a test-particle problem as applied in an electrostatic plasma. The motivation of the present study is then exposed. Arguments are forwarded in two directions. First, a collision term should a priori explicitly depend on an external force field, if one is present. Second, the existence of distribution function inhomogeneities should be taken into account. Both aspects should be considered in a correct collision term, which should be properly derived from first principles and not obtained by simple extrapolation from the homogeneous case. Finally, the content of forthcoming chapters is briefly outlined.

With classical thermodynamics,
one can calculate almost everything crudely;
with kinetic theory,
one can calculate fewer things, but more accurately;
and with statistical mechanics,
one can calculate almost nothing exactly.

`Eugene Wigner`
in *A Critical Review of Thermodynamics*

The aim of this chapter is to provide the general background for forthcoming chapters, by briefly presenting the framework of plasma kinetic theory.

1.1 Kinetic theory - prerequisites

The description of the macroscopic behaviour of matter has been a long-standing problem. Since the original work of Boltzmann, who set the founding blocks of the field that would later be identified as *Non-Equilibrium (NE) Statistical Mechanics (SM)*, a plethora of works have been devoted to the study of ensembles of particles, in an attempt to relate the macroscopic evolution in time of a real system - *not* in equilibrium - to the microscopic dynamics of its constituent particles.

Given the large number of particles in real physical systems (say N, typically of the order of 10^{23} to 10^{27} particles per liter, depending of the state of matter) the exact description of a real system via Newton's laws of dynamics would require the simultaneous solution of $6N$ equations of motion: an inconceivable task, even with today's (or the near future's) 'powerful' computing tools[1]. From a statistical-mechanical point of view, one overcomes this problem by defining an appropriate *phase space*, say Γ, representing the set of possible states of the system in terms of the combined values of the variables characterizing the state of the system, e.g. particle positions and momenta $\{\mathbf{q_j}, \mathbf{p_j}\}\,, \quad j = 1, 2, ..., N$ (i.e. $2d$ variables per particle in a $d-$dimensional physical space, $d = 1, 2, 3$). A (probability) distribution function (*pdf*, *df*), say $\rho(\Gamma)$, is thus defined, describing the way the system's states are distributed over all possible configurations. The evolution in time of ρ is governed by the LIOUVILLE equation[2]:

$$\frac{\partial \rho}{\partial t} = L\, \rho$$

However, even though this equation can *a priori* be solved *formally*:

$$\rho(t) = e^{L\,t}\, \rho$$

(so the system's state at an instant t is 'known') this description is rather *abstract*, given the large number of particles in the system (and variables in the argument of ρ).

1.1.1 Kinetic theory

The standard way to cope with the above problem consists in a *reduction* of the information contained in the system's phase-space, by actually limiting our ambition to the study of the evolution of the configuration space of a small number of (one, if possible) particle(s), which is (are) assumed to be representative of the system as a whole. In a general manner, such a theory aims at the

[1] This 'technical' problem lies in the very foundation of the field of Statistical Mechanics.
[2] or VON-NEUMANN equation in a quantum-mechanical system.

derivation of a *kinetic equation*:

$$\frac{\partial f}{\partial t} = \mathcal{K} f$$

describing the evolution in time of a single-particle reduced distribution function (*rdf*) in phase-space (i.e. a function $f(\mathbf{x}, \mathbf{v};\, t)$ of particle position and velocity). $\mathcal{K}$ formally denotes a kinetic evolution operator. The subtle point is *how* $\mathcal{K}$ is obtained, thus passing from the rigorous $N-$body Liouville equation to the $(1-$body) reduced kinetic equation.

According to the standard picture, a system's evolution may be divided, roughly speaking, into three distinct stages. In the *first* stage, the short (*dynamic*) stage (i.e. while t remains lower than a *correlation* time t_{cor}) particles evolve under their microscopic dynamic laws, resulting in initial inter-particle correlations being damped very fast In the *second* stage, the (long) *kinetic stage*, the single-particle distribution function is modified, obeying the kinetic evolution laws of the system (and approaching equilibrium under the influence of collisions). In the *final*, *hydrodynamic* stage, only (slow) changes in local hydrodynamic parameters of the system take place.

Kinetic theory, as defined above, aims in describing the system's evolution during the kinetic stage.

1.1.2 Transport theory

Once a kinetic operator has been obtained, the link to the macroscopic world is made by defining observable quantities as being average values of appropriate microscopic variables, and then deriving the corresponding evolution equations by using the kinetic equation as a starting point. To be more specific, let $A(t)$ be an observable associated to a microscopic quantity $a = a(\mathbf{x}, \mathbf{v};\, t)$:

$$A = \int d\mathbf{x}\, d\mathbf{v}\; a\, f$$

The evolution of A in time will obey:

$$\begin{aligned} \frac{\partial A}{\partial t} &= \frac{\partial}{\partial t} \int d\mathbf{x}\, d\mathbf{v}\; a f = \int d\mathbf{x}\, d\mathbf{v}\, \frac{\partial a}{\partial t}\, f \\ &= \int d\mathbf{x}\, d\mathbf{v}\; a\, \frac{\partial f}{\partial t} = \int d\mathbf{x}\, d\mathbf{v}\; a\, \mathcal{K} f \end{aligned}$$

[3]; transport of matter and energy is therefore modeled by a system of evolution laws, related to the microscopic dynamics of particles.

[3] We have used a well-known postulate of Statistical Mechanics [4] to pass from the third to the fourth step; namely, it is assumed that:

$$\langle f,\, \partial_t a \rangle = \langle \partial_t f,\, a \rangle \qquad ,$$

provided that the product $\langle \cdot\,, \cdot \rangle$ is non-degenerate (i.e. $\langle\, f,\, \cdot\, \rangle = 0 \Longrightarrow f = 0$ and $\langle\, \cdot\,, a \rangle = 0 \Longrightarrow a = 0$).

1.1.3 Collisions

A very important role in this description is played by inter-particle interactions. Such interactions are often called *'collisions'* even though, properly speaking, particles may interact via long-range (e.g. gravitational or electrostatic) forces and thus trajectories may be essentially different from the hard-sphere ('billiard - ball') intuitive image (see figure); this means that neither do particles *crash into* each other, nor are interactions necessarily instantaneous and point-like (i.e. localized in time and space). Collisions are taken into account through an appropriate *collision term*, say $\mathcal{C}$, in the kinetic equation. Such a term first appeared in the original BOLTZMANN equation and has ever since been a widely discussed subject in literature, since it is related to the *irreversible* character of dynamics of matter.

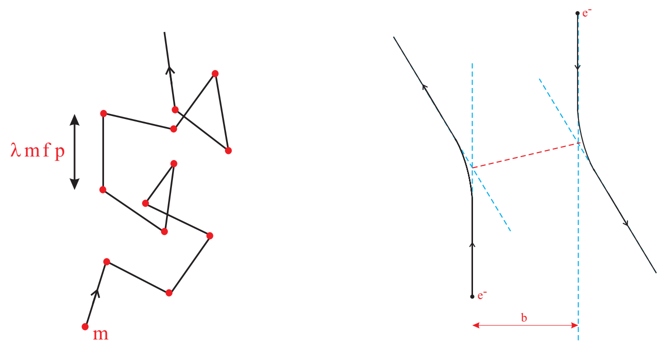

Figure 1.1: (a) Collisions in a hard-sphere model (point-like interactions), as compared to (b) charged-particle collisions (long-range interactions).

1.1.4 Influence of an external field on collisions

The derivation of a collision term for a given physical system should be carried out carefully, since the nature of particle interactions and the geometry of particle trajectories in between them need to be carefully taken into consideration.

A point that needs to be made is the following. In the majority of previous studies, particle motion between collisions is taken to be *free*: particles are essentially assumed to move on a rectilinear trajectory at a uniform velocity, as if no force were present. However, this is *not necessarily* (or, rather, *never*) true. First of all, motion is in principle affected by interactions with all other particles[4]; this fact, too difficult to handle analytically, is a priori neglected in the range of validity of the *weak-coupling approximation* (to be defined in the following chapter and adopted from there and on).

[4]More precisely, particles suffer forces due to either the cumulative effect of numerous two-body interaction forces or, in the case of charged particles, electro-magnetic forces generated by moving charges (via Maxwell's laws).

Furthermore, even at zeroth order in the interaction, the presence of an external force field may strongly modify 'free' particle trajectories between collisions. For instance, in the widely discussed paradigm of weakly-coupled oscillator gases (i.e. large ensembles of weakly-interacting particles, each one subject to a parabolic potential), periodic motion is by no means rectilinear and collisions definitely depend on the harmonic potential parameters (e.g. characteristic frequency). In the particular case of magnetized plasma, where 'free' particle motion is known to be helicoidal (spiral-shaped), this effect may be quite important, depending on the relative magnitude of gyration to interaction scales (see figure 2.4).

Finally, collisions in principle depend on intrinsic physical parameters of the system, for instance temperature and density of the thermostat, in a test-particle problem.

En resumé, a collision term is *a priori* expected to depend on all these parameters and should bear a form which takes into account, in particular, the exact effect of the external field on particle dynamics. As we shall see in forthcoming sections, the latter is simply absent from most kinetic equations widely used in literature. In particular, the field is often taken into consideration in the free Liouville part, i.e. the left-hand-side (*lhs*) of the kinetic equation, but *not* in the collision term (see e.g. in [5]).

The general ideas outlined so far were necessary to define the theoretical context of our study. However, these issues are basically standard, so supplying further details is not truly necessary in what will follow; details may be found in textbooks of Statistical Mechanics (see e.g. [4], [23], [24], [26], [29]). Nevertheless, the specific theoretical aspects to be considered in *our* particular problem will be exposed in forthcoming sections.

1.2 Plasma

Ensembles of electrically charged particles interacting with each other, referred to as *p*lasma[5], have always been present in space and in the earth's atmosphere; furthermore, laboratory plasma is today widely produced on earth, for fundamental research purposes as well as for practical applications. Plasma, which is often quoted as *'the fourth state of matter'* due to its omnipresence in real world, has long been studied, both theoretically and experimentally, and is known to present very rich dynamics, well above the 'usual' level of complexity of neutral matter.

In the last few decades, the aim of controlling thermonuclear fusion (the sun's power generation mechanism!) for man-oriented applications and the big amount of relevant experimental research run on fusion devices has provided a vast field of application for plasma transport theories, as well as a real challenge

[5]The term *p*lasma (from the greek word '$\pi\lambda\acute{\alpha}\sigma\mu\alpha$', meaning *'moulded form'*, *'creature'*) was first used by I. Langmuir in 1928 to describe the positive ion column in an experimental glow discharge (see reference in [123] - ch. 18, [22]).

in terms of demand for theoretical ground for prediction of measurable quantities (hence e.g. transport coefficients).

1.2.1 Plasma-kinetic theory

From a statistical mechanical point of view, the *long-range* character of electrostatic interactions actually distinguishes plasma from neutral gases (described by Boltzmann-type theories) so a different approach is needed. Since the original work of Landau [75] in 1936, plasma has often been used as a test-bed for kinetic theories and a variety of methods have been elaborated, in an attempt to relate macroscopic plasma transport phenomena to the microscopic nature of charged matter. Outstanding contributions have been the works of Vlasov [119] in 1938, who was the first to include the reciprocal interaction between a particle and the mean surrounding field in a "self-consistent" description and that of Balescu, Lennard and Guernsey [55], who included collective effects in an elegant description. In the 1960s, the so-called Brussels' school provided a new formal framework (in general *and* specifically in plasma research) and microscopic theories of dissipative phenomena were associated to *master equations* [4], [40]. This new formalism, under which all previous works were rigorously recovered and thoroughly studied, allowed for a broad discussion, concerning the paradox of the (obvious) irreversibility of the macroscopic world (already evoked by, and widely opposed since, Boltzmann's work) with respect to (time-reversible) microscopic dynamics.

In particular, the need to manipulate the dynamics of plasma in *magnetic confinement fusion devices* (*'tokamak'* reactors)[45] has motivated research on the influence of external electro-magnetic (EM) fields on plasma. A number of works carried out in this direction will be cited later on, for the sake of reference.

1.2.2 Plasma classification

As a matter of fact, the rich variety of plasmas existing in nature (whose complex behaviour may differ essentially from one case to another) imposes a certain classification. Starting from a fundamental level and focusing on and on, one has to distinguish, for instance, *classical* from *quantum-mechanical* plasmas (depending on the mean inter-particle distance, the latter corresponding to very high density values), *non-relativistic* from *relativistic* plasma (depending on the average particle energy, i.e. temperature, the latter being found in gravitationally collapsing stars in space), *weakly-coupled* plasmas from *strongly-coupled* electrostatic systems (depending on the average potential-to-kinetic energy ratio; the former are closer to rarefied gases and were studied first, while the latter, rich in collective phenomena, are closer to a solid state physical system[6]), *fully*

[6] According to certain authors, only the former is traditionally called *'plasma'* (see discussion in [11]); however, the latter seem to gain increasing interest in the last decade, since the theoretical prediction and subsequent discovery of 'plasma-crystals' in both laboratory plasma devices and tokamak walls (see for instance, the discussion in [118]; also a series of articles in [126]).

from *partially* ionized plasma and so on. Furthermore, modern literature in plasma-kinetic/transport theory often makes the distinction between: (i) *classical* transport theories (plasma in an infinite cubic vessel, possibly in the presence of external fields), (ii) *neoclassical* transport (new rich behaviour of plasma submitted to a toroidal geometry in *tokamaks*) and (iii) *anomalous* transport (fluctuation - induced transport, due to the random microscopic nature of both interactions and EM-fields). Finally, if the bulk plasma is in a highly spatially-correlated state (closer to the hydrodynamic picture of a common fluid) it is often identified as *quiescent* plasma, in contrast with *turbulent* plasma [14], [33].

The exact definitions of (certain among) the above terms will be given in the text, where necessary. In order to delimit the area of our work, let us mention that this study concerns a *classical, non-relativistic, fully-ionized, weakly-coupled plasma.* It mainly focuses on *classical transport*, though it also aims in establishing an intuitive link towards anomalous transport via the inclusion of interaction-force correlations in a classical kinetic picture.

1.3 Kinetic equations - an overview

1.3.1 Basics

Let us consider a particle (mass m), moving (in a $d-$dimensional physical space, $d = 1, 2, 3$). At the instant t, the particle is located at position $\mathbf{x}$ with a velocity $\mathbf{v}$. The equations of motion read:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}\,, \qquad \frac{d\mathbf{v}}{dt} = \mathbf{a} = \frac{\mathbf{F_{ext}}}{m}$$

where $\mathbf{F_{ext}}$ is due to an external force field accelerating the particle at an acceleration $\mathbf{a}$.

Now let $\Gamma = \{\mathbf{x}, \mathbf{v}\}$ be the ($2d$-dimensional) phase-space associated to this system. Defining the ($2d-$) position vector of the particle state in phase space: $\mathbf{X}(t) \equiv (\mathbf{x}(t), \mathbf{v}(t))$, the above dynamical problem takes the form:

$$\frac{d\mathbf{X}}{dt} = \mathcal{F}$$

where $\mathcal{F} = \dot{\mathbf{X}} = (\mathbf{v}, \mathbf{a})$ is the *generalized force* (a $d-$dimensional vector).

In the absence of interactions with other particles, the function $f(\mathbf{X})$ describing probability distribution in $\Gamma-$ space obeys a *continuity equation*[7]:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{X}}(\mathcal{F} f) = 0$$

[7] To be precise, continuity holds if: (i) a *Koopman operator* is defined by the solution of the given dynamical motion problem $S_t\mathbf{x} = \mathbf{x}(t)$ (assumed non-singular), and (ii) we assume that states evolve under the action of its *adjoint* (*Frobenius-Perron*) operator (under suitable boundary conditions). The evolution of observables is then described by: $U_t\,A(\mathbf{x}) = A(S_t\mathbf{x})$. Therefore, if dynamic law S_t is *volume preserving* (*Liouville th.*), then it may be proved that: $P_t\,f(\mathbf{x}) = f(S_{-t}\,\mathbf{x})$, where P_t is the operator of temporal evolution of states in phase space Γ. For details, see in [27] (pp. 36-37, 42-43, 185-186 therein), [96]; also p. 2-2 in [46].

If (and only if)[8]:

$$\frac{\partial \mathcal{F}_j}{\partial X_j} = 0 \qquad i.e. \qquad \frac{\partial F_{ext,j}}{\partial v_j} = 0$$

the above equation takes the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}\frac{\partial f}{\partial \mathbf{v}} \equiv \frac{\partial f}{\partial t} - L_1\, f = 0 \tag{1.1}$$

where the index 1 is used to distinguish the *single-particle* ('free') Liouville operator L_1 from the $N-$body Liouvillian, $L = L_N$ (mentioned in the beginning). Notice that this equation is *time - reversible*[9], as is the Hamiltonian dynamics related to it.

Once interactions with other particles are taken into consideration, the above equation will suffer a modification of the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{df}{dt}\right|_{coll} \tag{1.2}$$

where the *rhs* accounts for interactions (collisions) with other particles. This equation is *not* a closed equation in the d.f. f; as we shall see below, it introduces a coupling to a hierarchy of equations for higher order distribution functions, the whole of which contains no less information than the complete ($N-$body) Liouville equation. In order to obtain a closed equation in f, a truncation is considered, in order for the description to be compatible with thermodynamics and yet fulfil certain conditions (conservation laws). Collisions, assumed to lead the system towards a final state of maximum entropy (according to the second law of thermodynamics) are thus taken into account by a *collision term*, entering the *rhs* of (1.2). The final (kinetic) equation obtained in this way has the structure of (1.2) but is now closed in f and irreversible in time.

A simple case of a collision term is the *Bhatnagar-Gross-Krook* (*BGK*) collision term:

$$\left[\frac{df}{dt}\right|_{coll} = -\frac{f(\mathbf{x},\mathbf{v};t) - f_0}{\tau}$$

where τ is a phenomenological relaxation time and f_0 is a local equilibrium function[10]. This approach, discussed in [102], [123][11] (and criticized in [48]), has been adopted in various studies e.g. [73], as it reproduces the qualitative aspects expected (namely in terms of conservation laws; see discussions in the references).

A brief survey of kinetic equations of relevance to our problem will be presented in a forthcoming section.

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[8] The system is then said to behave as an *incompressible fluid*, see e.g. in [17].

[9] It remains unchanged upon setting: $t \to -t$, $\mathbf{v} \to -\mathbf{v}$ (leaving $\mathbf{x}$ unchanged).

[10] e.g. a Maxwellian with space- and time-dependent density and temperature parameters, in the most elaborate version of the model (see the discussion in [102]).

[11] Also see references in Elliott's paper therein (including the original BGK paper).

1.3.2 Generic form of a rigorous kinetic equation

Regardless of the method used for its derivation, a kinetic equation will obey the general form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + m^{-1}\mathbf{F}\frac{\partial f}{\partial \mathbf{v}} = \mathcal{C}\{f\} \tag{1.3}$$

As defined above, $f = f(\mathbf{x}, \mathbf{v};\, t)$ is a single-particle reduced distribution function (*rdf*) (a function of time t, particle velocity $\mathbf{v}$ *and* particle position $\mathbf{x}$, in a non-uniform system). $\mathbf{F}$ denotes the *total* force exerced on the particle; in a *self-consistent* description, it should correspond to:

$$\mathbf{F} = \mathbf{F_{mf}} + \mathbf{F_{ext}}$$

where $\mathbf{F_{ext}}(\mathbf{x}, \mathbf{v})$ is due to an *external* force field[12] and $\mathbf{F_{mf}}(\mathbf{x}, \mathbf{v};\, f)$ is the result of the *mean-field potential* due to charge screening effects[13]. Finally, the *collision term* $\mathcal{C}$ in the right-hand-side (*rhs*) accounts for particle interactions. In principle, the collision operator depends on the instantaneous value of f itself: $\mathcal{C} = \mathcal{C}\{f\}$; a kinetic equation is therefore nonlinear in f. In some cases, however, $\mathcal{C}$ may be a *linear* operator acting on f (so the kinetic equation describing a class of problems is a linear differential equation). Such is the case in a test-particle problem (to be presented and discussed below).

1.3.3 A brief account of existing plasma-kinetic equations

All of the studies mentioned above have led to an equation in the form of (1.3). Certain among them rely on a phenomenological description of particle collisions (related to *stochastic* mathematical theories) while others adopt a more rigorous approach, either based on perturbation theory [4], [7] or formal projection-operator methods (see e.g. [15], [46], [68]). In order to point out the relation of our work to its background, a brief summary of previous results (for *classical* systems) will be given in the following paragraphs. This account aims in roughly sketching the theoretical background of our study and is by no means exhaustive. For a more detailed report of the properties of the equations presented in the following paragraphs, see in [4], [5], [7], [28], [34], [43].

Boltzmann **equation (BE)**

In his original work in 1872 [14], Ludwig Boltzmann studied a *dilute gas* of colliding particles (*in the absence of an external field*) and derived the equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\,\nabla f \;=\; \int d\mathbf{v_1} \int d\omega\, g\, \sigma_d\, [f(\mathbf{x}, \mathbf{v}'; t) f(\mathbf{x}, \mathbf{v}_1'; t) - f(\mathbf{x}, \mathbf{v}; t) f(\mathbf{x}, \mathbf{v_1}; t)]$$

[12] Notice that $\mathbf{F_{ext}}$ corresponds to $m\,\mathbf{a}$ in the previous paragraph; it is (only) due to the external field (and has nothing to do with interactions). However, $\mathbf{F_{mf}}$ is due to interactions with the 'cloud' of particles surrounding our specific particle.

[13] $\mathbf{F_{mf}}$ in a specific position $\mathbf{x}$ depends on the value of f in the vicinity of $\mathbf{x}$, hence the notion of '*self-consistence*'.

[14] We will not burden this text with reference to the - widely cited - work of L. Boltzmann; an exhaustive list of relevant sources can be found in [4].

$$\equiv \mathcal{C}_{Bol}\{f\} \tag{1.4}$$

where $f = f(\mathbf{x}, \mathbf{v}; t)$ is the probability distribution function in phase space. Note the definitions:

- $\mathbf{v}$ denotes particle velocity before a collision;
- $\mathbf{v}'$ denotes particle velocity after a collision;
- $g = |\mathbf{g}| = |\mathbf{v} - \mathbf{v_1}|$;
- ω denotes the solid angle around a collision event,

and, finally,

- σ_d is the *differential cross-section*.

The kinetic evolution is therefore taken to obey the combined action of a *gain* and *loss* terms (in the *rhs*), plus a *flow* term (in the *lhs*). In a *uniform* system, the latter cancels and the equation becomes:

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= n \int d\mathbf{v_1} \int d\omega\, g\, \sigma_d \left[\phi(\mathbf{v}';t)\phi(\mathbf{v_1}';t) - \phi(\mathbf{v};t)\phi(\mathbf{v_1};t)\right] \\ &\equiv \mathcal{C}_{Bol}\{\phi\} \end{aligned} \tag{1.5}$$

where $\phi = \phi(\mathbf{v}\,; t)$ is now the probability distribution function in velocity space; n denotes particle density.

Vlasov equation (VE)

In 1938, A. Vlasov [119] studied a *weakly-coupled* system of charged particles interacting via (long-range) electrostatic forces (*in the absence of an external field*, once more) and derived an equation in the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + m^{-1}\mathbf{F_{mf}}(\mathbf{x})\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{1.6}$$

($f = f(\mathbf{x}, \mathbf{v}\,; t)$) where the role of interactions is limited to the appearance of

$$\mathbf{F_{mf}}(\mathbf{x}) = -\frac{\partial}{\partial \mathbf{x}} \int d\mathbf{x_1} \int d\mathbf{v_1}\, V(\mathbf{x} - \mathbf{x_1})\, f(\mathbf{x_1}, \mathbf{v_1}) \equiv -\frac{\partial V_{mf}}{\partial \mathbf{x}}$$

in the so-called *mean-field (mf)* (Vlasov) term, which accounts for interactions with surrounding particles (charge screening). Notice that this term is related to space inhomogeneities, and thus disappears in the case of a uniform system. The Vlasov equation is reversible; it describes an *inhomogeneous (collisionless) plasma.*

Landau equation (LE)

In 1936, Lev D. Landau [75] studied a *uniform* gas of weakly-interacting particles (in *no* external field) and derived the equation:

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= n \int d\mathbf{v_1}\, \frac{\partial}{\partial \mathbf{v}}\, \mathbf{G}(\mathbf{v}, \mathbf{v_1}) \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v_1}}\right) \phi(\mathbf{v};t)\, \phi(\mathbf{v_1};t) \\ &\equiv \mathcal{C}_{Landau}\{\phi(\mathbf{v}), \phi(\mathbf{v_1})\}\end{aligned} \tag{1.7}$$

($\phi = \phi(\mathbf{v}\,; t)$). G_{rs} denotes the Landau tensor:

$$G_{rs}(\mathbf{g}) = \frac{B}{m^2}\,\frac{g^2\delta_{rs} - g_r g_s}{g^3}$$

where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$; B is an integral in Fourier space involving the interaction potential (to be discussed in detail later on). Notice that the physical system described here is isotropic; therefore, not surprisingly, this tensor is symmetric.

In order to describe a *non-uniform* system (in the absence of external fields), the following generalization of the LANDAU equation was later considered [4]:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}\,\nabla f &= \\ &= \int d\mathbf{v_1} \int d\mathbf{x_1}\,\delta(\mathbf{x} - \mathbf{x_1})\,\frac{\partial}{\partial \mathbf{v}}\,\mathbf{G}\left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v_1}}\right) f(\mathbf{x}, \mathbf{v}; t), f(\mathbf{x_1}, \mathbf{v_1}; t) \\ &\equiv \mathcal{C}_{Landau}\{f(\mathbf{x}, \mathbf{v}), f(\mathbf{x_1}, \mathbf{v_1})\} \qquad (1.8) \end{aligned}$$

While trying to evaluate the LANDAU collision integral in the case of an infinite-range (e.g. Coulomb) potential, one encounters the well-known problem of *divergence* of the Fourier integral involved in it (i.e. B), at large as well as short distances. The former is attributed to the long-range character of electrostatic interactions while the latter is due to the weak binary interaction hypothesis, which fails at short distances. More details will be given later, where appropriate.

BALESCU-LENNARD-GUERNSEY **(BLG) equation**

In 1960-61, R. Balescu, A. Lenard and R. Guernsey [55], working independently, went one step further, including collective effects and furnishing a dynamical dielectric constant which depends on the phase-space distribution itself. Plasma polarization effects, not included in the Landau picture, were thus taken into account. We do not provide the specific form of the BLG collision term. Let us only remark that it was derived for a uniform system in the absence of external fields and yields a symmetric form which is very close in structure to the LANDAU collision term (see in [3], [5] for details).

1.3.4 Master equation

It is very interesting to notice that the original work of Landau in the 1930s consisted in actually solving the two-body problem for (long-range) electrostatic interactions in order to obtain an approximation of the Boltzmann equation for electrostatic plasma. Vlasov worked in a similar way. However, in the following two decades, these equations were derived in a quite different (and more elegant) manner, by establishing a hierarchy of coupled equations for reduced $p-$body

($p = 1, 2, 3, ...$) distribution functions: the *BBGKY hierarchy.* Assuming interactions to be weak (of the order, say, of λ), the hierarchy may be expressed in terms of powers of λ; the VLASOV and LANDAU equations are thus formally recovered in order λ^1 and λ^2 respectively [3], [4], [8], [23], [24].

In a generic manner, at second order in[15] λ the above procedure gives the *Generalized Master Equation (GME)* [4], [40]:

$$\frac{\partial f(\mathbf{X_1};\, t)}{\partial t} = \int_0^t d\tau \int d\mathbf{X_2}\, L_I\, L_I(\tau)\, f(\mathbf{X_2};\, t-\tau)\; f(\mathbf{X_1};\, t-\tau) \qquad (1.9)$$

where L_I the Liouvillian of interaction between particles 1 and 2. Note that initial correlations were neglected. The GME is a *non-Markovian* integro-differential equation: the value of f at the instant t depends on the system's 'history' i.e. its state for all values of $\tau \in [0,\, t]$.

The effect of 'non-Markovianity' (non-locality in time) is very subtle to handle and has most often been overcome either via formal manipulations[16] or by making certain 'Markovianization assumptions'; we will comment on this point later. Nevertheless, let us mention that some recent studies have considered non-locality effects; see e.g. [58], [121]. This is beyond our scope here.

1.3.5 Kinetic equations for plasma in an external field

A number of works have focused on the kinetic description of an electrostatic plasma in the presence of an electric and/or magnetic field(s). Let us limit ourselves to citing the most important contributions. A detailed discussion of the relation of our work to these studies will be carried out later.

Homogeneous plasma

The equivalent of the LANDAU equation in the presence of a uniform stationary magnetic field has been derived, independently and via different analytical methods, in the 1960s by N. Rostoker [101], M. Haggerty and co-workers[17] [69], V. P. Silin and co-workers [64], [105] and P. Schram [103]. These authors derived a 'Landau-type' kinetic equation for magnetized plasma and obtained a complicated set of expressions for the diffusion tensor therein, in terms of the magnitude of the magnetic field and particle velocity.

The problem was later revisited - still for a *homogeneous* distribution function - by D. Montgomery et al. [85], [86], who tried to remove the notorious Fourier divergences by splitting the interaction sphere in two parts, one of which is assumed not to be affected by the field. The Landau tensor is then plainly

[15] Remember that: $L = L_0 + L_{int}$, see above.

[16] See for instance §17.2 in [4], where the formal solution of the Liouville equation is used to provide a closed equation in $f(t)$.

[17] Haggerty's work was based on Prigogine and Balescu's diagram technique [4], [40] quite popular at that time, yet very hard to follow. His analytical results seem to confirm (and generalize) Rostoker's formal - and very lengthy - calculation, who had rather described a test-particle problem (similar to the one we define below).

recovered (despite the intrinsic cylindrical symmetry of the system!), and a simple modification of the Coulomb logarithm is proposed, in order to take account of the field. The influence of the external field is somewhat under-estimated in this method; see the discussion carried out in [86], as well as relevant criticism in [16].

In the same time, a BLG-like equation for magnetized plasma was derived, independently, by Hassan *et al.*[18] [70].

The results of all these studies involved an infinite series of Bessel functions[19]. Nevertheless, the convergence of this series was never questioned, and neither was the - rather doubtful - utility of these complicated expressions for practical (either analytical or computational) purposes. As we shall see later on, these expressions are confirmed in the basis of our work (see in Chapter 6). However, our calculation will try to go one step further in analytical tractability, by deriving exact closed computable expressions for diffusion coefficients (i.e. involving *no* infinite special function series; see in Chapter 8 and on).

Inhomogeneous plasma

A non-uniform plasma embedded in an external magnetic field (taken to be uniform and stationary) was considered in various approximations by Øien in a series of papers since the late 1970s. Having already studied the *unmagnetized* inhomogeneous case [87], Øien actually refined the picture by adding an external electric field, which he assumed to be either uniform [88], slightly inhomogeneous [90] or slowly time-periodic [93][20]. In practically all these studies, however, a linearization of force-correlations around the *unmagnetized* case was considered, for the sake of analytical tractability (and yet against rigour); particle trajectories between collisions were therefore calculated as if *no* field were present.

A little later, P. Ghendrih [65], [16] rigorously obtained a generalization of the previous LANDAU collision operator in the non - uniform magnetized case. However, space-gradients appearing in the (spatially inhomogeneous part of the) collision term were straightforward neglected, through physical arguments. The remaining part of the collision term that was then used for the rest of that study was identical to Montgomery's [85].

1.4 Test-particle formalism - random processes

In the theoretical framework depicted above, a paradigm of particular interest among statistical physicists consists in the study of a small subsystem weakly interacting with a large heat bath (*'reservoir'*) in thermal equilibrium (a *ther-*

[18] Hassan's work, actually very interesting and yet no so well-known, is presented in [20]; also see references cited therein for details).

[19] This was rather expected, due to the cylindrical symmetry of the problem.

[20] Øien's rich contribution even includes the derivation of a Boltzmann-like collision operator [89], a study of toroidicity effects [91] and much more, of less importance here.

mostat). Both sub-systems may be subject to an external force field. The small system relaxes towards equilibrium under the influence of the bath.

In a *test-particle* problem, such a system studied consists of a particle[21] denoted by[22] σ, which is (are) singled out and 'tagged' from a large electrostatic background (the *reservoir* $\mathcal{R}$, consisting of $N_{\mathcal{R}}$ particles) in equilibrium. As $N_\sigma \ll N_{\mathcal{R}}$, the test-particle(s) (*t.p.*) is (are) assumed *not to* affect the reservoir equilibrium state, say $\rho_{\mathcal{R}}$ $(\partial_t \rho_{\mathcal{R}} = 0)$[23]. The t.p., whose distribution is initially not in (and not necessarily close to) equilibrium, will presumably relax towards an equilibrium state $\rho_{eq} \approx \rho_R$ under the influence of the bath[24]. Information drawn from such a model includes relaxation times (as well as their dependence on physical parameters), velocity or space diffusion-related phenomena etc.

The system described here is a well-known paradigm of a (so-called) *Open System*, term used for statistical-mechanical systems which are allowed to exchange energy with the exterior. It is typically modeled by a Hamiltonian of the form:

$$H = H_\sigma \, + \, H_R \, + \, \lambda \, H_I$$

where H_σ, H_R denote the Hamiltonian of the two sub-systems (separately) and H_I is an interaction Hamiltonian term. The resulting Liouvillian is:

$$L = L_\sigma \, + \, L_R \, + \, \lambda \, L_I \equiv \, L_0 \, + \, \lambda \, L_I$$

where the definition of all terms is obvious. Interactions are tagged by λ (which is assumed to be small in the weak-coupling approximation).

The physical mechanism of interaction depicted in this paragraph may also be considered from a different point of view. As interactions with the *reservoir* are completely erratic, they constitute a *random* (or *stochastic*) process. Phenomenological theories of stochastic processes most often use probabilistic arguments in order to account for lack of microscopic information. Therefore, one may envisage to establish a link between such theories and our work (aiming to relate microscopic dynamics to macroscopic randomness); we will attempt to sketch such a relation in the following paragraphs.

1.4.1 Theory of Brownian motion

The first historical paradigm that drew attention to random processes was *Brownian motion*, referring to the motion of a heavy colloidal particle immersed in a fluid of much lighter particles. It was defined, rather 'accidentally', in 1827, when the British botanist Robert Brown observed - and tried to explain - the random motion (ever since named after him) of pollen grains immersed in a quantity of liquid at rest. All sorts of arguments and ideas were advanced by

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[21] i.e. one, or *'a few'*, say N_σ, particle(s). The underlying hypothesis is that the test-particles interact with the environment, but not between one another, since $N_\alpha \ll N_R$.

[22] The letter σ, from the greek word '$\sigma\omega\mu\alpha\tau\iota\delta\iota o$' (= 'particle') will henceforth denote the test-particle in this text.

[23] This is often quoted as the 'zeroth' law of statistical mechanics; see e.g. [72].

[24] This is not to be taken for granted. It is a desired property, which should be a consequence of the theory used to model the physical problem.

himself and many others, in order to explain the phenomenon; as a matter of fact, the first dynamical theories of Brownian motion were *'vitalistic'* ones [37]: they suggested that motion was due to the fact of pollen grains being alive![25]

It was only long after Brown's report, in 1905, that Albert Einstein[26] formulated his theory about Brownian motion [63]. Einstein defined a coarse-graining time interval Δt, which is much shorter than the observation time step, but much longer than the typical correlation time. Considering the probability of a grain moving at a certain distance over time Δt, he obtained the well-known diffusion equation of macroscopic physics, which describes motion of particles suspended in a medium.

Let us remark that Einstein's intuitive attempt for an explanation was very efficient but still phenomenological. A little later, Von Smoluchowski [107] studied the same problem from a detailed study of the underlying microscopic physics. It is interesting to notice that *these two different approaches (or should I say different mentalities?) can still be traced in much of the research on fluctuations that has developed since then: those who argue from general principles and those who delve into the microscopic physics*[27].

It should be noted that the problem of Brownian motion proper, as modeled by the equations to be presented below, refers to a *heavy* particle surrounded by erratically moving light particles. This assumption will not be made in our study of particle motion in magnetized plasma.

Kramers **equation**

The theory proposed by Einstein and, independently, Von Smoluchowski, was later formulated in phase space by Kramers [74]. Denoting by $W(x, v)$ the probability distribution function at point (x, v) in phase space[28], Kramers derived the equation:

$$\frac{\partial W}{\partial t} = \{H, W\} + \zeta \frac{\partial (vW)}{\partial v} + \frac{\zeta k_B T}{m} \frac{\partial^2 W}{\partial v^2} \tag{1.10}$$

where ζ is the friction constant, T is the fluid temperature and m is the mass of the particle. This equation is in agreement with the Langevin equation of motion (see below). However, it is more convenient to manipulate: if the initial value-problem is solved, the resulting distribution function allows us to calculate any average value of (a function) velocity v, by simple quadrature. Furthermore, the Maxwell distribution identically cancels the right-hand-side, and is thus immediately seen to be an equilibrium distribution.

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[25] Remember that the atomic hypothesis was not yet widely accepted at that early time.

[26] In the same year, A. Einstein published his study of the photo-electric effect, which he was later awarded the Nobel prize for.

[27] We quote N. G. van Kampen from a recent paper inaugurating a new journal on fluctuation phenomena [116].

[28] The one-dimensional case is considered here, for simplicity.

LANGEVIN equation

Two years after Eistein's paper, Langevin [77] advanced a description which is more familiar to the physicist's image than Eistein's, since based on the laws of motion. He assumed that the particle's motion (mass m, position $\mathbf{x}$) obeys the equation [50]:

$$m \frac{d^2\mathbf{x}}{dt^2} = -\zeta \frac{d\mathbf{x}}{dt} + \xi(t) \tag{1.11}$$

where ζ is a friction constant and $\xi(t)$ is a Gaussian random process with zero mean value and covariance $\langle \xi_i(t)\xi_j(t') \rangle = \mu\, \delta_{ij}\, \delta(t-t')$ (a *stationary* process). This equation of motion can be exactly solved for the particle velocity $\mathbf{v}(t)$ and the average values of velocity and its moments can be analytically calculated, as functions of time t. The equipartition theorem of statistical mechanics (associating the average kinetic energy of a particle in thermal equilibrium at temperature T to $\frac{3}{2}k_B T$ - k_B is Boltzmann's constant) then provides a link to the asymptotic value of the mean-square-velocity, thus 'imposing' a value for ζ.

Uhlenbeck & Ornstein [114] later integrated the LANGEVIN equation, given the statistics of ξ, and showed that the average position and mean-square-displacement are given by:

$$\langle \mathbf{x}(t) \rangle = \mathbf{x}(0) \qquad \langle [\mathbf{x}(t) - \mathbf{x}(0)]^2 \rangle = 2\,\mu\, t$$

This procedure allowed to gain insight on the microscopic mechanism of noise, based on macroscopic measurement.

It should be pointed out that the theory outline here is only a semi-phenomenological theory. The final thermal equilibrium has been *imposed* on the theory, instead of being *derived* from it. The dissipation constant ζ (which is characteristic of the time scale of the evolution) a priori represents (and yet, also, here *'hides'*) all the complicated dynamical processes involved in the interaction (*"collision"*) mechanism.

FOKKER-PLANCK equation

The KRAMERS equation is a special case of the general FOKKER-PLANCK equation (FPE):

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial v}\Big[A(v)\, W \Big] + \frac{\partial^2}{\partial v^2}\Big[B(v)\, W \Big] \tag{1.12}$$

where $W = W(v;t)$[29]. The coefficients $B(v)$, $A(v)$ account for *diffusion* and *dynamical friction*, respectively, suffered by the particle[30]. The coefficients $A(v)$ and $B(v)$ are respectively related to the first and second moments of velocity v [15]. Of course, this is the form of the FPE in the one-dimensional case. In a $d-$ dimensional problem ($d = 1, 2, 3$), where $W = W(\mathbf{x}, \mathbf{v}; t)$, it should

[29] More rigorously, a term '$-\{H, W\}$' should be added to the *left-hand-side*; cf. (1.10).

[30] In the case of *Kramers* equation (1.10) Brownian motion (and, in fact, in *any 'Ornstein-Uhlenbeck' process*, see e.g. §3.8.4 in [15]), the *diffusion coefficient* B is constant, while the *drift coefficient* A is linear in the velocity v.

generalize appropriately to a $(2d+1)$-variable second order linear parabolic partial-differential-equation (PDE). The diffusion and drift (friction) coefficients then take the form of a square matrix and a vector, respectively.

The FOKKER-PLANCK equation arises in a variety of problems, in Physics, Chemistry and Biology. Without going into further details here, let us just say that our work aims in relating the form of the *FPE* to the microscopic 'collision' mechanism[31]. For a more general coverage of the subject, Risken's book [42] is a genuine treasure of reference and a mine of analytical tools. The notions of stochastic calculus, to the extent of mathematical rigor needed by a statistical physicist, are covered in [15], [21], [30], [49]; also in [44] for mathematical tools. Deeper insight in fluctuation-induced phenomena, as well as a wealth oh historical data, can be sought in [37] for Brownian motion and in [81] for the Kramers problem; also in monograph series [35] for *"everything"*.

1.4.2 'Landau-Fokker-Planck' (LFP) equation for plasma

It is interesting to see that the LANDAU equation (1.7) can be re-arranged into the form of a three-dimensional FOKKER-PLANCK equation. However, the coefficients will then depend on the distribution function, i.e. the solution of the equation itself, at any instant t. This is a *nonlinear* FPE.

Following the outline of a test-particle problem, as described above, one may consider a charged particle moving against a thermalized background of electrons and ions (*bulk* plasma) assumed to be in homogeneous equilibrium. Taking the reservoir distribution $\phi(\mathbf{v_1}; t)$ in (1.7) to be Maxwellian, one may derive exact expressions for the coefficients in the kinetic equation, which then becomes a linear 'Landau-Fokker-Planck' (LFPE) equation. This kind of physical situation was first studied in a completely different context by S. Chandrasekhar [60], who considered a 'test-star' in a stellar population (*cluster*) in equilibrium (!); Chandrasekhar *assumed* its evolution to be governed by an FP equation and solved the two-body problem for a central potential in order to obtain the exact form of the coefficients in it. This type of treatment was first applied to a classical plasma by Landau [76] and L. Spitzer [109] and was later reformulated in various forms by M. N. Rosenbluth et al. [99] and others[32].

A similar calculation has been carried out for a homogeneous *magnetized* plasma by Montgomery et al. [85] and, independently, by Baldwin [54] in 1977 and Hassan [71] in 1978. However, to our knowledge, no analogous study has been carried out in the *inhomogeneous* plasma case, in the presence of an external field. This type of calculation is part of the aim of this thesis, as will be discussed below.

[31] In a similar spirit, note the paper by J. Lebowitz and E. Rubin [78], who derived a Fokker-Planck equation from dynamical principles, in an attempt attempt to bridge the gap between microscopic laws and macroscopic behaviour, as related to stochastic Brownian-type motion. The authors assume that the test-particle is well heavier than surrounding field particles: $M \gg m_R$, and thus proceed on perturbation in the smallness parameter $\gamma = m_R/M \ll 1$. This expansion scheme is different from the one adopted later in this text.

[32] Also see references mentioned in §38 in [3], where an analytical calculation, rather close to ours here, is presented.

As a matter of fact, a *charged t.p.* problem may refer to situations not as simple as an electron moving against an ion background. We refer to the situation appearing when *heavy* charged particles (typically α particles, i.e. helium nuclei) are injected in compressed pellet plasmas, in inertial fusion-related experiments, with the aim of deposing energy to the fuel material in both the initial (cold and compressed) state and during evolution towards full ignition and burn). α particle stopping is typically modeled by a *FPE* [10], [59], [79]. This picture is, in fact, closer to the original Brownian image of a heavy particle erratically moving inside a medium of light particles.

Here we will present the exact calculation of the coefficients of the FPE in both magnetized and *unmagnetized* electrostatic plasma case, based on the formalism that will be developed later in this thesis. The older results (in the latter case) are thus exactly recovered.

Let us close this section with an important remark. Sometimes, the 'LANDAU - FOKKER - PLANCK' equation[33] is referred to as the 'linearized Landau equation'. This may lead to the (erroneous) impression that it is obtained by setting $f = f_0 + \epsilon\, \Delta f$ in the Landau equation. This is wrong: *the fact that (the* LFP *equation) is linear does not imply that the distribution ϕ is close to equilibrium. The linearization has been achieved here only by the assumptions that the medium has a stationary distribution and that the test-particles do not interact among themselves but only with the medium. Another type of linearization of (the* LANDAU *equation) would be obtained if it were assumed that the distribution of* all *the particles was close to equilibrium...* (In spite of its similarity with the equation one would obtain in this case, the LFP equation) *describes a completely different physical situation* [34] .

1.4.3 Master equation in a test-particle problem

The general form of the master equation for a test-particle problem, as described above, is obtained by substituting the reservoir distribution in the Generalized Master Equation (1.9) with the homogeneous equilibrium *df*, i.e. setting $f(\mathbf{X_2};\, t) = n\phi_{eq}(\mathbf{v_2})$ (satisfying: $\partial_t \phi_{eq} = 0$):

$$\mathcal{C}_0\{f(\mathbf{x}, \mathbf{v};\, t),\, \phi(\mathbf{v_1})\} = n \int_0^t d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, L_I\, L_I(\tau)\, \phi_{eq}(\mathbf{v_1})\; f(\mathbf{x}, \mathbf{v};\, t - \tau) \tag{1.13}$$

where the indices were re-arranged appropriately; the index *'zero'* is used to distinguish this collision term from the one defined in (1.9).

The properties of this 'linearized' Master Equation (setting the founding blocks of our study) will be discussed in the following section.

[33] sometimes also quoted, more correctly, as the RMJ (Rosenbluth-McDonald-Judd) equation.

[34] We simply chose to quote R. Balescu [3], here.

1.5 Motivation of our study - discussion

The above introduction was necessary for the context of our work to be defined. We may now discuss the aim of this study.

So far, the description of a test-particle problem was seen to obey a kinetic equation of the form of (1.3), where the collision term $\mathcal{C}_0$ is given by (1.13) above. Let us draw our attention to some issues of particular importance, involved in the evaluation of $\mathcal{C}_0$.

1.5.1 Which way to a Markovian equation ?

As pointed out before, this Master Equation is a *Non-Markovian* equation: see the appearance of $f(\mathbf{x}, \mathbf{v}; t-\tau)$ in the *rhs*. Nevertheless, we are interested in deriving a closed (kinetic) equation for $f(t)$.

In principle, the solution of the (Liouville) evolution equation in phase-space can be formally obtained [4], so:

$$f(t-\tau) = e^{-L\,\tau}\, f(t)$$

where $e^{-L\,\tau} \equiv U(t)$ is the time-evolution operator (propagator) appearing in the formal solution of the *complete* Liouville equation, i.e. taking into account free particle motion *inside* the external field *and* interactions with other particles. However, as we have already said, this formal solution is practically useless, since its explicit computation would demand detailed knowledge of the solution of the $N-$body problem of motion itself!

A common way to overcome this problem is to assume that the zeroth order solution of the problem of motion (i.e. in the absence of interactions) *should suffice* in this order, since corrections due to the (weak) interactions should a *priori* enter *higher* orders in λ (do not forget that this is a λ^2 theory). This *'Markovian' assumption* amounts to considering:

$$f(t-\tau) \approx e^{-L_0\,\tau}\, f(t)$$

Therefore, if the free (i.e. collisionless) problem of motion has an explicit analytical solution, substituting into the master equation (1.13) will provide us with an explicit *linear* kinetic operator, acting on the distribution function $f(t)$.

As a matter of fact, the method outlined so far provides a linear differential operator of 2nd order, with respect to the phase-space variables $\{\mathbf{x}, \mathbf{v}\}$. In general, the coefficients appearing in this operator are expected to be time-dependent functions of $\{\mathbf{x}, \mathbf{v}\}$. A common procedure at this stage is an asymptotic evaluation of the kernel of the master equation (1.13), i.e. setting $t \to \infty$ in the upper limit of the time integral (thus essentially obtaining time-independent coefficients).

So far, we have defined the linear operator:

$$\mathcal{C}_\Theta\{f\} = n \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, L_I\, e^{\tilde{L}_\sigma^0\,\tau}\, e^{\tilde{L}_1^0\,\tau}\, L_I\, e^{-\tilde{L}_\sigma^0\,\tau}\, e^{-\tilde{L}_1^0\,\tau}\, \phi_{eq}(\mathbf{v_1})\, f \tag{1.14}$$

where $f = f(\mathbf{x}, \mathbf{v};\, t)$ [35]. The 'tilde' in $\tilde{L}_j^0$, introduced in this expression (only), is meant to point out that the external field *is* taken into account.

This kinetic operator, which was defined as the '$\Theta-$*operator*' in [46], [68], was shown therein *not to* possess the desired mathematical and physical properties. In specific, its action *does not preserve the positivity* of the probability distribution function $f(\mathbf{x}, \mathbf{v};\, t)$. This nuisance was not noticed for a long time, since most studies were limited to the reduced velocity space $\{\mathbf{v}\}$, where this problem does not arise.

Part of this thesis will be devoted to the explicit construction of this operator for electrostatic plasma and the study of its properties.

1.5.2 Influence of the external field

Remember that the action of the propagator in the *GME* has to be calculated in the presence of the external field. However, this point has not been paid due attention in the past[36]; the influence of the field on particle trajectories has quite often been neglected through physical arguments or even plainly omitted, apparently because of the complicated expressions it leads to. Notice, as an exception, the formal studies in [46][37], [110][38]. Nevertheless, as we have previously argued, the field *has to* be taken into consideration in the rigorous derivation of a kinetic equation, even if its influence may be less important in certain physical regimes (a fact which should then be rigorously justified, case by case).

1.5.3 Inhomogeneity effects

One more subtle point has to be discussed here. It should be noted, once more, that the $\Gamma-$space distribution function f may not only depend on particle velocity $\mathbf{v}$, but on position $\mathbf{x}$ as well. In the former case, i.e. for *homogeneous systems*, one generally obtains precisely a $(d+1)-$ (velocity + time) variable FP equation (in a $d-$dimensional problem, $d = 1, 2, 3$), while in the latter case, i.e. for *inhomogeneous systems*, one comes up with a $(2d+1)-$variable equation.

The delicate manipulations leading to the collision term in the latter case differ widely from one method to another and are still a matter of (controversial) discussion. Some authors prefer to start from the initial *GME* (1.9) and develop $\mathbf{x_2}$ around $\mathbf{x_1}$, while others plainly omit the non-homogeneous part in the collision term. In this thesis, we shall rather adopt a procedure which was first introduced in formal theories of *quantum open systems*. Further details will be given in the text (see Chapter 5).

[35]Remember that $L_I(\tau) = e^{\tilde{L}_\sigma^0 \tau}\, e^{\tilde{L}_1^0 \tau}\, L_I$ in (1.13); also, ϕ_{eq} is identically equal to $e^{-\tilde{L}_1 \tau}\, \phi_{eq}(\mathbf{v_1})$ since $\partial_t \phi_{eq} = \tilde{L}\, \phi_{eq} = 0$.

[36]See e.g. [85], [86].

[37]See the discussion in 1-13 therein.

[38]See the discussion in p. 233 therein.

1.5.4 Kinetic description of a test-particle problem

In an attempt to draw a rough picture of the purpose of this study, let us consider, in a test-particle problem, the collision term $\mathcal{C}_0\{f;\{\Pi\}\}$ acting on the distribution function $f = f(\mathbf{x}, \mathbf{v};\, t)$. $\{\Pi\}$ formally denotes the set of parameters which are characteristic of the dynamical problem, given the existence of external fields, e.g. cyclotron frequency Ω for a uniform magnetic field, characteristic frequency ω for a linear oscillator etc. or plainly *zero* if no field is present. Therefore, for instance[39],

$$\mathcal{C}_0\{\phi(\mathbf{v});\, 0\} = \mathcal{C}_0^{(V)}\{\phi\}$$

denotes the *homogeneous* free-of-field case, while

$$\mathcal{C}_0\{f(\mathbf{x}, \mathbf{v});\, 0\} = \mathcal{C}_0^{(V)}\{f\} + \mathcal{C}_0^{(X)}\{f\}$$

denotes its *inhomogeneous* analogue[40], and so forth.

Focusing on the description of plasma, let us clearly make our point concerning the structure of the kinetic equation. The 'Landau - Fokker - Planck' equation presented previously obeys the form:

$$\partial_t\, \phi = \mathcal{C}_0^{(V)}\{\phi(\mathbf{v});\, 0\}$$

in the unmagnetized electrostatic plasma case, or:

$$\partial_t\, \phi \,+\, m^{-1}\, \mathbf{F_{ext}} \frac{\partial \phi}{\partial \mathbf{v}} = \mathcal{C}_0^{(V)}\{\phi(\mathbf{v});\, \Omega\}$$

in the presence of a uniform external magnetic field. The external force $\mathbf{F_{ext}}$ is obviously the Lorentz force here. Both of these equations are mathematically sound. However, when an external EM field is present, some authors extrapolate to an equation of the form:

$$\partial_t\, \phi \,+\, m^{-1}\, \mathbf{F_{ext}} \frac{\partial \phi}{\partial \mathbf{v}} = \mathcal{C}_0^{(V)}\{\phi(\mathbf{v});\, \Omega = 0\} \tag{1.15}$$

(i.e. *not* taking the field into account in the collision term). Furthermore, in order to take into account space inhomogeneities and/or geometry, certain works introduce phenomenological generalizations of the above equations in the form:

$$\partial_t\, f \,+\, \mathbf{v}\nabla\, f \,+\, m^{-1}\, \mathbf{F_{ext}} \frac{\partial f}{\partial \mathbf{v}} = \mathcal{C}_0^{(V)}\{f;\, \Omega\} \tag{1.16}$$

(i.e. keeping only the *homogeneous* part in the collision term and adding a drift term to the *rhs*), *or* (even worse) of the form:

$$\partial_t\, f \,+\, \mathbf{v}\nabla\, f \,+\, m^{-1}\, \mathbf{F_{ext}} \frac{\partial f}{\partial \mathbf{v}} = \mathcal{C}_0^{(V)}\{f;\, \Omega = 0\} \tag{1.17}$$

[39] Remember that $f = n\, \phi(\mathbf{v};\, t) = \mathcal{C}_0^{(V)}$ in a uniform system.

[40] Since this is a *linear* collision operator, the modification in the in-homogeneous (space-dependent) case will consist of a differential operator involving space gradients (∇_i). Obviously: $\mathcal{C}_0\{\phi(\mathbf{v})\} = \mathcal{C}_\mathbf{0}^{(\mathbf{V})}\{\phi(\mathbf{v})\}$ since $\mathcal{C}_0^{(X)}\{\phi(\mathbf{v})\} = 0$.

(i.e. not taking the field into account in the collision term).

We therefore argue that the generalization of the above kinetic equations for a *non-uniform* plasma (which is exactly our target, here) should definitely bear the structure:

$$\begin{aligned} \partial_t f + \mathbf{v}\nabla f + m^{-1}\,\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} &= \mathcal{C}_0\{f(\mathbf{x},\mathbf{v});\,\Omega\} \\ &= \mathcal{C}_0^{(V)}\{f(\mathbf{x},\mathbf{v});\,\Omega\} + \mathcal{C}_0^{(X)}\{f(\mathbf{x},\mathbf{v});\,\Omega\} \end{aligned}$$

This is the form of the kinetic equation we are after.

1.6 Outline of the thesis

The outline of this thesis goes as follows.

First, in the next Chapter, this introduction is completed with a brief account of notions and definitions specifically concerning the description of plasma as a statistical-mechanical system.

The main body of this text is divided into four parts.

The first, *Part A*, is devoted to the general formalism to be used in the kinetic description of a test-particle problem. In Chapter 3, we study the dynamics of a small subsystem off-equilibrium weakly interacting with a large heat bath. Given an explicit dynamical problem (particle inside a force field), our aim is the derivation of a kinetic equation, describing the evolution in time of the phase-space density function. As a starting point we take the microscopic equations of motion and we derive the associated *BBGKY* hierarchy of evolution equations for appropriate reduced distribution functions. A Non-Markovian Generalized Master Equation (GME) is obtained and discussed. In Chapter 4, a Fokker-Planck-type equation is obtained from the *GME* as a "markovian" approximation. This kinetic operator (the '$\Theta-$*operator*') is constructed explicitly. All coefficients in it are explicit functions of the dynamical variables $\{\mathbf{x},\mathbf{v}\}$ *and* the external field. Furthermore, all coefficients in it explicitly depend on one's choice of (i) the form of the inter-particle interaction potential $V(r)$ and (ii) the form of the homogeneous equilibrium distribution function of the reservoir state ϕ_{eq} (typically, yet *not necessarily*, a Maxwellian state). We show that, in general, such an equation does not preserve the positivity of the distribution function (d.f.) $f(\mathbf{x},\mathbf{v};\,t)$. This problem, which is generic - regardless, that is, of the particular dynamical problem considered - has been pointed out in the theory of open quantum-mechanical systems where possible remedy to the situation was suggested. An analytical procedure introduced therein, essentially amounting to time-averaging the evolution operator with respect to free-particle motion, defines the '$\Phi-$ *operator*', which we construct in Chapter 5, for an arbitrary dynamical problem.

The second part of the thesis, *Part B*, consists in the application of the formalism in the kinetic description of *plasma*, which is embedded in an external magnetic field. Considering a uniform magnetic field, the first ($\Theta-$) kinetic operator defined previously, is explicitly constructed in Chapter 6. The equation thus obtained is shown to be in agreement with previous results, where coefficients are expressed in the form of infinite series involving Bessel functions of the first kind. All coefficients are functions of particle velocity *and* the cyclotron frequency Ω. Non-preservation of the distribution function positivity is again demonstrated in this case. The construction of the second ($\Phi-$) operator, in Chapter 7, yields a new kinetic equation for plasma, including space gradients and, in particular, a new diffusion term. This equation is thus suggested as the basis of the detailed study of the properties of magnetized plasma, as compared to the *unmagnetized* (Landau) case, presented in chapter J. In Chapter 8, an alternative method of derivation of the coefficients is proposed, by explicitly assuming the reservoir state ϕ_{eq} to be Maxwellian and the interaction potential $V(r)$ to be of Debye type (and leaving the time $\tau-$ integration for the end). This procedure leads to a simpler computable set of exact expressions (*no* infinite series are involved); furthermore, this calculation (also provided for a multiple species plasma) is also valid for a finite upper τ-integration limit t.

In *Part C*, the analytical results of the previous part are thoroughly analyzed. In Chapter 9, the coefficients involved in the description of magnetized plasma are studied with respect to the physical parameters they depend upon, *including* the magnitude of the magnetic field. In Chapter 10, we attempt to solve the kinetic equation exactly, treating the charged-particle collision mechanism as a 3-dimensional *Ornstein - Uhlenbeck random process*. The evolution of observable quantities (i.e. average values of microscopic quantities) in time, under the action of the new kinetic operator, is considered in Chapter 11. Finally, the results obtained are summarized and discussed in the concluding chapter.

Finally, a critical discussion of related literature is carried out and some concluding remarks are gathered in Chapter 12, ending this thesis.

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Chapter 2

Characteristic scales in magnetized plasma

Summary

We briefly review notions and definitions involved in the description of plasma as a statistical-mechanical system. Space and time scales are defined and different plasma regimes are discussed, with respect to an external magnetic field.

It appears that the radical element responsible
for the continuing thread of cosmic unrest
is the magnetic field.

`Eugene Newman Parker`
in *Cosmical Magnetic Fields*

2.1 Introduction

From a statistical-mechanical point of view, plasma is quite particular a system, as compared to the standard image of rarefied gases. Charged particles interact with external fields and with themselves. Long-range Coulomb-type interactions are responsible for a realm of new phenomena (e.g. collective effects) but also for a great deal of mathematical complication involved in a kinetic study of plasma. The most common example of the latter is the notorious divergence of the momentum transfer integral (Coulomb logarithm), which already appears once one tries to solve the two-body problem for Coulomb interactions.

Plasma is characterized by different space and time scales, the relative magnitude of which depends on physical parameters e.g. density, temperature, external fields etc. These parameters may vary over a wide range of values (see figure 2.1), whose combination defines several plasma regimes. A theoretical study of non-equilibrium phenomena therefore imposes a precise definition of the region of validity of (and assumptions underlying) one theory or another.

The aim of this chapter is to provide a brief account of notions and physical quantities which are necessary for the study of plasma as a statistical-mechanical system. Notions introduced below are standard in plasma theory, and details can be found in relevant textbooks (see e.g. [3], [22]); [38].

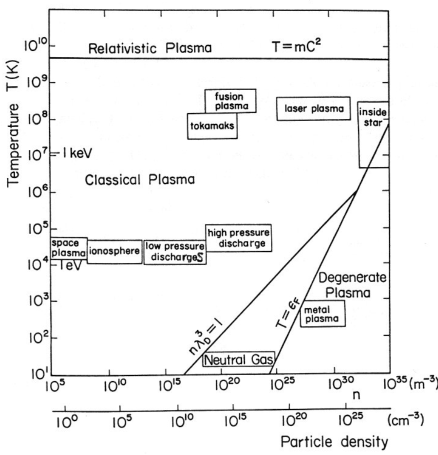

Figure 2.1: Plasma classification, according to the values of density n and temperature T .

2.2 Of dimensions and scales in plasma

In order to gain insight to the system studied, let us start by a simple dimensional analysis. Since electrostatic plasma is an ensemble of interacting charged particles (charge e i.e. $-|q_e|$ for electrons, $+Z\,|q_e|$ for positive ions etc.), we would expect any quantity defined to depend on particle mass m, charge[1] e, density n and temperature (mean kinetic energy)[2] $k_B T$. Finally, certain scales may be imposed by external fields; this issue will be addressed a little later.

Let us assume that any quantity $\mathcal{Q}$ will be of the form:

$$\mathcal{Q} = m^\alpha \, n^\beta \, T^\gamma \, e^\delta$$

The dimensions of $\mathcal{Q}$ sought will impose the values of the exponents.

2.2.1 Length scales

First, let us try to define a quantity, say $\mathcal{L}$, with the dimensions of *length*. Simple dimensional arguments show that it should be equal to:

$$\mathcal{L} = \left(\frac{e \, n^{1/6}}{T^{1/2}} \right)^\delta n^{-1/3} \tag{2.1}$$

Mean inter-particle distance $\langle r \rangle$: The simplest choise would be to set $\delta = 0$ in the above relation; we obtain the mean inter-particle distance:

$$\langle r \rangle \; = \; n^{-1/3}$$

This length is characteristic of the average repartition of particles in space and takes, of course, higher values for lower densities.

Distance of closest approach: Let us try to obtain a length which is independent of particle density n. By setting $\delta = 2$ in the above relation; we obtain the *Landau length* [2]:

$$r_L = \frac{e^2}{T}$$

This length can be interpreted (up to a numerical factor) as the distance between two point charges which corresponds to a potential energy which is equal in value to the mean kinetic (thermal) energy. It is also related to the value of the collision impact parameter which would result in a deflection by 90 [5]. Particles in real charged particle 'gases' *never* get that close (not even a few orders of magnitude as far!), so the inverse of this distance has been proposed as a realistic alternative to infinity in the upper boundary of the (diverging) Coulomb integral. Clearly, the small value of r_L is related to the validity of the weak-coupling hypothesis (to be discussed below) .

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[1] Notice, for the sake of rigor, that charge e should be replaced by $e/\sqrt{\epsilon_0}$ everywhere in the following, for consistence with the SI system of units (i.e. $e^2 \doteq e^2/\epsilon_0$); ϵ_0 denotes the dielectric constant in vacuum.

[2] Boltzmann's constant k_B will be omitted throughout this text.

Mean-free-path λ_{mfp}: Setting $\delta = -4$ in the above relation, we obtain the *mean-free-path*:

$$\langle \lambda_{mfp} \rangle = \frac{T^2}{n\, e^4}$$

This length represents the average distance covered between two successive collisions. It is related to the *relaxation time*, or *inverse collision frequency*; see below.

Debye length r_D: Setting $\delta = -1$, we obtain - up to a numerical factor - the *Debye length*:

$$r_D = \left(\frac{T}{4\pi\, n\, e^2} \right)^{1/2}$$

This is a very important parameter for the study of plasmas, since it sets the range of correlations between charged particles. In specific, the rigorous consideration of the Poisson equation, taking into account charge distribution in space, results in interactions being *'screened'* by a factor e^{-r/r_D} (see e.g. [22], [38]). This cumulative charge screening effect therefore defines, roughly speaking, an interaction sphere, the *Debye sphere*, around the particle; r_D will be the radius of this sphere.

For all real plasmas, the Debye length takes values which are very small, compared to macroscopic scales of the system; see figure 2.2. Furthermore, in most plasmas, the order of magnitude of r_D is well below that of λ_{mfp}; this fact expresses the plausibility of the weak-coupling hypothesis (to be defined below).

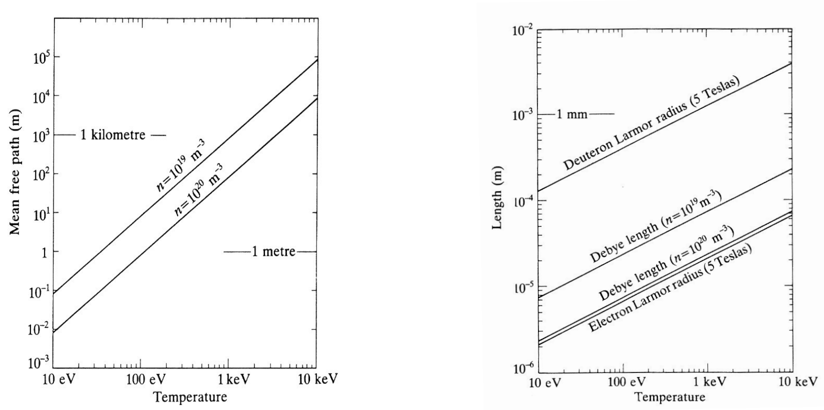

Figure 2.2: The mean-free-path λ_{mfp}, the Debye length r_D and the Larmor radius ρ_L, plotted against temperature T. Notice that the values of λ_{mfp} for a given density may range from a few millimeters to a few kilometers, depending on temperature T. Also notice that r_D may be quite close to ρ_L, for higher values of density and magnetic field.

2.2.2 Time scales - characteristic frequencies

Let us now construct a quantity, say $\mathcal{T}$, with the dimensions of *time*. We are thus seeking time scales, though the traditional plasma description rather consists in defining typical frequencies, instead. These will be defined as the inverse of our characteristic times.

Once more, simple dimensional arguments show that it should be equal to:

$$\mathcal{T} = \left(\frac{e\, n^{1/6}}{T^{1/2}} \right)^{\delta} n^{-1/3} \left(\frac{m}{T} \right)^{1/2} \tag{2.2}$$

Upon simple inspection, we see that the right-hand-sides of formulae (2.1), (2.2) are related by a factor which is precisely the *thermal velocity*, defined by:

$$v_{th} = \left(\frac{T}{m} \right)^{1/2}$$

Therefore, not surprisingly, *all* typical time scales $\mathcal{T}$ will generally be equal to corresponding length scales $\mathcal{L}$ over the thermal velocity v_{th}.

Proceeding as above, we may impose certain values on the δ exponent.

Inverse collision frequency ν: Setting $\delta = 0$ in the above relation, we obtain the *mean time-interval* between collisions (in a hard sphere picture), most often defined through its inverse, which is intuitively interpreted as a *collision frequency* ν:

$$\nu = \frac{T^{1/2}\, n^{1/3}}{m^{1/2}} = \frac{v_{th}}{\lambda_{mfp}}$$

It is related to the average time it takes the particle to cover a distance equal to the *average inter-particle distance*.

Rigorously speaking, ν has the meaning of a collision frequency only in a hard-sphere system. A different definition of the collision frequency is more familiar from studies of charged particle ensembles; it corresponds to the choice of $\delta = -4$:

$$\nu_{Coulomb} \sim \frac{n\, e^4}{m^{1/2}\, T^{3/2}} = \frac{v_{th}}{T^2/ne^4} \tag{2.3}$$

This expression (up to a numerical factor) can be rigorously derived either by combining the Coulomb momentum transfer integral with the mean velocity provided by a Maxwellian distribution [22] of from kinetic theory [5], [51]. We shall later see that this expression is indeed recovered in our study.

Plasma oscillation period / plasma frequency: Setting $\delta = -1$, we obtain (up to a numerical factor) the *period of plasma oscillation* [3], [22] i.e. the inverse of the *plasma (Langmuir) frequency* ω_p:

$$\omega_p = \left(\frac{4\pi\, n\, e^2}{m} \right)^{1/2} = \frac{v_{th}}{r_D}$$

This is a widely discussed intrinsic parameter in plasmas, so we will not go into extended comments. Let us remark, however, that its inverse ω_p^{-1} roughly expresses the time the particle needs to cross a Debye sphere.

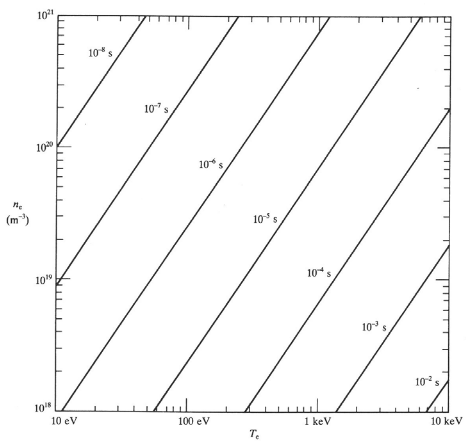

Figure 2.3: The plasma oscillation period (inverse plasma frequency ω^{-1}) for a given density n and temperature T.

'Landau' characteristic time: Setting $\delta = 2$, we obtain a characteristic time related to the distance of closest approach between particles (see above); it is the only characteristic time which is independent of density n:

$$\tau_L = \frac{m^{1/2}\, e^2}{T^2} = \frac{v_{th}}{r_L}$$

This is a very small time, compared to all the others (see scaling below).

2.2.3 Magnetic field-related scales

The existence of external magnetic fields may impose an additional set of space-time- scales. In the case of a uniform magnetic field $\mathbf{B}$, only its magnitude B is relevant, so we need to define:
- the *Larmor* (or *cyclotron* or *gyro-*) *frequency*:

$$\Omega_c = \frac{eB}{mc} = \frac{2\pi}{T_c}$$

where T_c denotes the gyration period,
and
- the *Larmor radius*

$$\rho_L = \frac{v_{th}}{\Omega_c}$$

representing the radius of the particle's gyrating motion.

2.2.4 Dimensionless parameters

To end this dimensional investigation, let us seek the general form of *non-dimensional* quantities, say Π, characterizing an electrostatic plasma. Using the same method as above, we find that all such quantities appear to be powers of a fundamental one:

$$\Pi_\delta = \left(\frac{e\, n^{1/6}}{T^{1/2}} \right)^\delta$$

Plasma parameter: Setting $\delta = 2$ we find the *plasma parameter*[3] g:

$$g = \frac{e^2\, n^{1/3}}{T}$$

This parameter expresses the ratio of the average potential energy $e^2/\langle r \rangle$ to the average kinetic (thermal) energy $k_B\, T$ (see discussion below).

Some authors prefer to set $\delta = 3$ in the above relation for Π_δ, thus obtaining a different (small) plasma parameter:

$$\mu_p = \left(\frac{e\, n^{1/6}}{T^{1/2}} \right)^3 \sim \frac{1}{n\, r_D^3} \equiv \left(\frac{\langle r \rangle}{r_D} \right)^3$$

We see that this parameter expresses the inverse number of particles inside a Debye sphere; in most plasmas, it takes a very small value $\mu_p \ll 1$, which makes it a convenient candidate for a perturbation expansion parameter[4]. Clearly, the two smallness parameters are related: $\mu_p = g^{3/2}$.

A magnetic-field-related dimensionless parameter: As obvious, the ratio of *all* of the above mentioned characteristic lengths (or frequencies), combined by two, provides a meaningful dimensionless parameter. Of particular importance to us is the parameter, say λ, defined as the ratio of the Larmor radius to the Debye length[5]:

$$\lambda = \frac{\rho_L}{r_D} \equiv \frac{\omega_p}{\Omega_c}$$

The magnitude of this parameter is related to the influence of the external magnetic field on all quantities involved in the description of plasma; not quite surprisingly, it will often spontaneously appear in this text.

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[3] The definition and notation used in literature with respect to this parameter may vary by a multiplication constant or an exponent; here, we chose an expression which can be conveniently expressed as the ratio of length/time scales defined above.

[4] Notice that μ_p (for $\delta = 3$) can also be interpreted as: $\mu_p \sim \frac{r_D}{\lambda_{mfp}} = \frac{\nu}{\omega_p}$ (check by making use of previous definitions); this implies, by the way: $r_D^4 = \langle r \rangle^3\, \lambda_{mfp}$, as can be easily verified.

[5] The dimensionless parameter λ denoted here should not be confused with the coupling 'weakness' parameter mentioned elsewhere.

Notice that, from its very definition, a familiar reader may be tempted to relate λ, defined above, to the ratio of the speed of light c to the Alfvén velocity v_A, related to ion (denoted by 'i') frequencies:

$$v_A = \frac{\Omega_c^i}{\omega_{pi}} c = 2.18 \times 10^{16} \frac{B}{(n_i/A)^{1/2}} \qquad m/s$$

where B is the field, in $Tesla$, n is particle density (expressed in m^{-3}) and A is the atomic mass [51]. One might thus erroneously deduce that λ is obviously well above unity: $\lambda_i = \frac{\omega_{pi}}{\Omega_c^i} \gg 1$. For instance, for a hydrogen plasma ($A = 1$) with $n = 10^{20}\, m^{-3}$, $B = 3\,T$, v_A is of the order of $0.015\,c$ so λ_i is roughly close to 45. This would immediately imply that motion is *hardly* curved within a Debye sphere, since $\rho_L \approx 45\, r_D$. This may be virtually true for heavy *ions* (only), in practical terms; nevertheless, in a fully ionized electron-ion plasma, such that $n_e \approx n_i$, the value of λ_e for *electrons* will be lower than λ_i by a factor equal to $\sqrt{m_e/m_i}$, so it may be around *unity* or even less.

2.2.5 Plasma regimes

Keeping the plasma parameter μ_p as the elementary *dimensionless* parameter of relevance in electrostatic plasma, we may express all of the above quantities (e.g. lengths/ times) in terms of a basic one (i.e. characteristic length/time, respectively) times a power of μ_p. The following scaling naturally appears:

$$r_L \ll \langle r \rangle \ll r_D \ll \lambda_{mfp}$$

or the equivalent:

$$\tau_L \ll \nu^{-1} \ll \omega_p^{-1} \ll \nu_{Coul}$$

in the following orders of magnitude, respectively:

$$\mu_p^{2/3} \ll \mu_p^0 \ll \mu_p^{-1/3} \ll \mu_p^{-4/3}$$

As we see, the relative magnitude of the various characteristic quantities is *fixed* by the value of μ_p (i.e. for a specific value of n and T). In presence of a magnetic field, the appearance of one more space (time) scale, representing the order of magnitude of the Larmor radius ρ_L (the cyclotron period $\sim \Omega_c^{-1}$, respectively) defines certain regimes, depending that is on the position occupied by this new scale in the above hierarchy.

A thorough discussion of magnetized plasma regimes can be found in [6] (see Ch. 10 therein). Of particular importance to us, here, will be the relative magnitude of magnetic field-related to charge-screening (Debye) characteristic scales, expressed by λ, defined above. Three distinct plasma *regimes* may be defined, in terms of λ (see figure 2.4 [6]).

[6] Of course, even though the heuristic picture in figure 2.4 conveys the qualitative idea of our argument, it is not quantitatively correct, since large-angle scattering, such as depicted, is excluded in the weak-coupling approximation.

(a) $\lambda \gg 1$ (vanishing-field-limit): the Larmor radius ρ_L is far larger than r_D. In consequence, particle motion between collisions is practically rectilinear, so the (unmagnetized) Landau-type approach may be sufficient in this case.

(b) $\lambda \approx 1$ (finite magnetic field): the Larmor radius ρ_L is comparable to r_D and particle motion between collisions is *strongly* curved. Spiral motion should be rigorously taken into account in the calculation of particle correlations: this regime is of particular interest to us, here[7].

(c) $\lambda \ll 1$ (infinite magnetic field): the Larmor radius ρ_L is well below the value of r_D; particles gyrate a number of times between successive collisions. The gyrophase plays no role, so a *guiding center approximation* [6] should be sufficient in this case.

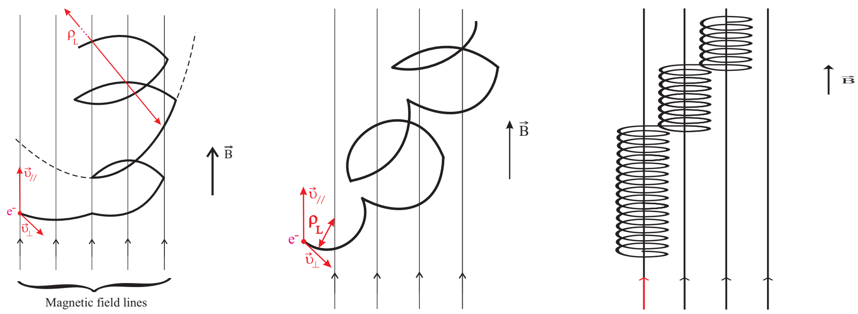

Figure 2.4: A heuristic sketch of the trajectory of colliding (negatively charged) particles, in the presence of a uniform magnetic field. In between successive collisions, particles rotate around the magnetic field lines. The typical length scale of interaction (Debye length r_D) is compared to the gyration scale (Larmor radius ρ_L). Three distinct regimes are depicted: (a) $\rho_L \gg r_D$, (b) $\rho_L \approx r_D$ and (c) $\rho_L \ll r_D$.

It should be noted that *all* of these situations may be present in today's real plasmas. As a matter of fact, different cases may simultaneously apply to electrons and positive ions within the *same* plasma, as pointed out in the previous paragraph.

2.3 The weak-coupling assumption (w.c.a.)

The dimensionless *plasma parameter* g defined above has a physical meaning of major importance in plasma kinetic theory. Consider the ratio of the average potential to the average kinetic energy of a charged particle in the system described so far:

$$\frac{\langle E_{pot} \rangle}{\langle E_{kin} \rangle} = \frac{\frac{e^2}{2\langle r \rangle}}{\frac{3}{2}T} \sim g \tag{2.4}$$

[7]This regime corresponds to *'Regime E'* defined in §10.1 in [6].

We see that g precisely expresses the ratio of mean values of inter-particle interaction (*'coupling'*) to kinetic energy. In an alternative, physically transparent, picture, we see that g is inversely proportional to $(n\, r_D^3)^{2/3}$; remember that $N = n\, r_D^3$ expresses the number of particles within a Debye sphere; therefore, the more these particles, the lower the value of g.

The vast majority of classical plasmas studied today *are* indeed *weakly-coupled* (sometimes called *'ideal'*), meaning that the value of g is very small. In fusion plasmas, for instance, we have typically a few hundreds particles inside a Debye sphere, leading to a value of g of the order of 10^{-3} or less[8].

For the sake of rigor, let us mention that *strongly*-coupled plasmas *do* exist (see e.g. [126] for an exhaustive recent report): a strong ratio of potential-to-kinetic energy may result in the formation of plasma lattices primarily in fusion device walls [120]; such plasma 'crystals' have indeed been reproduced in laboratory and their formation is currently thought to be enhanced by the existence of charged dust grains in the plasma (see e.g. [118]).

2.4 Non-dimensional kinetic description - relaxation time

It should be pointed out that the physical system described above presents a *cylindrical* symmetry, induced by the presence of the magnetic field. We therefore expect this symmetry to be reflected in the form of the kinetic equation. Furthermore, the system is naturally characterized by a number of intrinsic parameters (*particle density* n, *temperature* T), it is 'equipped' with an (long-range) electrostatic *interaction law* and is subject to an *external magnetic field* $\mathbf{B}$. As extensively discussed above, these facts imply a set of space- and time-scales characterizing our system. Therefore, the interplay between these scales, reflecting the relative strength, for instance, of charge screening and gyration phenomena, is expected to appear in all formulae. As mentioned previously, different regimes may be defined and discussed, in terms of the value of these parameters.

These considerations may be supported by a non-dimensional kinetic description of a test-particle problem. The generic - dimensionless - form of a kinetic equation may be obtained by scaling all variables over appropriate characteristic quantities (these may vary from one system to another); it is explicitly presented in the Appendix. One more time scale appears naturally as a result of this dimensional analysis: the *relaxation time* τ_R, representing the typical time evolution scale of the collision term. As argued in the Appendix (see §A), the general form of τ_R takes into account all the problem's fundamental physical

[8] It is worth mentioning that certain authors have derived and investigated the properties of binary collision integrals for so-called moderately dense or dense plasma, i.e. for small values of the Coulomb logarithm Λ (beyond the weak-coupling hypothesis) [61], [95], [104], often treating the notorious divergences by considering alternative cutoffs, derived from dynamical arguments (see e.g. [79]); it has even been suggested that the w.c.a. may be the source of the divergence; see e.g. [98]. We do not discuss those works here.

parameters, *including* the external field. In the case of plasma, we obtain:

$$\tau^{\alpha,\alpha'}_{R,plasma} = \frac{m_\alpha^2\, v_{th,\alpha}^2}{n_{\alpha'}\, e_\alpha^2 e_{\alpha'}^2\, k_D\, \tau_0} = \cdots = \frac{m_\alpha\,(T_\alpha)^{3/2}}{n_\alpha^{3/2}\, e_\alpha^5\,(4\pi)^{1/2}}\frac{1}{\tau_0} \tag{2.5}$$

where τ_0 is a time-scale, which may be related to the field, if any (e.g. Ω_c^{-1} for magnetized plasma, ω_p^{-1} for *non*-magnetized plasma and so forth).

It is interesting to see that the inverse of the relaxation time, and therefore the order of magnitude of the collision term (see above), is related to the Coulomb *collision frequency* $\nu_{Coulomb}$ *and* to the external field, through:

$$(\tau_R^{\alpha,\alpha})^{-1} = \frac{n\, e^4\, k_D\, \tau_0}{m^2\, v_{th}^2} = \cdots \sim \frac{n\, e^4}{m^{1/2}\, T^{3/2}}\,(k_D\, v_{th}\, \tau_0) \sim \nu_{Coulomb}\, \xi \tag{2.6}$$

(we have set α, α') where ξ is defined as the quantity within parenthesis; see in the Appendix for details.

2.5 Focusing on the system studied

2.5.1 Description

The physical system which we will focus on, consists of a charged test-particle, say σ, moving against (and weakly-coupled to) a homogeneous background plasma consisting of a large number, say N, of particles (the 'reservoir' R), embedded in an external magnetic field, taken to be uniform for simplicity. Interactions (typically long-range) are assumed to be weak.

In an equivalent manner, one may consider *a few* (test-)particles, say N_σ, weakly-interacting with a background reservoir of N particles, in thermal equilibrium. Due to fluctuations, these particles have found themselves in a dynamic state (possibly inhomogeneous) off equilibrium; they are therefore relaxing back to thermal equilibrium under the influence of the bath. Both subsystems are subject to the external magnetic field. As $N_\sigma \ll N$, the test-particles are assumed to interact with the bath, but *not* within themselves. Furthermore, one assumes that these particles are so rarely distributed all around the (homogeneous) background plasma, that they will not influence the global equilibrium state.

Throughout this text, α will denote the *'species'* of the *test-particle(s)*[9] (mass $m_\alpha = m$, charge $e_\alpha = e$), while α' will denote that of the *reservoir-particles*[10].

A plasma may consist of several populations belonging to 'species', say $\{\alpha'\} \equiv \{\alpha, \beta, \gamma, ... \in \{e, i, ...\}\}$, i.e. electrons, ions, ... , of mass $m_{\alpha'}$ and

[9] i.e. electron, ion, ...

[10] So, α may, or may (generally) not, be the same as α'. To make things clear, this notation only helps in describing (single-)particle dynamics (e.g. via the value of Ω'_α, in plasma). It should not be forgotten that the test-particle subsystem forms a *'population'* apart, since its distribution (temperature, mean) is different from that of R-particles of the same species.

charge $e_{\alpha'}$. We obviously have:

$$\sum_{\alpha'} N_{\alpha'} = N$$

while overall charge neutrality implies:

$$\sum_{\alpha'} N_{\alpha'}\, e_{\alpha'} = 0$$

For instance, in a two component fully ionized ${}^{A}_{Z}X$ plasma (X being some element e.g. hydrogen ${}^{1}_{1}H$, helium ${}^{4}_{2}He$, ...), consisting of electrons ($m_1 = m_e$, $e_1 = -e$) and ions (nuclei, $m_2 = Z\, m_p + (A - Z)\, m_n$, $e_2 = +Z\, e$), the above relations read[11]:

$$N_1 + N_2 = N\,, \qquad N_1(-e) + N_2(+Z\,e) = 0$$

implying:

$$N_1 = Z\, N_2 = \frac{Z}{Z+1} N$$

Such is the general plasma picture. Nevertheless, in the name of simplicity in description, plasma is often modeled as a collection of electrons moving against a *'frozen'* ion background. In this simplified picture, only collisions between *same-species* particles are present, i.e. $\alpha = \alpha'$, since all particles obey the same dynamics. From a technical point of view, the formulae for a single component system (e.g. electron plasma) correspond to the equal-species term in the general case [12].

2.5.2 Characterization

Let us focus on a set of parameters in order to clarify the region of validity of this study. We have chosen a temperature of $T = 10\, KeV$ and a particle density of $n = 10^{14}\, cm^{-3} = 10^{20}\, m^{-3}$; these values are typical of fusion plasmas [9], [43], [51] (see figure 2.1).
This choice implies:

- a *mean inter-particle distance*:

$$\langle r \rangle = n^{-1/3} = 2.15 \cdot 10^{-7}\, m$$

and

- an electron *Debye length*:

$$r_D^e = v_{th}/\omega_p \approx 7.43 \cdot 10^3\, T^{1/2}\, n^{-1/2}\, [m] \approx 7.43 \cdot 10^{-5}\, m$$

[11] More rigorously, N should be replaced by $N_{tot} = N + N_\sigma$ in these formulae (however, $N_{tot} \approx N$ since $N_\sigma \ll N$).

[12] In a multiple species system the collision term $\mathcal{C} = \mathcal{C}^\alpha$ has the form of a sum over particle populations α':

$$\mathcal{C} = \mathcal{C}^\alpha = \sum_{\alpha'} \mathcal{C}^{\alpha,\alpha'} = \mathcal{C}^{\alpha,\alpha} + \sum_{\alpha' \neq \alpha} \mathcal{C}^{\alpha,\alpha'}$$

(T in eV, n in m^{-3}). As a consequence, the *plasma parameter* is:

$$\mu_p = \left(\frac{\langle r \rangle}{r_D^e}\right)^3 \approx 2.44 \cdot 10^{-8}$$

(or $g = \mu_p^{2/3} = 8.4\,10^{-6}$) , so this is a *weakly coupled* plasma.

The electron thermal velocity is:

$$v_{th}^e \approx 4.19\ 10^5\ T^{1/2}\ [m/s] \approx 4\ 10^7 m/s \approx 0.1\,c$$

(T in eV) so a *non-relativistic* treatment should suffice.

Furthermore, the *de Broglie length*[13] $\lambda_{dB} = h/\sqrt{2\pi m_e k_B T}$ is :

$$\lambda_{dB} \approx 2.76\,10^{-8}\,T^{-1/2}\ [cm] \approx 3\ 10^{-10} cm \approx 10^{-5}\,\langle r \rangle$$

(T in eV; h is Planck's constant) so this is indeed a *classical* plasma.

Finally, the electron *Larmor radius* is

$$\rho_L^e \approx 2.38\,10^{-6}\,T_e^{1/2}\,B^{-1} \quad [m] \quad \approx 2.38\,10^{-4}\,B^{-1} \quad [m]$$

(B in Tesla, T in eV). For a value of, say, $B = 1\,T$, we have $\rho_L^e \approx 2.38\,10^{-2}\,cm$; this implies a Larmor radius to Debye length ratio of only 3.2, which suggests a rather non-negligible particle trajectory curvature within the size of a Debye sphere (hence the importance of taking into account the magnetic field in calculating trajectories between collisions); see figure 2.4[14]. The corresponding quantities for ions would be:

$$\rho_L^i \approx 1.02\,10^{-4}\,Z^{-1}\,A\,T_e^{1/2}\,B^{-1} \quad [m] \quad \approx 1.02\,10^{-2}\,Z^{-1}\,A\,B^{-1} \quad [m]$$

(B in Tesla, T in eV; Z: positive charge, A: atomic number, e.g. $Z = 2$, $A = 4$ in ${}^4_2He$.); this implies a Larmor radius to Debye length ratio of

$$\frac{\rho_L^i}{r_D^i} = 1.36\,10^{-8} Z^{-1}\,A\,n^{-1/2}\,B^{-1}$$

(B in Tesla, n in m^{-3}), say typically $\rho_L^i/r_D^i \approx 136$ for protium nuclei. This ratio is related to the characteristic parameter dimensionless λ that we will define later in the text (ch. 8). Notice that λ is independent of temperature:

$$\lambda \equiv \sqrt{2}\,\frac{\rho_L^\alpha}{r_D^\alpha} = \sqrt{2}\,\frac{\omega_p^\alpha}{\Omega_c^\alpha} = (8\,\pi\,m_\alpha\,c^2)^{1/2}\,n_\alpha^{1/2}\,B^{1/2} \tag{2.7}$$

See that $\lambda_i/\lambda_e = \sqrt{m_i/m_e} \approx 43$, so positive ions are practically inside the 'unmagnetized' region depicted in figure 2.4a. For the parameter values listed here, for instance, we have:

$$\lambda_e = 4.531 \times B^{-1}\,, \qquad \lambda_i = 194.1531 \times B^{-1}$$

[13] Remember that the condition for quantum effects to be of relevance is: $\lambda_{dB} \simeq \langle r \rangle$; see e.g. in [127]. This condition is only fulfilled in exotic situations like astrophysical plasmas, occurring in regions of gravitationally collapsing stars.

[14] Qualitatively speaking, we are in case (b) therein.

(B in Tesla).

In conclusion, this study refers to a *weakly coupled, fully ionized, multiple-species, non-relativistic classical plasma.*

2.5.3 Assumptions of our study

From the discussion carried out so far, it has been made clear that kinetic theory is a rather complex domain of physics and, so to speak, a very *"ambitious"* one: it aims at drawing as much information as possible from a system derived via a *reduction* of the complete data space provided by a real system to a smaller sub-space one can manipulate. Information loss 'ON THE WAY' is inevitable and what one desires is to compensate this loss with analytical tractability and, ideally, plausibility of results via experimental confirmation.

The scope of this study definitely lies within the range of validity of specific assumptions, more or less inevitable, some quite technical, yet all physically sound.

First, only *binary* interactions are considered; three-body effects are neglected, so *plasma polarization* is not accounted for, here[15].

Second, inter-particle interactions are assumed to be *weak*. Physically speaking, this excludes the occurrence of large angle deviations during particle collisions (see fig. 1.1b) and relies on the assumption of sufficiently low densities and/or high enough temperatures (in a classical system[16]). This weak-coupling-approximation (*w.c.a.*) is indeed well-founded in fusion plasmas, as has already been said; see the discussion e.g. in [7]. Exceptions to this *w.c.a.* (actually manifested via small values of the plasma parameter or high values of the Coulomb logarithm) are densely coupled plasmas in (i) laboratory, (ii) fusion device walls (formation of dust quasi-lattices), (iii) *some* astrophysical situations (in strong gravitational fields). These issues will not be treated here.

Third, *correlations* between particles at *'the beginning of time'* ($t = 0$) will be neglected. This is a kind of initial *molecular chaos* hypothesis, which is assumed[17] in most kinetic studies.

2.6 Conclusions

We have examined the characteristic lengths and times for magnetized plasma, and defined different plasma regimes in terms of magnetic field scales as related to intrinsic plasma parameters. This analysis was necessary for the dissemination of results to be presented in forthcoming chapters.

---

[15]Some attempts to take magnetized plasma polarization into account (via a BLG-like equation) [54], [70] are quite remarkable, yet limited to velocity space (they have not investigated space-related phenomena at all). Note a study of the dielectric constant as related to Brownian motion, in [56], [108].

[16]The image is somewhat different in the quantum case; see the pedagogical discussion in [127].

[17]and sometimes criticized, see e.g. the discussion in [58].

Part A
The Formalism

When you go out for the truth,
leave elegance to the tailor...
Albert Einstein

Chapter 3

Statistical-mechanical description - the Generalized Master Equation

Summary

Relying on first microscopic principles, a general formalism is presented, describing the evolution in time of the distribution function of a test-particle weakly interacting with a large heat bath in homogeneous equilibrium. This method is valid for any particular dynamical problem, provided that an explicit solution of the 'free' (collisionless) problem of motion is given.

Nature, it seems, is the popular name
for milliards and milliards and milliards
of particles playing their infinite game
of billiards and billiards and billiards.

Piet Hein
in *Atomyriades*

3.1 The model

Let us consider a test-particle (t.p.), denoted by σ, surrounded by (and weakly coupled to) a homogeneous background system of N particles (the 'reservoir' R). Both subsystems are subject to an external force field. Interactions will be assumed to be weak (see in the previous chapter for definitions).

Let $\{\mathbf{X_j}\} \equiv \{\mathbf{x_j}, \mathbf{v_j}\}$ be the microscopic variables denoting the position and velocity of the $j-$th particle ($j = 1, 2, ..., N, \sigma$). The Hamiltonian of the system is of the form:

$$H = H_R + H_\sigma + \lambda\, H_{int} \tag{3.1}$$

H_σ denotes the Hamiltonian of the test-particle, while H_R denotes the Hamiltonian of the reservoir:

$$H_R = \sum_{j=1}^{N} H_j + \sum_{j<n} \sum_{n=1}^{N} V_{jn} \tag{3.2}$$

H_j is the free (*single-particle*) Hamiltonian corresponding to the $j-$th particle. For any single particle, being *either* the t.p. σ *or* an $R-$ particle (so $j = 1, 2, ..., N$ *or* σ here), the one-body Hamiltonian H_j is, *in principle*, of the form:

$$H_j = \frac{1}{2} m_j v_j^2 + \Phi(\mathbf{x_j})$$

where Φ is a potential energy function accounting for the external field. In case of a magnetic field, the field is "hidden" in the *non-canonical* phase-space variable $\{\mathbf{x_j}, \mathbf{v_j}\}$, instead, through the standard transformation [17]:

$$\{\mathbf{x_i}, \mathbf{v_i}\} \equiv \{\mathbf{x_i}, \frac{1}{m_i}[\mathbf{p_i} - \frac{e_i}{c}\mathbf{A}(\mathbf{x_i})]\}$$

(Jacobian: $J = m_i^3$) where $\mathbf{A}(\mathbf{x_i})$ is the vector magnetic potential:

$$\mathbf{B}(\mathbf{x_i}) = \nabla \times \mathbf{A}(\mathbf{x_i})$$

so that:

$$H_i(\mathbf{x_i}, \mathbf{p_i}) = \frac{1}{2m_i} |\mathbf{p_i} - \frac{e_i}{c}\mathbf{A}(\mathbf{x_i})|^2 \equiv \frac{1}{2} m_i v_i^2$$

H_σ is of the same form. Finally, H_{int} stands for the interaction between the two subsystems:

$$H_{int} = \sum_{n=1}^{N} V_{\sigma n}$$

where $V_{ij} \equiv V(|\mathbf{x_i} - \mathbf{x_j}|)$ $(i, j = 1, 2, ..., N, \sigma)$ denotes the interparticle potential energy, say typically related to a long-range, Coulomb-type interaction potential. The random interactions between the t.p. and reservoir particles surrounding it are 'tagged' by λ, which may later be used as an expansion parameter.

The resulting equations of motion for the test-particle are:

$$\dot{\mathbf{x}} = \mathbf{v} \ ; \qquad \dot{\mathbf{v}} = \frac{1}{m} \left[\mathbf{F}^{(0)}(\mathbf{x}, \mathbf{v}) + \lambda \, \mathbf{F_{int}}(\mathbf{x}, \mathbf{v}; \mathbf{X_R}) \right] \tag{3.3}$$

where we used the notation: $\{\mathbf{x}, \mathbf{v}\} \equiv \{\mathbf{x}_\sigma(t), \mathbf{v}_\sigma(t)\}\}$. The zeroth-order force $\mathbf{F}^{(0)}$ is due to the external field, e.g. $\mathbf{F}^{(0)} = -\frac{\partial}{\partial \mathbf{x}} \Phi(\mathbf{x})$ (or the Lorentz force $\mathbf{F_L} = \frac{e}{c}(\mathbf{v} \times \mathbf{B})$ in the magnetized plasma case). The *interaction* force

$$\mathbf{F_{int}} = -\frac{\partial}{\partial \mathbf{x}} \sum_{j=1}^{N} V(|\mathbf{x} - \mathbf{x_j}|) = \sum_{j=1}^{N} \mathbf{F_j}(\mathbf{r_{\sigma j}}) \tag{3.4}$$

is actually the sum of interactions between σ and $R-$ particles surrounding it; it is a *random* process, since the reservoir is assumed to be in homogeneous equilibrium. Remember that the Coulomb potential is a *central* potential i.e. $V(\mathbf{r}) = V(-\mathbf{r}) = V(r)$.

In this model, the test-particle is free to exchange energy and momentum with the reservoir. Therefore, the open subsystem represented by the test-particle is neither conservative nor autonomous.

3.1.1 Single-particle dynamics

Consider the zeroth-order ('free') problem of motion (i.e. (3.3) for $\lambda = 0$) in d dimensions ($d = 1, 2, 3$). We will look for an explicit analytical solution of the form[1]:

$$\begin{aligned} \mathbf{v}^{(0)}(t) &= \mathbf{v} + \frac{1}{m} \int_0^t dt' \ \mathbf{F}^{(0)}(t') = \mathbf{M}'(t)\, \mathbf{x} + \mathbf{N}'(t)\, \mathbf{v} \\ \mathbf{x}^{(0)}(t) &= \mathbf{x} + \int_0^t dt' \ \mathbf{v}(t') = \mathbf{M}(t)\mathbf{x} + \mathbf{N}(t)\mathbf{v} \end{aligned} \tag{3.5}$$

with the initial condition $\{\mathbf{x}, \mathbf{v}\} \equiv \{\mathbf{x}^{(0)}(0), \mathbf{v}^{(0)}(0)\}$.

This solution can be cast in the form:

$$\mathbf{X}^{(0)}(t) = \begin{pmatrix} \mathbf{x}^{(0)}(t) \\ \mathbf{v}^{(0)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{M}(t) & \mathbf{N}(t) \\ \mathbf{M}'(t) & \mathbf{N}'(t) \end{pmatrix} \begin{pmatrix} \mathbf{x}^{(0)}(0) \\ \mathbf{v}^{(0)}(0) \end{pmatrix} \equiv \mathbf{E}(t)\, \mathbf{X}^{(0)}(0) \tag{3.6}$$

implying

$$\mathbf{E}(0) = \mathbf{I}$$

[1] Properly speaking, one has:

$$x_i(t) = M_{ij}\, x_j(0) + N_{ij}\, v_j(0) \quad , \qquad v_i(t) = M'_{ij}\, x_j(0) + N'_{ij}\, v_j(0)$$

i.e.

$$\begin{pmatrix} M_{ij}(t) & N_{ij}(t) \\ M'_{ij}(t) & N'_{ij}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial x_i^{(0)}(t)}{\partial x_j} & \frac{\partial x_i^{(0)}(t)}{\partial v_j} \\ \frac{\partial v_i^{(0)}(t)}{\partial x_j} & \frac{\partial v_i^{(0)}(t)}{\partial v_j} \end{pmatrix}$$

(the derivatives are evaluated at $\{\mathbf{x}, \mathbf{v}\}$); this is rather obvious in a *linear* problem. In general, (3.6) may be viewed as a linearized solution of the (possibly nonlinear) problem of motion.

i.e.

$$\mathbf{M}(0) = \mathbf{N}'(0) = \mathbf{I} \qquad \mathbf{M}'(0) = \mathbf{N}(0) = \mathbf{0}$$

(obviously, the *prime* denotes differentiation with respect to time).

The form of the $d \times d$ matrices $\{\mathbf{M}(t), \mathbf{N}(t)\}$ depends on the particular aspects of the dynamical problem taken into consideration (provided that an explicit solution is known); therefore, these matrices definitely depend on the external field.

For the sake of clarity, a few explicit examples are given in the following.

(i) Free motion

In the *free-motion limit* (in the absence of external field), there is *no* external force: $\mathbf{F}^{(0)} = \mathbf{0}$ (cf. (3.3)) so

$$\{x_i(t), v_i(t)\} = \{x_i + v_i t, v_i\} \qquad i = 1, \cdots, d$$

(with the initial condition $\{x, v\} \equiv \{x(0), v(0)\}$) i.e.

$$M_{ij} = \delta_{ij}, N_{ij} = \delta_{ij}\, t$$

and thus

$$M'_{ij} = 0, N'_{ij} = \delta_{ij}$$

(cf.(3.6)).

(ii) Harmonic oscillator in 1d

Consider a particle moving in a harmonic potential. The force felt by the particle is: $F^{(0)} = -m\omega^2 x$ so the free equation of motion (i.e. (3.3) for $\lambda = 0$) yields the solution:

$$\begin{pmatrix} x^{(0)}(t) \\ v^{(0)}(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \omega^{-1}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \equiv \mathbf{E}(t) \begin{pmatrix} x(0) \\ v(0) \end{pmatrix}$$

Comparing this formula to (3.6) the meaning of $M(t), N(t)$ (simple real functions of time, in 1 dimension) is quite obvious:

$$M(t) = \cos\omega t \quad , \qquad N(t) = \omega^{-1}\sin\omega t$$

Note that taking $\omega \to 0$ one recovers exactly the above free-motion limit.

(iii) Charged particle motion in a uniform magnetic field

Consider a charged particle (of species $\alpha \in \{e, i, ...\}$, mass m_α, charge e_α) moving in a uniform magnetic field $\mathbf{B}$ assumed to lie along the $z-$ direction i.e. $B_i = B\delta_{i3}$.
$\mathbf{F}^{(0)}$ is now the Lorentz force

$$\mathbf{F}_L = \frac{e_\alpha}{c}(\mathbf{v} \times \mathbf{B}) \equiv s\, m\, \Omega\, (\mathbf{v} \times \hat{z})$$

where we defined:

$$\Omega = \Omega_\alpha = \frac{|e_\alpha| B}{m_\alpha c}, \qquad\qquad s = s_\alpha = \frac{e_\alpha}{|e_\alpha|} = \pm 1 \tag{3.7}$$

The problem of motion:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\end{pmatrix} \equiv \begin{pmatrix}\mathbf{v}\\ s\,\Omega\,(\mathbf{v}\,\times\hat{z})\end{pmatrix}$$

yields the well-known helicoidal solution:

$$\begin{aligned}
x(t) &= x + \Omega^{-1} v_x \sin\Omega t + s\,\Omega^{-1} v_y (1 - \cos\Omega t)\\
y(t) &= y - s\,\Omega^{-1} v_x (1 - \cos\Omega t) + \Omega^{-1} v_y \sin\Omega t\\
z(t) &= z + v_z\, t\\
v_x(t) &= v_x \cos\Omega t + s\, v_y \sin\Omega t\\
v_y(t) &= -s\, v_x \sin\Omega t + v_y \cos\Omega t\\
v_z(t) &= v_z = const.
\end{aligned}$$

which can be expressed as

$$\begin{pmatrix}\mathbf{x}^{(0)}(t)\\ \mathbf{v}^{(0)}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{N}(t)\\ \mathbf{0} & \mathbf{R}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} \equiv \mathbf{E}(t)\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} \tag{3.8}$$

Therefore, we have:

$$\mathbf{M}(t) = \mathbf{I}, \qquad \mathbf{M}'(t) = \mathbf{0}$$

$$\mathbf{N}'^{\alpha}(t) = \mathbf{R}^\alpha(t) = \begin{pmatrix}\cos\Omega t & s\ \sin\Omega t & 0\\ -s\ \sin\Omega t & \cos\Omega t & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\mathbf{N}^\alpha(t) = \int_0^t dt'\ \mathbf{R}^\alpha(t) = \Omega^{-1}\begin{pmatrix}\sin\Omega t & s\ (1-\cos\Omega t) & 0\\ s\ (\cos\Omega t - 1) & \sin\Omega t & 0\\ 0 & 0 & \Omega t\end{pmatrix} \tag{3.9}$$

(compare with (3.6)). Notice that (3.9) satisfy:

$$\mathbf{R}^{-1}(t) = \mathbf{R}(-t) = \mathbf{R}^T(t) \qquad \forall t \in \Re$$

however,

$$\mathbf{N}^{-1}(t) \neq \mathbf{N}(-t) = -\mathbf{N}^T(t) = -\mathbf{N}(t)\mathbf{R}^{-1}(t) \qquad \forall t \in \Re$$

Once more, notice that for $\Omega \to 0$ we obtain the correct free-motion limit (see in §3.1.1).

Note the way the gyro-frequency Ω is defined here: $\Omega_\alpha \geq 0$; the influence (if any) of the *sign* of the charge e will therefore be traced through its signature 's_α' throughout the formulae.

Group properties

In a generic manner, the $2d \times 2d$ matrix $\mathbf{E}(t)$ in (3.6) satisfy the condition:

$$\mathbf{E}(t)\mathbf{E}(t') = \mathbf{E}(t+t') \quad \forall t, t' \in \Re \tag{3.10}$$

implying

$$\mathbf{E}(-t) = \mathbf{E}^{-1}(t)$$

The group property (3.10) implies a number of relations for the $d \times d$ sub-matrices; in particular, if $\mathbf{M}(t) = \mathbf{I}$ (as in the first and third cases above) we have:

$$\mathbf{N}'(t)\,\mathbf{N}'(t') \;=\; \mathbf{N}'(t+t') \quad , \qquad \mathbf{N}(t) + \mathbf{N}(t')\mathbf{N}'(t) \;=\; \mathbf{N}(t+t') \qquad \forall t, t' \in \Re \tag{3.11}$$

thus, setting $t' = -t$:

$$\mathbf{N'}^{-1}(t) \;=\; \mathbf{N}'(-t)$$

yet

$$\mathbf{N}(-t) \;=\; -\mathbf{N}(t)\mathbf{N}'(-t) \neq \mathbf{N}^{-1}(t) \qquad \forall t \in \Re$$

The explicit examples provided above fulfill all these requirements, as can be readily checked.

3.2 Statistical formulation

Let us define the $(6N+6) \times d$ (total) $\Gamma-$ space of the system:

$$\Gamma \equiv \{\mathbf{x_i}, \mathbf{v_i}, i = 1, 2, ..., N, \sigma\} \equiv \{\mathbf{X}_\sigma, \mathbf{X_R}\}$$

and let D be the total $((N+1)-$particle) phase-space distribution function:

$$D = D(\Gamma_R \cup \Gamma_\sigma) \equiv D(\{\mathbf{X_j}, j = 1, 2, ..., N, \sigma\}$$

which is normalized to unity: $\int d\Gamma\, D(\Gamma) = 1$.

3.2.1 Liouville equation

The *equation of continuity* in phase space reads:

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial \mathbf{X_j}}(\mathcal{F}_\mathbf{j}\, D) \;=\; 0$$

where $D = D(\mathbf{X_j})$ ($\mathbf{X_j} \equiv \{\mathbf{x_j}, \mathbf{v_j}\}$) and $\mathcal{F}_\mathbf{j}$ denotes $\{\frac{d\mathbf{x_j}}{dt}, \frac{d\mathbf{v_j}}{dt}\}$ or, precisely

$$\frac{\partial D}{\partial t} + \mathbf{v_j}\frac{\partial D}{\partial \mathbf{x_j}} + \frac{\partial}{\partial \mathbf{v_j}}(\frac{1}{m}\,\mathbf{F_j}\, D) \;=\; 0 \tag{3.12}$$

(a summation over j is understood). For a system where $\frac{\partial \mathcal{F}_{\mathbf{j}}}{\partial \mathbf{X}_{\mathbf{j}}} = 0$ (an *'incompressible fluid'*, e.g. *all* examples in §3.1.1) this linear equation (the LIOUVILLE equation) can be cast in the form:

$$\frac{\partial D}{\partial t} = L\, D = (L_R + L_\sigma + \lambda\, L_{int})\, D \tag{3.13}$$

which reflects the form of the Hamiltonian (3.1). The operators in it are defined by:

$$L_R = \sum_{n=1}^{N} L_n^{(0)} + \sum_{j<n} \sum_{n=1}^{N} L_{jn}, \qquad L_{int} = \sum_{n=1}^{N} L_{\sigma n} \tag{3.14}$$

where $L_j^{(0)}$ is the *single particle* 'free'-Liouville operator *in the presence of the field*:

$$L_j^{(0)} = -\mathbf{v}_j \frac{\partial}{\partial \mathbf{x}_j} - \frac{1}{m_j} \mathbf{F}_j^{(0)} \frac{\partial}{\partial \mathbf{v}_j} \tag{3.15}$$

Note, once more, that j denotes *either* an $R-$ particle: $j = 1, 2, ..., N$ *or* the t.p. σ. L_{ij} is the *binary interaction* term:

$$\begin{aligned} L_{ij} &= \mathbf{F_{int}}(|\mathbf{x_i} - \mathbf{x_j}|) \left(\frac{1}{m_i} \frac{\partial}{\partial \mathbf{v_i}} - \frac{1}{m_j} \frac{\partial}{\partial \mathbf{v_j}} \right) \\ &\equiv \frac{\partial V(|\mathbf{x_i} - \mathbf{x_j}|)}{\partial \mathbf{x_i}} \left(\frac{1}{m_i} \frac{\partial}{\partial \mathbf{v_i}} - \frac{1}{m_j} \frac{\partial}{\partial \mathbf{v_j}} \right) \end{aligned} \tag{3.16}$$

3.2.2 Reduction of the Liouville equation

The total distribution function D contains the amount of information one needs to describe the system, so complete knowledge of it implies complete knowledge of the trajectory of the system in its phase space. However the full Liouville equation is practically impossible to solve for D (either analytically or numerically, given the number of particles in a real system) so one has to resort to information reduction methods (averaging techniques, projection operations, ...) through which the loss of information is compensated by simplicity and tractability in description; this problem actually lies in the heart of kinetic theory.

Reduced distribution functions - definitions

By averaging D over the coordinates of all but one, two, ..., p particles we obtain the $1-, 2-, ..., p-$ particle *reduced distribution functions* (*rdf*). This is a standard procedure which can be found in literature (see e.g. [4]). Nevertheless, the procedure adopted here is different from the standard one, in the sense that *two* kinds of such functions are defined, say $\sigma-$ and $R-$ *rdf*, depending on whether or not the test-particle is included within the p-tuplet considered (i.e. whether $\Gamma_\sigma \subset \Gamma_p$ or not). For instance, the σ-1-particle *rdf* reads:

$$f_1(\mathbf{X}) = \int d\Gamma_R D(\Gamma)$$

whereas the R-1-particle *rdf* is:

$$F_1^{\alpha}(\mathbf{X_1}_{\alpha}) = N_\alpha \int d\Gamma^c_{1^\alpha} D(\Gamma)$$

As our system may consist of different distinct populations e.g. electrons, ions, ... in the case of a multi-component plasma, we used the superscript $\alpha_j \in \{e, i, ...\}$ to denote the species of the $j-$th particle. In a similar manner the σ-2-particle *rdf* reads:

$$f_2^{\alpha}(\mathbf{X}, \mathbf{X_1}^{\alpha}) = N_\alpha \int d\Gamma^c_{1^\alpha,\sigma} D(\Gamma)$$

while the R-2-particle *rdf*(s) read:

$$F_2^{\alpha,\alpha}(\mathbf{X_1}^{\alpha}, \mathbf{X_2}^{\alpha}) = N_\alpha (N_\alpha - 1) \int d\Gamma^c_{1^\alpha,2^\alpha} D(\Gamma)$$

$$F_2^{\alpha,\beta}(\mathbf{X_1}^{\alpha}, \mathbf{X_2}^{\beta}) = N_\alpha N_\beta \int d\Gamma^c_{1^\alpha,2^\beta} D(\Gamma)$$

(for R-particles of the same or different species respectively) i.e. more concisely:

$$F_2^{\alpha,\alpha}(\mathbf{X_1}^{\alpha}, \mathbf{X_2}^{\beta}) = N_\alpha (N_\alpha - \delta_{\alpha,\beta}) \int d\Gamma^c_{1^\alpha,2^\beta} \, D(\Gamma)$$

$\Gamma^c_{1^{\alpha_1},2^{\alpha_2},...,N^{\alpha_n}}$ denotes the part of phase-space $\Gamma \equiv \Gamma_R \cup \Gamma_\sigma$ which is *complementary* to $\Gamma_{1^{\alpha_1},2^{\alpha_2},...,N^{\alpha_n}}$, i.e. $\Gamma^c_{1^{\alpha_1},2^{\alpha_2},...,n^{\alpha_n}} = \Gamma - \Gamma_{1^{\alpha_1},2^{\alpha_2},...,n^{\alpha_n}}$. In general the σ-p-body *rdf* is defined as:

$$f_p(\mathbf{X}, \mathbf{X_1}^{\alpha_1}, ..., \mathbf{X}_{p-1}^{\alpha_{p-1}}) = C_{p_e,p_i} \int d\mathbf{X}_p^{\alpha_p} d\mathbf{X}_{p+1}^{\alpha_{p+1}} ... d\mathbf{X}_N^{\alpha_N} \, D(\{\mathbf{X}_j^{\alpha_j}\})$$

for a p-tuplet containing p_e electrons and p_i ions $(p_e + p_i = p - 1)$ whereas the R-p-body *rdf* is defined as:

$$F_p^{\alpha}(\mathbf{X_1}^{\alpha_1}, ..., \mathbf{X}_{p-1}^{\alpha_{p-1}}, \mathbf{X}_p^{\alpha_p}) = C_{p_e,p_i} \int d\mathbf{X}_{p+1}^{\alpha_p} d\mathbf{X}_{p+2}^{\alpha_{p+1}} ... d\mathbf{X}_N^{\alpha_N} \, D(\{\mathbf{X}_j^{\alpha_j}\})$$

for a p-tuplet containing p_e electrons and p_i ions $(p_e + p_i = p)$. Note that f_1 is normalized to unity, while all other *rdf*'s are normalized to the corresponding combinatorial factors:

$$C_{p_e,p_i} = \Delta_{p_e}^{N_e} \Delta_{p_i}^{N_i} \equiv \frac{N_e!}{(N_e - p_e)!} \frac{N_i!}{(N_i - p_i)!}$$

Since f_1 is a *real* function satisfying

$$0 \le f_1 \quad , \qquad \int d\Gamma_1 f_1 = 1$$

it can be interpreted as a probability function density. Obtaining a closed equation with respect to f_1 is our final goal.

Evolutions equations for the *rdf* - the BBGKY Hierarchy

The $p-$particle reduced distribution functions are governed by a system of (N+1) coupled evolution equations, which are readily obtained by appropriately integrating the complete Liouville equation. One needs to take into account the conditions:

$$\begin{aligned}\int d\mathbf{X_j}\, L_j^{(0)}\, D = 0\\ \int d\mathbf{X_i}\, d\mathbf{X_j}\, L_{ij}\, D = 0\\ \int d\mathbf{x_j}\, \frac{\partial}{\partial \mathbf{x_j}}\, D = 0\\ \int d\mathbf{v_j}\, \frac{\partial}{\partial \mathbf{v_j}}\, D = 0\end{aligned} \tag{3.17}$$

as well as the normalization conditions in the previous paragraph. Even though, as explained above, our definition of *rdf*s (for a test-particle problem) is not identical to the standard one appearing in textbooks of Statistical Mechanics, the method of the calculation is quite similar and can be found therein (see for example in [4], [22]) so we omit details here.

The first members of the hierarchy obtained read:

$$\begin{aligned}(\partial_t - L_\sigma^{(0)})\, f_1(\mathbf{X}) &= \lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_1^{\alpha'}\, L_{\sigma 1}\, f_2^{\alpha,\alpha'}(\mathbf{X}^\alpha, \mathbf{X_1}^{\alpha'})\\ (\partial_t - L_\sigma^{(0)} - L_1^{(0)})\, f_2^{\alpha,\beta}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta) &= \lambda\, L_{\sigma 1}\, f_2^{\alpha,\beta}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta)\\ &+\lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_3^{\alpha'}\, (L_{\sigma 2} + L_{12})\, f_3^{\alpha,\beta,\alpha'}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta, \mathbf{X_2}^{\alpha'})\end{aligned} \tag{3.18}$$

for the $\sigma-$*rdf*s, whereas for the $R-$*rdf*s we obtain:

$$\begin{aligned}&(\partial_t - L_1^{(0)})\, F_1^\beta(\mathbf{X}^\beta) =\\ &\quad \lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_2^{\alpha'}\, L_{12}\, F_2^{\beta,\alpha'}(\mathbf{X}_1^\beta, \mathbf{X_2}^{\alpha'}) + \lambda \int d\mathbf{X}_\sigma L_{\sigma 1}\, f_2^{\beta,\alpha'}(\mathbf{X}_\sigma^\alpha, \mathbf{X_1}^\beta)\end{aligned}$$

$$\begin{aligned}(\partial_t - L_1^{(0)} - L_2^{(0)})\, F_2^{\beta,\gamma}(\mathbf{X}_1^\beta, \mathbf{X_2}^\gamma) &= \lambda\, L_{12}\, F_2^{\beta,\gamma}(\mathbf{X}_1^\beta, \mathbf{X_2}^\gamma)\\ &+ \quad \lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_3^{\alpha'}\, (L_{13} + L_{23})\, F_3^{\alpha,\beta,\alpha'}(\mathbf{X}^\alpha, \mathbf{X_2}^\beta, \mathbf{X_3}^{\alpha'})\\ &+ \quad \lambda \int d\mathbf{X}_\sigma^\alpha (L_{1\sigma} + L_{2\sigma})\, f_3^{\alpha,\beta,\gamma}(\mathbf{X}^\alpha, \mathbf{X_2}^\beta, \mathbf{X_3}^\gamma)\end{aligned} \tag{3.19}$$

The *BBGKY hierarchy* thus obtained is strictly equivalent to the initial Liouville equation and, inevitably, just as difficult to solve. We therefore have to adopt an appropriate approximation, which will allow us to truncate the hierarchy equations at some point and thus obtain a closed equation for f.

3.2.3 Cluster expansion - truncation of the BBGKY hierarchy

At this stage it is appropriate to define a set of $p-$body correlation functions, which contain the information of the *deviation* from the non-interacting-particle image. For instance, the $\sigma-$2-body (i.e. $\sigma^\alpha + 1_R^\beta$) correlation function reads: $g_2 = g^{\alpha,\beta}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta)$ reads:

$$g^{\alpha,\beta}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta) = f_2^{\alpha,\beta}(\mathbf{X}^\alpha, \mathbf{X_1}^\beta) - f_1^\alpha(\mathbf{X}) F_1^\beta(\mathbf{X_1}) \tag{3.20}$$

An analogous expression holds for the $R-$2-body (i.e. $2_R^\beta + 1_R^\gamma$) correlation function $G_2 = G^{\beta,\gamma}(\mathbf{X_1}^\beta, \mathbf{X_2}^\gamma)$. In the same way we define $\sigma-$ and $R-$ 3$-$body correlation functions and so forth. This *cluster expansion* (often quoted as *the Mayer expansion* [32]) allows us to express to obtain a modified form of the lowest members of the BBGKY hierarchy, in terms of *rdf*s and correlations. Distinct orders in λ may thus be separated and then corresponding equations may successively be solved.

A standard procedure at this stage consists in assuming that the mutual interaction between particles is very small, as compared to their kinetic energy. This implies the existence of distinct orders of magnitude, in terms of powers of a smallness parameter, which is related to the value of the plasma parameter $\mu_p = \frac{e^2 n^{1/3}}{k_B T}$ [2]. The condition $\mu_p \ll 1$ is indeed met in a majority of plasmas of interest, as we mentioned before. The weak-coupling approximation is extensively discussed in [5], [46]; here, we shall adopt the truncation scheme suggested in pp. 101-102 of the former (also see pp. 56-58 in [7]).

We saw that, as implied by the form of our Hamiltonian (reflected in the structure of the Liouville equation, as well), the free and the interaction Liouville operators ($L^{(0)}$, L_{jn}) appear in orders λ^0 and λ^1 respectively. We will now assume that the 1-body *rdf*s are of the order λ^0. The correlation function $g^{\alpha,\beta}$ is expected to be of the same order of magnitude as the interaction potential, i.e. $g_2 \sim \lambda$, while higher-order $p-$body correlations ($p > 2$) are assumed to be of higher order i.e. $g_p \sim \lambda^{2+}$. The hierarchy equations can therefore be re-formulated using a scaling of the form:

$$G_2 \sim \lambda^1 \tilde{G}_2 \quad , \qquad g_2 \sim \lambda^1 \tilde{g}_2$$

(the *tilde* will be dropped in the following).

We shall now seek the equation obtained in order λ^2, so terms of higher order will be neglected. The truncated version of the first two members of the hierarchy now read:

$$(\partial_t - L_\sigma^{(0)})\, f(\mathbf{X}) = \lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_1^{\alpha'}\, L_{\sigma 1}\, F^{\alpha'}(\mathbf{X}_1^{\alpha'})\, f(\mathbf{X})$$
$$+\lambda^2 \sum_{\alpha'=e,i} \int d\mathbf{X}_1^{\alpha'}\, L_{\sigma 1}\, g^{\alpha'}(\mathbf{X}, \mathbf{X_1}^{\alpha'})$$

---

[2] See in the previous chapter for definitions. Also see the discussion about smallness parameters in plasmas defined through simple dimensional arguments, in [3], ch. 2.

$$(\partial_t - L_\sigma^{(0)} - L_1^{(0)})\, g^{\alpha\beta}(\mathbf{X}, \mathbf{X_1}) = L_{\sigma 1}\, F^\beta(\mathbf{X_1})\, f_1^\alpha(\mathbf{X}) \tag{3.21}$$

For the sake of simplicity, we dropped all function indices, so f, F, g henceforth denote f_1, F_1, $\tilde{g}_2$, respectively. Contributions of order higher than λ^2 were omitted in this system of equations. In particular, 2-body correlations were truncated at $\sim \lambda^1$ since higher orders do not enter the equation for f.

According to the statement of our test-particle problem, we shall consider the reservoir distribution function F in the above formulae to be in a stationary state, i.e. $\partial_t F = 0$. So, $F^{\alpha'}(\mathbf{X_1})$ will be taken to be equal to $\frac{N_{\alpha'}}{V}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \equiv n_{\alpha'}\, \phi_{eq}^{\alpha'}(\mathbf{v_1})$. The homogeneous state $\phi_{eq}(\mathbf{v_1})$ is typically, yet not necessarily, a Maxwellian state. Rigorously speaking, ϕ may be some other function of the conserved quantities in a specific problem, i.e. $\{v_i\}$ $(i = 1, ..., d)$ in free motion or $\{v_\perp, v_\parallel\}$ in magnetized plasma.

At this order, the lowest two members of the hierarchy are therefore decoupled from the rest. We may therefore proceed by solving the second of the above equations and substituting into the first. This will provide us with a closed equation with respect to f. Note that only *binary* (i.e. 2-body) interactions are retained in this picture. Accounting for higher-order (collective) contributions and dynamical screening effects would impose keeping at least 3-body terms (thus obtaining a BALESCU-LENARD-GUERNSEY-type equation [3]). Such a level of description goes beyond the scope of our work.

3.3 Solution in successive orders in λ

3.3.1 Zeroth order - free LIOUVILLE equation

The *"free"* (1−body) Liouville equation (i.e. (3.21a) for $\lambda = 0$):

$$\partial_t\, f(\mathbf{X}) = L_\sigma^{(0)}\, f(\mathbf{X}) \tag{3.22}$$

or (cf. (3.15))

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F}^{(0)}\frac{\partial}{\partial \mathbf{v}} = 0$$

can be solved formally. The solution reads:

$$f(t) = e^{L_0\, t}\, f(0) \equiv U^{(0)}(t)\, f(0) \tag{3.23}$$

The *'propagator'* $U(t)$ satisfies:

$$U(t_1)\, U(t_2) = U(t_1 + t_2)$$

and, since

$$U(0) = I$$

setting $t_2 = -t_1 = -t$ gives

$$U^{-1}(t) = U(-t)$$

The above relations provide the trajectory of the system in phase space, given the existence of the external field. The solution for $f(t)$ satisfies *Liouville's theorem*[3], so the value of the distribution at $\{\mathbf{x}(0), \mathbf{v}(0)\}$ at the instant t equals its value at $\{\mathbf{x}(-t), \mathbf{v}(-t)\}$ for $t = 0$:

$$\begin{aligned} U(t)\, f(\mathbf{x}, \mathbf{v}) &\equiv U(t)\, f(\mathbf{x}, \mathbf{v}; 0) = f(\mathbf{x}, \mathbf{v}; t) = f(\mathbf{x}(-t), \mathbf{v}(-t); 0) \\ &\equiv f(\mathbf{x}(-t), \mathbf{v}(-t)) \end{aligned} \tag{3.24}$$

The propagator formalism is extensively discussed in [4], [5] in general; an exhaustive study of the generator of *charged-particle* motion in an external field, in particular, exists in [82].

3.3.2 First-order in λ: VLASOV term

In order $\sim \lambda^1$, relations (3.21a, b) are still decoupled. (3.21a) now reads:

$$(\partial_t - L_\sigma^{(0)})\, f(\mathbf{X}) = \lambda \sum_{\alpha'=e,i} \int d\mathbf{X}_1^{\alpha'}\, L_{\sigma 1}\, F_1^{\alpha'}(\mathbf{X}_1^{\alpha'})\, f(\mathbf{X}) \tag{3.25}$$

Using definition (3.16) and the relation (3.17d), the right-hand-side can be re-arranged into the form:

$$\begin{aligned} rhs &= \left(\frac{\partial}{\partial \mathbf{x}} \sum_{\alpha'=e,i} \int d\mathbf{X}_1^{\alpha'}\, V(\mathbf{x} - \mathbf{x_1})\, F_1^{\alpha'}(\mathbf{X}_1) \right) \frac{1}{m} \frac{\partial}{\partial \mathbf{v}} f(\mathbf{X}) \\ &\equiv \frac{1}{m} \frac{\partial V_{mf}(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{v}} f(\mathbf{X}) \\ &= -\frac{1}{m} \mathbf{F}_{mf}(\mathbf{x}) \frac{\partial}{\partial \mathbf{v}} f(\mathbf{X}) \end{aligned}$$

In the case of electrostatic interactions, this results in a mean (*Vlasov*) electric field $\mathbf{E}_{mf}(\mathbf{x}) = \mathbf{F}_{mf}(\mathbf{x})/e$, induced by dynamical charge-screening.

Equation (3.25) thus becomes the celebrated VLASOV equation [3]:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m} \left(\mathbf{F}_{mf}(\mathbf{x}) + \mathbf{F}^{(0)} \right) \frac{\partial}{\partial \mathbf{v}} = 0$$

In a test-particle problem, however, note that the mean-field potential, giving rise to this extra (*'self-consistent'*) term at order $\sim \lambda^1$, *disappears*[4] once the reservoir is assumed to be in a uniform state:

$$\int d\mathbf{v_1} d\mathbf{x_1} V(\mathbf{x} - \mathbf{x_1})\, \frac{N}{V}\, \phi_1(\mathbf{v_1}) = 0$$

so *no* contribution to the dynamics of our test-particle is obtained at this order.

[3] See e.g. p. 109 in [5]. As a matter of fact, continuity in phase space is ensured, under certain conditions, by the foundation of statistical evolution laws; see detailed footnote 7 in §1.3.

[4] In order to see this, shift the integration variable $\mathbf{x_1}$ to $\mathbf{r} = \mathbf{x} - \mathbf{x_1}$; the space integral then readily vanishes, for reasons of symmetry (provided that the potential V is not divergent, as *is*, in fact, the case in the Coulomb potential; see elsewhere in this text).

3.3.3 Second-order ($\sim \lambda^2$): the Generalized Master Equation

In order $\sim \lambda^2$, one may solve the second equation in (3.21) and then substitute into the first in order to obtain a closed equation in terms of f. This is a more or less standard procedure (cf. [5]), which we will outline briefly.

Equation (3.21b) contains a homogeneous linear part (left-hand-side) and a driving (source) term (right-hand-side). The formal solution for $g(t) = g^{\alpha\beta}(\mathbf{X}, \mathbf{X_1};\, t)$ reads:

$$g(t) \;=\; U^{(0)}_{\sigma,1}(t)\, g(0) \;+\; \lambda \int_0^t d\tau\, U^{(0)}_{\sigma,1}(t) L_{\sigma,1}\, F(\mathbf{X_1};\, t-\tau)\, f(\mathbf{X};\, t-\tau) \qquad (3.26)$$

where $U^{(0)}_{\sigma,1}(t)$ is the two-body propagator:

$$U^{(0)}_{\sigma,1}(t) \;=\; \exp(L^{(0)}_{\sigma} + L^{(0)}_{1})t \;=\; U^{(0)}_{\sigma}(t)\, U^{(0)}_{1}(t)$$

in the presence of the field.

Neglecting initial correlations (i.e. setting $g(t=0)=0$) and substituting into (3.21a), we obtain the *non-markovian Generalized Master Equation* (GME):

$$\partial_t f(\mathbf{x}, \mathbf{v}; t) = L_{(0)}\, f(\mathbf{x}, \mathbf{v}; t)$$

$$+\lambda^2\, n \int_0^t d\tau \int d\mathbf{x_1}\, d\mathbf{v_1}\, L_I\, U^{(0)}_{\sigma,1}(\tau)\, L_I\, \phi_{eq}(\mathbf{v_1})\, f(\mathbf{x}, \mathbf{v}; t-\tau) \qquad (3.27)$$

or

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F}^{(0)}\frac{\partial}{\partial \mathbf{v}} =$$

$$= \; n \int_0^t d\tau \int d\mathbf{x_1}\, d\mathbf{v_1}\, L_I\, U^{(0)}_{\sigma,1}(\tau)\, L_I\, \phi_{eq}(\mathbf{v_1})\, f(\mathbf{x}, \mathbf{v}; t-\tau)$$

$$\equiv \; \mathcal{C}\{f\}$$

(3.27-bis)

(λ was dropped for simplicity). A summation over particle species α' is understood in the *rhs* if appropriate. By $f = f^{\alpha}_1(\mathbf{x}, \mathbf{v})$, $F^{\alpha'}_1(\mathbf{x_1}, \mathbf{v_1})$ $(= n_{\alpha'}\phi^{\alpha'}(\mathbf{v_1})$ here) we denote the distribution functions of the test-particle and one (*any*) particle from the reservoir (of species, say, α and α' respectively); $n = n_{\alpha'} = \frac{N_{\alpha'}}{V}$ is the particle density; $L_0 \equiv L^{(0)}_{\sigma}$ is the *"free"* Liouville operator defined previously (see (3.15)); the *binary interaction* Liouville operator $L_I \equiv L_{\sigma 1}$ was defined in (3.16). Remember, once more, that the mean-field *(Vlasov)* term, in order λ^1, disappeared since we assumed the reservoir state to be in a homogeneous equilibrium state.

A point we want to stress is that, in principle, the external field appears in the *collision term* $\mathcal{C}\{f\}$ (through the action of the propagator, cf. (3.24)) and not only the left-hand-side (i.e. the zeroth-order Liouville operator). Therefore, one *a priori* expects the field to enter the expressions for the coefficients in the final kinetic equation. As discussed in the Introduction, this fact has often been neglected in previous studies.

Equation (3.27-bis) provides the generic form of a *Generalized Master Equation* (GME) equation for a test-particle problem as formulated above. It is a *Non-Markovian* integro-differential equation: the evolution of f at the instant t depends of its value at all previous times through $f(t-\tau)$. Certain 'markovianization' procedures have been proposed in the past, leading to a closed equation in terms of $f(t)$. In the following chapters, we shall construct - and compare - two of these kinetic operators, both leading to a 'Markovian' kinetic equation. This equation will be a *linear* partial differential equation (*PDE*) of second-order in the phase-space variables $\{\mathbf{x}, \mathbf{v}\}$, since the reservoir distribution function was taken to be stationary.

Equation (3.27-bis) is therefore the final result of this section, and will serve as the basis of the analysis that will follow. However, in order to clarify the features of our test-particle model, we will grasp the opportunity to present, in the next section, a brief study of the statistical properties of the random interactions between the test-particle and the reservoir particles surrounding it. One may thus attempt to sketch an analogy between our formalism and existing stochastic theories based on force autocorrelations [7], [49].

3.4 Properties of inter-particle interactions

Let A be an arbitrary function of the microscopic variables: $A = A(\mathbf{x}, \mathbf{v}; \mathbf{X}_R)$. We define its average over the (homogeneous) reservoir state $\sigma_R(\{\mathbf{x_i}\})$:

$$\boldsymbol{E}\{\ A\ \} = (A,\, \sigma_R)_R = \int_{\Gamma_R} d\mathbf{X_R}\ \sigma_R(\mathbf{X_R})\ A \tag{3.28}$$

[5]. Let us now consider the random interaction force $\mathbf{F_{int}}(t)$ felt by our test-particle, due to the existence of the thermalized environment around it.

3.4.1 Interactions as a random process

Interactions between a particle and its thermalized environment are purely *random*. Even though they were defined in a deterministic manner, in (3.4), the test-environment interaction mechanism can be viewed as a *'stochastic'* process. Nevertheless, information on the statistics of this process may be drawn from our microscopic model, and does not have to be added via phenomenology.

[5]If A is a sum over R- particles of single-particle-terms i.e. $A(\mathbf{x}, \mathbf{v}; \mathbf{X}_R) = \sum_{j=1}^{N} \tilde{A}(\mathbf{x}, \mathbf{v}; \mathbf{X_j})$, this ensemble-average becomes equal to

$$n \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1})\ A\ \equiv \langle\ A\ \rangle_R \tag{3.29}$$

(n is *particle density*); we have taken: $\sigma_R = \frac{1}{V^N} \prod_{j=1}^{N} \phi_{eq}(v_j)$.

Interactions are characterized by a vanishing mean-value:

$$\langle \mathbf{F_{int}}(t) \rangle_R \equiv \boldsymbol{E}\{\mathbf{F_{int}}(\mathbf{x}, \mathbf{v};\ t)\} = 0$$

The covariance $\mathbf{C}$ reads:

$$\begin{aligned}
\mathbf{C} &= \mathbf{C}(\mathbf{x}, \mathbf{v}; t_1, t_2) = \langle \mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2) \rangle_R \equiv \boldsymbol{E}\left\{\mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2)\right\} \\
&\equiv \int_{\Gamma_R} d\mathbf{X}_R\ \sigma_R(\mathbf{X}_R)\, \mathbf{F_{int}}(t_1)\, \mathbf{F_{int}}(t_2) \\
&= n \int_{\Gamma_R} d\mathbf{x_1} d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1})\ \mathbf{F_{int}}(|\mathbf{x}(t_1) - \mathbf{x_1}(t_1)|)\ \mathbf{F_{int}}(|\mathbf{x}(t_2) - \mathbf{x_1}(t_2)|) \\
&= n \int_{\Gamma_R} d\mathbf{x_1} d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1}) \left(-\frac{\partial}{\partial \mathbf{x}(t_1)} V(|\mathbf{x}(t_1) - \mathbf{x_1}(t_1)|)\right) \\
&\qquad \left(-\frac{\partial}{\partial \mathbf{x}(t_2)} V(|\mathbf{x}(t_2) - \mathbf{x_1}(t_2)|)\right) \\
&= n \int d\mathbf{r} \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1})\ \frac{\partial V(r(t_1))}{\partial \mathbf{r}(t_1)} \otimes \frac{\partial V(r(t_2))}{\partial \mathbf{r}(t_2)}
\end{aligned} \tag{3.30}$$

In the last part we have shifted from $\mathbf{x}$ to $\mathbf{r} \equiv \mathbf{x} - \mathbf{x_1}$ [6] [7].

3.4.2 Force auto-correlations - the general case

Let us introduce the Fourier transform (*F.T.*) of $V(r)$:

$$\tilde{V}_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int d\mathbf{r}\ V(\mathbf{r}(t))\ e^{-i\mathbf{k}\mathbf{r}(t)} \tag{3.31}$$

so that

$$V(\mathbf{r}(t)) = \int d\mathbf{k}\ \tilde{V}_{\mathbf{k}}\ e^{i\mathbf{k}\mathbf{r}(t)}$$

Note that V is a *real* function $V(r) \in \Re$, so its *F.T.* is a real function $\tilde{V}_k \in \Re$ itself. Furthermore, notice that V is an *even* function of $\mathbf{r}$ (it actually depends on its module r, only) :

$$V(-\mathbf{r}) = V(\mathbf{r}) = V(|\mathbf{r}|) = V(r)$$

Therefore, the *F.T.* has the property:

$$\tilde{V}_{\mathbf{k}} = \tilde{V}_{-\mathbf{k}} = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$$

[6] Generally, taking the reservoir to be in a homogeneous equilibrium state $\sigma_R = \frac{1}{V^N} \prod_{j=1}^{N} \phi_{Max}(v_j)$ (typically a Maxwellian state) and using definition (3.4) - remember that $F(\mathbf{r_{\sigma j}}) = F(-\mathbf{r_{\sigma j}})$- and expression (3.32), one may prove that *odd* moments of $\mathbf{F_{int}}$ vanish (i.e. $\langle \mathbf{F_{int}}(t_1) ... \mathbf{F_{int}}(t_{2n+1}) \rangle_R = 0$), from symmetry arguments (in the Fourier integrals involved), while *even* moments ($\langle \mathbf{F_{int}}(t_1) ... \mathbf{F_{int}}(t_{2n}) \rangle_R$) can be expressed as symmetric cyclic combinations of the correlations $\mathbf{C}(t_i, t_j)$ $(i, j = 1, 2, ..., 2n)$. Interactions represent a *Gaussian* process [15] with zero mean.

[7] For a 'pedagogical' review of the formalism involved in the statistical description of random functions see [47] (chapter 2 therein).

The interaction forces can now be expressed as follows:

$$\begin{aligned}\mathbf{F_{int}}(t_1) &= \mathbf{F_{int}}(\mathbf{r}(t_1)) = -\frac{\partial}{\partial \mathbf{r}(t_1)} V(\mathbf{r}(t_1)) = -\frac{\partial}{\partial \mathbf{r}(t_1)} \int d\mathbf{k}\ \tilde{V}_{\mathbf{k}}\ e^{i\mathbf{k}\mathbf{r}(t_1)} \\ &= \int d\mathbf{k}\ (i\mathbf{k})\, \tilde{V}_{\mathbf{k}}\ e^{i\mathbf{k}\mathbf{r}(t_1)} \end{aligned} \tag{3.32}$$

In a similar manner:

$$\begin{aligned}\mathbf{F_{int}}(t_2) &= \int d\mathbf{k}'\ (i\mathbf{k}')\, \tilde{V}_{\mathbf{k}'}\ e^{i\mathbf{k}'\mathbf{r}(t_2)} = \int d\mathbf{k}'\ (i\mathbf{k}')\, \tilde{V}_{\mathbf{k}'}\ e^{i\mathbf{k}'\mathbf{r}(t_1)}\ e^{i\mathbf{k}'(\mathbf{r}(t_2)-\mathbf{r}(t_1))} \\ &= \int d\mathbf{k}'\ (i\mathbf{k}')\, \tilde{V}_{\mathbf{k}'}\ e^{i\mathbf{k}'\mathbf{r}}\, e^{-i\mathbf{k}'\mathbf{\Delta r}} \end{aligned}$$

where we have set $\mathbf{r} = \mathbf{r}(t_1)$ and

$$\mathbf{\Delta r}(t_1, t_2) = \mathbf{r}(t_1) - \mathbf{r}(t_2) \tag{3.33}$$

Notice that only the *real* part contributes to the above formulae, since the interaction potential forces (its derivatives) is a real function.
Substituting in (3.30), we have:

$$\mathbf{C} = n \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1}) \int d\mathbf{r} \int\ d\mathbf{k} \int\ d\mathbf{k}'\, (i\,\mathbf{k}) \otimes (i\,\mathbf{k}')\tilde{V}_{\mathbf{k}}\tilde{V}_{\mathbf{k}'} e^{-i(\mathbf{k}+\mathbf{k}')\mathbf{r}}\ e^{i\mathbf{k}'\mathbf{\Delta r}} \tag{3.34}$$

Note that $\mathbf{C} = \mathbf{C}(\mathbf{v}; field)$ is a *symmetric* real matrix, i.e. $C_{ij} = C_{ji}$.

A very important comment has to be made here. The relative displacement $\mathbf{\Delta r}(t_1, t_2)$, appearing in the exponential, was defined in (3.33). It contains all the information regarding the influence of the external field on the auto-correlation function. Its exact form can be evaluated in terms of the solution of the problem of motion, by making use of (3.6):

$$\begin{aligned}\mathbf{\Delta r}(t_1, t_2) &= \left[\mathbf{x}_\sigma(t_1) - \mathbf{x}_\sigma(t_2)\right]\ - \left[\mathbf{x}_1(t_1) - \mathbf{x_1}(t_2)\right] \\ &= \left\{ \left[\mathbf{M}_\sigma^\alpha(t_1)\ - \mathbf{M}_\sigma^\alpha(t_2)\right] \mathbf{x}_\sigma + \left[\mathbf{N}_\sigma^\alpha(t_1)\ - \mathbf{N}_\sigma^\alpha(t_2)\right] \mathbf{v}_\sigma \right\} \\ &\quad - \left\{ \left[\mathbf{M}_\mathbf{1}^{\alpha'}(t_1)\ - \mathbf{M}_\mathbf{1}^{\alpha'}(t_2)\right] \mathbf{x_1} + \left[\mathbf{N}_\mathbf{1}^{\alpha'}(t_1)\ - \mathbf{N}_\mathbf{1}^{\alpha'}(t_2)\right] \mathbf{v_1} \right\}\end{aligned}$$

As the second pair of brackets contains exactly the same combination of variables as the first, but concerning the $R-$particle '1', we will henceforth express this kind of expression as:

$$\mathbf{\Delta r}(t_1, t_2) = \left\{ \left[\mathbf{M}_\sigma^\alpha(t_1)\ - \mathbf{M}_\sigma^\alpha(t_2)\right] \mathbf{x}_\sigma + \left[\mathbf{N}_\sigma^\alpha(t_1)\ - \mathbf{N}_\sigma^\alpha(t_2)\right] \mathbf{v}_\sigma \right\} \left(1 - \mathcal{P}_{\sigma_\alpha, 1_{\alpha'}}\right) \tag{3.35}$$

where we defined the *permutation operator* $\mathcal{P}_{1,2}$:

$$\mathcal{P}_{1,2}\, f(X_1, X_2)\ =\ f(X_2, X_1)$$

(for any function f whose argument depends on particles $1, 2$).

The appearance of $\mathbf{\Delta r}$ in (3.34) is actually the signature of the external field. A simplifying procedure, quite often used in the past (and which will *not* be adopted here!), consists in evaluating the exponential along free trajectories, that is plainly neglecting the field in the collision term (see, for instance, in [90], where a relation similar to (3.37) was derived in the presence of a field and was then computed as if the latter were not there).

It is interesting to point out that since $\mathbf{C}(\tau)$ is by definition a *real* matrix, only the *real* part ($\cos \mathbf{k}\, \mathbf{\Delta r}$) of the exponential is relevant; furthermore, C_{ij} is *a priori even* in the time argument τ:

$$C_{ij}(-\tau) = C_{ij}(\tau)$$

[4]. We shall see that this condition is satisfied in the magnetized plasma case.

The evaluation of the form of (3.30) from microscopic laws is strongly simplified if two special (yet not so restricting) assumptions are made, independently.

(i) Force auto-correlations - simplification 1

First, let us assume that $\mathbf{M}(t)$ is the unit matrix $\mathbf{I}$, i.e.

$$\partial x_i(t+t')/\partial x_j(t) = \delta_{ij}$$

($t, t' \in \Re$) which *is* indeed the case for a variety of systems of interest, for instance in the first and third examples considered in §3.1.1 (i.e. plasma in a uniform magnetic field *and* free motion).

In this case, we may carry out the $\mathbf{r}-$ integration in (3.34):

$$\begin{aligned}
\mathbf{C} &= n \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1}) \int d\mathbf{r} \int\ d\mathbf{k} \int\ d\mathbf{k}'\, (i\,\mathbf{k}) \otimes (i\,\mathbf{k}')\tilde{V}_{\mathbf{k}}\tilde{V}_{\mathbf{k}'}\, e^{-i(\mathbf{k}+\mathbf{k}')\mathbf{r}}\, e^{i\mathbf{k}'\mathbf{\Delta r}} \\
&= n \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1})(2\pi)^3 \int d\mathbf{k} \int\ d\mathbf{k}'\, (i\,\mathbf{k}) \otimes (i\,\mathbf{k}')\tilde{V}_{\mathbf{k}}\tilde{V}_{\mathbf{k}'}\delta(\mathbf{k}+\mathbf{k}')\, e^{-i\mathbf{k}'\mathbf{\Delta r}} \\
&= n \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1})(2\pi)^3 \int d\mathbf{k}\, (i\,\mathbf{k}) \otimes (-i\,\mathbf{k})\tilde{V}_{\mathbf{k}}^2\, e^{i\mathbf{k}\mathbf{\Delta r}} \\
&= n\,(2\pi)^3 \int d\mathbf{v_1}\ \phi_{eq}(\mathbf{v_1}) \int\ d\mathbf{k}\ \mathbf{k} \otimes \mathbf{k}\, \tilde{V}_{\mathbf{k}}^2\, e^{i\mathbf{k}\mathbf{\Delta r}} \qquad (3.36)
\end{aligned}$$

where we have used the property:

$$\int d^3\mathbf{r}\, e^{i\mathbf{K}\mathbf{r}} = (2\pi)^3\, \delta(\mathbf{K})$$

($\mathbf{K} \in \Re^3$).

Now, substituting from the zeroth-order solution (3.6) we obtain:

$$\mathbf{C}^{\alpha,\alpha'} = (2\pi)^3 \sum_{\alpha'} n_{\alpha'} \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\ \tilde{V}_k^2\, \mathbf{k} \otimes \mathbf{k}\, e^{i\mathbf{k}[\mathbf{N}^{\alpha}(t_1)-\mathbf{N}^{\alpha}(t_2)]\mathbf{v}(0)}\, e^{-i\mathbf{k}[\mathbf{N_1}^{\alpha'}(t_1)-\mathbf{N_1}^{\alpha'}(t_2)]\mathbf{v_1}(0)} \qquad (3.37)$$

Setting $t_1 = t + \tau$, $t_2 = t$ in the above formula, so that $t_1 - t_2 = \tau$, and making use of (3.6), (3.11), we have:

$$[\mathbf{N_j}^{\alpha_j}(t+\tau) - \mathbf{N_j}^{\alpha_j}(t)]\mathbf{v_j}(0) = \mathbf{N_j}^{\alpha_j}(\tau)\, \mathbf{N_j'}^{\alpha_j}(t)\mathbf{v_j}(0) = \mathbf{N_j}^{\alpha_j}(\tau)\, \mathbf{v_j}(t)$$

($j = \sigma, 1$) so (3.35) simplifies as:

$$\begin{aligned} \mathbf{\Delta r}(t_1, t_2) &= \left[\mathbf{N}_\sigma^\alpha(t+\tau) - \mathbf{N}_\sigma^\alpha(t)\right] \mathbf{v}_\sigma \left(1 - \mathcal{P}_{\sigma_\alpha, 1_{\alpha'}}\right) \\ &= \mathbf{N}^\alpha(\tau)\, \mathbf{v}(t) \left(1 - \mathcal{P}_{\sigma_\alpha, 1_{\alpha'}}\right) \end{aligned} \tag{3.38}$$

and relation (3.37) above becomes:

$$\mathbf{C}^{\alpha,\alpha'}(\tau) = (2\pi)^3 \sum_{\alpha'} n_{\alpha'} \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, \mathbf{k} \otimes \mathbf{k}\, e^{i\mathbf{k}\,\mathbf{N}^\alpha(\tau)\,\mathbf{v}}\; e^{-i\mathbf{k}\,\mathbf{N_1}^{\alpha'}(\tau)\,\mathbf{v_1}} \tag{3.37-bis}$$

Therefore $\mathbf{C}(t_1, t_2) = \mathbf{C}(t_1 - t_2) = \mathbf{C}(\tau)$, so the interactions represent a *stationary* random process[8].

(ii) Force auto-correlations - simplification 2

A second simplifying hypothesis consists in considering interactions between particles of the same species (or a *single component* system e.g. an electron plasma, $\alpha = \alpha'$). In this case, all particles obey the same dynamics:

$$\mathbf{N_1}(t) = \mathbf{N}_\sigma(t)$$

so the integrand in the correlation function comes out to be a function of $\mathbf{v} - \mathbf{v_1} \equiv \mathbf{g}$:

$$\mathbf{C}^{(\alpha=\alpha')}(\mathbf{v}; \tau) = n(2\pi)^3 \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \int d\mathbf{k}\; \tilde{V}_k^2\, e^{i\mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v}-\mathbf{v_1})}\, \mathbf{k} \otimes \mathbf{k} \tag{3.39}$$

3.4.3 Comments

Notice, in all the above formulae, the explicit dependence of the correlations on the specific features of the dynamical problem, and namely on the external field.

Relation (3.36) (or its simplifications that follow i.e. (3.37-bis), (3.39)) provides a general formula for the study of a *random* (*'stochastic'*) multi-component open system with long-range inter-particle interactions. We see that this formalism establishes a link between the process statistics and the mechanism of

[8]If the above simplifying hypothesis is *not* satisfied, i.e. when $\mathbf{M_i}(t) \neq \mathbf{I}$ (cf. §3.1.1), a lengthier expression, of no interest here, replaces (3.37): $\mathbf{C} = \mathbf{C}(\mathbf{x}, \mathbf{v}; field)$; once more, that expression may be simplified for a single-component system. However, in this case, the process is presumably *not* stationary any more.

collisions itself, through parameters like:
- the external field (if present),
- physical parameters e.g. temperature T and density n, through ϕ_{eq}

and
- particle microscopic variables: velocity $\{\mathbf{v}\}$ (and, possibly, position $\{\mathbf{x}\}$).

For a given (*any*) specific dynamical problem, one should solve (if analytically possible) the equations of motion (3.6), substitute the solution in expression (3.36) for the correlation matrix and then evaluate the resulting quantities in an appropriate reference frame. This is exactly the procedure we shall later follow in the specific case of magnetized plasma.

3.5 Projection-operator approach

It seems appropriate to point out that the formalism presented here is essentially based on the projection:

$$\boldsymbol{P} \cdot = \sigma_R \int d\Gamma_R \cdot$$

where σ_R is the reservoir distribution function (*d.f.*). Notice that:

$$\boldsymbol{P}^2 = \boldsymbol{P}$$

Therefore, what we have done, basically, above was to construct the test-particle reduced *d.f.* f by applying this operator on the complete $(\sigma + R)$ *d.f.* D:

$$\boldsymbol{P}\, D = \sigma_R\, f$$

and then derive a kinetic equation in terms of $f = \sigma_R^{-1}\, \boldsymbol{P}\, D$. Following an idea suggested in the past [124], the kinetic equation could have been obtained by defining a pair of complementary projections, say P and $Q = I - P$, and then deriving a pair of equations of evolution of $P\,D$ and $Q\,D$ in time.

The idea has been rigorously formulated in order to describe the relaxation of a small sub-system weakly-interacting with a large heat-bath (i.e. our system) in [68], [110] and a thorough formal study has been carried out in [46], as well as a series of subsequent works (see e.g. [113] and references therein). We will not go into details here. Nevertheless, let us point out that the kinetic equations derived in the following (two) chapters coincide *formally* with the ones discussed in [46], for systems described by a free-Liouvillian with a discrete spectrum of eigenvalues, and actually represent the explicit construction of the kinetic operator(s) studied in [68] from a dynamical point of view.

3.6 Conclusions

Based on microscopic particle dynamics, we have presented a general formalism, decsribing the evolution in time of the distribution function of a test-particle weakly interacting with a large heat bath in homogeneous equilibrium when an external force field is present. This method, which is actually valid for any particular dynamical problem (provided that an explicit solution of the dynamical problem is given), takes into account the existence of long-range interactions *and* the field, as the latter may strongly modify particle trajectories between collisions.

Chapter 4

Kinetic equation in a pseudo-markovian approximation: the $\Theta-$operator

Summary

Based on the Generalized Master Equation obtained in the previous Chapter, we adopt a widely used 'markovian' approximation. Evaluating the kernel along the particle's trajectories (taking into account the dynamical problem, in general), a Fokker-Planck-type (FP) partial derivative equation is thus obtained. We show that this equation does not preserve the positivity of the distribution function $f(\mathbf{x}, \mathbf{v}; t)$. This problem does not appear in the case of a uniform system.

There is a goal but where is a way?
What we call the way might only be wavering...

`Frantz Kafka`
in *The Castle*

4.1 Introduction

In the previous chapter, we derived a *non-Markovian master equation*. In order to obtain a closed equation for $f(t)$, we should now express $f(t-\tau)$ in the ME. As we have already said, this formal task is very delicate to handle, since the details of the dynamic problem (in the presence of interactions *and* the field) have to be taken into account.

Various 'markovianization' procedures have been proposed in the past. One of them is defined and then constructed in the following. Even though the result of this approach will be shown to be mathematically unacceptable, its description is provided here, in order to set some definitions and fix the ideas underlying what will follow.

4.2 A 'Markovian' approximation: the Θ-operator

The *collision term* derived so far is valid up to *second* order in the interaction. Remember that the *free* problem of motion was formally solved (to *zeroth* - order, i.e. for a particle alone). Since interactions were assumed to be weak, corrections to the particle's free trajectory are expected to be of order λ or higher. Therefore, they might be expected *not* to influence the collision term (to second order), since they may only interfere in higher orders. We may therefore assume that substituting with the *zeroth*-order solution of the problem of motion, i.e. setting

$$f(t-\tau) \approx e^{-L_0 \tau} f(t) \equiv U^{(0)}(-\tau) f(t)$$

should suffice in this order.

Furthermore, as in the vast majority of relevant kinetic studies, one is essentially interested in the *asymptotic* form of the kernel of the master equation, i.e. for times larger than the typical time scale characterizing the kernel[1]. The standard method therefore consists in considering the asymptotic limit, by taking the upper time-integration boundary t to tend to *infinity*. One thus essentially expects to obtain a linear kinetic equation with *time-independent* coefficients.

This formalism defines the realisation (to order λ^2) of a specific kinetic evolution operator, referred to as *'the Θ-operator'* (we borrow the notation used in [46], [68]. In this chapter, it will be constructed *in the general case*. It will be explicitly adapted to electrostatic plasma, in a subsequent section.

[1] It is only recently that attention was paid to the derivation of time-dependent ('running') diffusion coefficients, related to studies of space-diffusion problems; see e.g. in [7].

4.2.1 A 'quasi-Markovian' (QM) master equation

Substituting $f(t-\tau) = U^{(0)}(-\tau)f(t)$ and taking $t \to \infty$ in (3.27), one obtains the *quasi-markovian master equation*:

$$\begin{aligned} \partial_t f - L_0 f &= n_{\alpha'} \int_0^\infty d\tau \int_\Gamma d\mathbf{X_1}\; L_{\sigma 1}\, U^{(0)}(\tau)\; L_{\sigma 1}\; U^{(0)}(-\tau)\; \phi_{eq}^{\alpha'}(\mathbf{v_1})\; f(\mathbf{X};t) \\ &= \int_0^\infty d\tau\, \mathcal{K}(\tau)\, f(\mathbf{X}) \quad \equiv \quad \Theta_2(t)\, f \qquad (4.1) \end{aligned}$$

A summation over particle species (populations) α' is understood where appropriate (namely applicable in the study of plasma). Note that the propagator $U^{(0)}$ is actually the product $U_\sigma^{(0)}\, U_1^{(0)}$; remember that $U_1^{(0)}$ leaves ϕ_{eq} unchanged. This relation is the realisation, to order $\sim \lambda^2$, of the Θ operator.

Note that the *rhs* of the above master equation can be expressed[2] as the velocity derivative of a *'probability current'* $\mathbf{J}$:

$$\begin{aligned} \partial_t f - L_0 f &= \frac{\partial}{\partial \mathbf{v}} \left[n \int_0^\infty d\tau \int_\Gamma d\mathbf{v_1}\, d\mathbf{v_1}\, U^{(0)}(\tau)\; L_{\sigma 1}\; U^{(0)}(-\tau)\; \phi_{eq}(\mathbf{v_1})\; f \right] \\ &= \frac{\partial}{\partial \mathbf{v}} \mathbf{J}(\mathbf{x}, \mathbf{v}) \qquad (4.2) \end{aligned}$$

4.3 Evaluation of the kernel in the QM master equation

In order to evaluate the kernel in (4.1) we need to recall definitions (3.15), (3.16) of the Liouville operators in the collision term, as well as property (3.24). Particular attention will be paid in the evaluation of the action of the propagator on the velocity gradients appearing in the collision term.

4.3.1 Generalized gradients in Γ-space

Any function of the dynamical variables $\{\mathbf{x}, \mathbf{v}\}$ has to be evaluated along trajectories in phase space, that is taking into account the external field. We *should* point out that:

$U(t)$ does not commute with Γ-space gradients $\frac{\partial}{\partial \mathbf{v}}$, $\frac{\partial}{\partial \mathbf{x}}$.

Indeed, applying the principle of (3.24) and combining with the solution (3.6) of the problem of motion, one may show that:

$$\begin{aligned} \mathbf{D}_{\mathbf{V_i}}(t) \equiv U^{(0)}(t) \frac{\partial}{\partial \mathbf{v_i}} U^{(0)}(-t) &= \mathbf{N_i^T}(t)\, \frac{\partial}{\partial \mathbf{x_i}} + \mathbf{N'}_\mathbf{i}^\mathbf{T}(t)\, \frac{\partial}{\partial \mathbf{v_i}} \\ \mathbf{D}_{\mathbf{X_i}}(t) \equiv U^{(0)}(t) \frac{\partial}{\partial \mathbf{x_i}} U^{(0)}(-t) &= \mathbf{M_i^T}(t)\, \frac{\partial}{\partial \mathbf{x_i}} + \mathbf{M'}_\mathbf{i}^\mathbf{T}(t)\, \frac{\partial}{\partial \mathbf{v_i}} \qquad i = \sigma,\ 1^R \end{aligned} \qquad (4.3)$$

[2]To see this, recall (3.16) and use: $\int d\mathbf{v_1} \frac{\partial}{\partial \mathbf{v_1}} \cdot = 0$, $\int d\mathbf{v_1} \frac{\partial}{\partial \mathbf{v}} \cdot = \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{v_1} \cdot =$.

The detailed calculation is provided in Appendix B. At the initial instant t:

$$\mathbf{D}_{\mathbf{V}_i}(0) = \frac{\partial}{\partial \mathbf{v_i}} \quad , \qquad \mathbf{D}_{\mathbf{X}_i}(0) = \frac{\partial}{\partial \mathbf{x_i}} \qquad .$$

Notice that, if one takes $\mathbf{M} = \mathbf{I}$ (as in §3.1.1-(i) and (iii) above) the second relation reduces to:

$$\mathbf{D}_{\mathbf{X}_i}(t) = \frac{\partial}{\partial \mathbf{x_i}}$$

Obviously, for a homogeneous function ϕ:

$$\mathbf{D}_{\mathbf{V_1}}(t)\ \phi(\mathbf{v_1}) = \mathbf{N}'^{\mathbf{T}}(t)\ \frac{\partial \phi(\mathbf{v_1})}{\partial \mathbf{v_1}}$$

Since we shall focus on magnetized plasma, in the following, let us mention that a formal study of the generator of *charged-particle* motion in the presence of external fields, in particular, exists in [82]. The above considerations, and specifically relations (4.3), are in full agreement with this reference, where relation (4.3a) appears just as it stands[3], yet derived in a different, formal manner. Notice that the propagator formalism as presented here has also been involved in quasi-linear theories for plasma turbulence; see e.g. [83].

4.4 The 'Quasi-markovian' Fokker-Planck equation (QM-FPE)

4.4.1 QM-FPE in the general case

In general, the distribution function f depends on particle velocity and position: $f = f(\mathbf{x}, \mathbf{v}; t)$. We then obtain a 2nd-order partial-differential equation (*pde*) of the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\big[\mathbf{A}\frac{\partial}{\partial \mathbf{v}} + \mathbf{G}\frac{\partial}{\partial \mathbf{x}} + \mu\,\mathbf{a}\big]\, f \qquad (4.4)$$

where μ denotes the mass ratio: $\mu \equiv m/m_1^{\alpha'}$. The exact form of the *velocity-* and the *cross-velocity-position-* diffusion matrices $\mathbf{A}$ will be given below.

Notice that

$$\frac{\partial}{\partial v_i} A_{ij} \frac{\partial}{\partial v_j} \cdot = \frac{\partial^2}{\partial v_i \partial v_j}(A_{ij} \quad \cdot) - \frac{\partial}{\partial v_i}(\frac{\partial A_{ij}}{\partial v_j} \quad \cdot)$$

Also,

$$\frac{\partial}{\partial v_i} G_{ij} \frac{\partial}{\partial x_j} \cdot = \frac{\partial^2}{\partial v_i \partial x_j}(G_{ij} \quad \cdot) - \frac{\partial}{\partial v_i}(\frac{\partial G_{ij}}{\partial x_j} \quad \cdot)$$

[3]Notice, however, that the formulae in [82] refer to an *electron* plasma, and $\Omega < 0$ therein, hence the difference in signs within the plasma dynamical matrices defined in §3.1.1-(iii).

Therefore, equation (4.4) takes the form of a *diffusion equation* in phase space:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{q}} : (\mathbf{D}^{\Theta}\ f) - \frac{\partial}{\partial \mathbf{q}}(\mathcal{F}^{\Theta}\ f) \tag{4.5}$$

where $\mathbf{q} \equiv (\mathbf{x}, \mathbf{v})$ is the position vector in Γ-space. Therefore, in a d-dimensional problem ($d = 1, 2, 3$), $\mathbf{q}$ consists of d position and d velocity variables: $\mathbf{q} \in \Re^{2d}$. Obviously, this equation will be a $2d$+1-variable[4] *'Fokker-Planck'*-type equation.

The *diffusion matrix* $\mathbf{D}$ is:

$$\mathbf{D}^{\Theta}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{0} & \frac{1}{2}\mathbf{G}^{\mathbf{T}} \\ \frac{1}{2}\mathbf{G} & \mathbf{A} \end{pmatrix} \tag{4.6}$$

The $2d$-dimensional vector $\mathcal{F}^{\Theta}$ has the form: $\mathcal{F}^{\Theta} = (\mathbf{0}, \mathbf{F})^T$, where the d-dimensional vector $\mathbf{F}$ accounts for *dynamical friction* suffered by a moving particle due to the presence of surrounding ones. In general, it is defined by:

$$F_i = -\mu\, a_i + \frac{\partial A_{ij}}{\partial v_j} + \frac{\partial G_{ij}}{\partial x_j} \qquad i, j = 1, 2, 3 \tag{4.7}$$

In a variety of problems, though, A, G only depend on velocity, so the last term cancels.

4.4.2 QM-FPE in the homogeneous case

Let us now consider the special case where the system is in a *uniform* state: $f = f(\mathbf{v}; t)$. By the same procedure, we obtain an equation of the form:

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left[\mathbf{A}\frac{\partial}{\partial \mathbf{v}} + \mu\, \mathbf{a}\right] f \tag{4.8}$$

where μ again denotes the mass ratio: $\mu \equiv m/m_1^{\alpha'}$.

Carrying out an algebraic manipulation, as described in the previous paragraph, the *rhs* can be cast in the form of a *'Landau-Fokker-Planck'*-type equation [3]:

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i}(F_i\ f) + \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}(A_{ij}\ f) \tag{4.9}$$

All coefficients (to be defined below) are the same as in the general case.

If the coefficients on the *rhs* (*and* $\mathbf{F_{ext}}$) are independent of position[5] $\mathbf{x}$, the same result is obtained by considering a space-averaged distribution function $\tilde{f}(\mathbf{v}; t) = \int d\mathbf{x}\, f(\mathbf{x}, \mathbf{v}; t)$. In that case, integrating equation (4.4) (or (4.5)) over space, one readily recovers (4.8) (or (4.9), respectively).

[4]i.e. including time t.

[5]This *is* exactly the case in our systems of interest here: magnetized plasma *and* free motion. Nevertheless, this is not true in the linear oscillator case (mentioned above).

4.4.3 Coefficients

The coefficients in (4.4), functions of $\{\mathbf{x}, \mathbf{v}\}$ in general, are given by:

$$\begin{aligned}
(\mathbf{A}, \mathbf{G}) &= \frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1}) \\
&\qquad \mathbf{F_{int}}(|\mathbf{x} - \mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau) - \mathbf{x_1}(-\tau)|)\, (\mathbf{N'}^T(\tau),\, \mathbf{N}^T(\tau)) \\
&\equiv \frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{C}(\mathbf{x}, \mathbf{v}; t, t-\tau)\, (\mathbf{N'}^T(\tau),\, \mathbf{N}^T(\tau)) \\
\mathbf{a} &= -\frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1} \\
&\qquad \mathbf{F_{int}}(|\mathbf{x} - \mathbf{x_1}|) \otimes \mathbf{F_{int}}(|\mathbf{x}(-\tau) - \mathbf{x_1}(-\tau)|)\, \mathbf{N'}_1^T(\tau)\, \frac{\partial \phi(\mathbf{v_1})}{\partial \mathbf{v_1}} \\
&\equiv -\frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{d}(\mathbf{x}, \mathbf{v}; t, t-\tau) \qquad (4.10)
\end{aligned}$$

($\mathbf{a} \otimes \mathbf{a}$ denotes the tensor product i.e. $(\mathbf{a} \otimes \mathbf{a})_{ij} \equiv a_i\, a_j \qquad \forall \mathbf{a} \in \Re^d$).

In a single species system one may prove that:

$$a_i = -\partial A_{ij} / \partial v_j \qquad (4.11)$$

(the proof is provided below) so that the drift vector $\mathbf{F}$ defined above becomes:

$$\mathcal{F}_i = (1 + \mu)\, \frac{\partial A_{ij}}{\partial v_j} \qquad i, j = 1, 2, ...d \qquad (4.12)$$

(we have assumed that $\partial \{A_{ij}, G_{ij}\} / \partial x_j = 0$, as discussed before).

Notice the explicit appearance of the *Kubo coefficients* for the interactions [7], [25] in the above expressions, as well as the dependence on the external force field through the $\mathbf{N}(t), \mathbf{N}'(t)$ matrices (*and* - implicitly - through $\mathbf{F}(\mathbf{x}(t))$), which are to be evaluated along particle trajectories).

The above formulae provide a 'recipe' relating diffusion coefficients to the (interaction) force auto-correlations. In studies based on stochastic calculus, this is often done via phenomenology, usually considering a *white noise* (i.e. a δ-correlated process) which is assumed to represent (and, rather instead, hides) information provided by the microscopic collision mechanism.

Proof of (4.11). From (4.10b):

$$a_i = -\frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, F_i\, F_k\, N'_{1jk}\, \frac{\partial \phi}{\partial v_{1j}}$$

where, obviously, $F_i = F_i(t)$, $F_k = F_k(t - \tau)$ and $N'_{1jk} = N'^T_{1kj}(\tau)$. After an integration by parts:

$$a_i = +\frac{n}{m^2} \int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi\, \frac{\partial}{\partial v_{1j}} F_i\, F_k\, N'_{1jk}$$

Now, recall that the quantity to the right of the velocity derivative is a function of $\mathbf{g} = \mathbf{v} - \mathbf{v_1}$ for a single component plasma, as shown in §3.4; therefore, we can set:

$$\frac{\partial}{\partial v_{1j}} = -\frac{\partial}{\partial g_j} = -\frac{\partial}{\partial v_j}$$

and the derivative can be carried outside the integration. The remaining quantity is precisely the *rhs* of (4.10a) (defining A_{ij}):

$$a_i = -\frac{\partial}{\partial v_j}\frac{n}{m^2}\int_0^\infty d\tau \int d\mathbf{x_1} \int d\mathbf{v_1}\, \phi\, F_i F_k N'_{1jk} = -\frac{\partial A_{ij}}{\partial v_j} \qquad .$$

4.5 Mathematical properties of the QM - FPE

A kinetic equation should possess a number of properties; namely, it should preserve

(i) the *reality*,

(ii) the *normalization*

and

(iii) the *positivity* of the (probability) distribution function.

Furthermore,

(iv) an *H-theorem* should be satisfied, ensuring entropy growth as the systems approaches an equilibrium state [41].

These requirements define a *Markov semi-group* [13], [113], [52].

Let us examine these requirements, briefly.

The *first* requirement is readily met, since all coefficients in the evolution equation are *real*. Indeed, if the imaginary part of the function f is null at $t = 0$, it is trivial to show that it will remain so at all times t.

The *second*, actually ensuring particle number conservation, one can be easily verified, by checking that the integration of the collision term over Γ-space gives *zero*[6].

The *fourth* one, somewhat more 'delicate', will be studied in a separate section.

Let us focus on the *third* requirement. In order for the probability distribution to be positive at any instant t under the action of the evolution operator $\Theta(t)$, the diffusion matrix $\mathbf{D}$ should be positive definite [13], i.e. one should have, for any[7] $\mathbf{a} \in \Re^{2d}$:

$$(\mathbf{a}, \mathbf{D}\,\mathbf{a}) = \mathbf{a}^T \mathbf{D}\,\mathbf{a} = \mathbf{a}^T \mathbf{D}^{SYM}\,\mathbf{a} \geq 0 \tag{4.13}$$

This criterion is definitely *not* satisfied here: notice that $det\,\mathbf{D}^{\Theta} = -(det\mathbf{C})^2 \leq 0$. As a consequence, the Quasi-Markovian F.P. equation (4.4) *does not guarantee the preservation of the positivity* of the probability d.f. f.

[6] Recall the form (4.2) of the equation; also that $\int d\mathbf{q}\, \frac{\partial \cdot}{\partial \mathbf{q}} = 0$ - see above.

[7] Once more, d is the dimensionality of the physical space considered.

In fact, the forementioned problem is *in principle* absent from the homogeneous *d.f.* case[8] i.e. when $f = f(\mathbf{v})$. Therefore, the problem of positivity preservation has not been stressed in the past since the effect of spatial inhomogeneity of the system on the collision term has often been neglected, through one argument or another, or even plainly omitted (as discussed in the Introduction). Inhomogeneity effects in the collision term have been considered in certain works, yet the second (cross-V-X) term in the *rhs* of eq.(4.4) has always been neglected, often by assuming on physical grounds that it is negligible (see the discussion about the *"hydrodynamic approximation"* in § 18.4 in [4]; also in § VIII.6 in [41]).

This problem was pointed out in the quantum theory of open systems [57] and was later examined with respect to classical systems; see in particular [46], [68]. These authors used formal operator methods to show that the problem was due to the very construction of the kinetic equation and actually suggested a possible solution, for systems with a *discrete* spectrum of the zeroth-order Liouville operator. That formalism will be adopted here (as applied to an electrostatic plasma).

4.6 A qualitative paradigm: stochastic acceleration in 1d

We have drawn the conclusion that the *Fokker-Planck-type equation* obtained in the usual *pseudo-'Markovian'* approximation (i.e. (4.4)) is *incorrect*, as d.f. positivity is not preserved in time. Indeed, the solution of equations of the form of (4.4) is ill-defined, as may be verified numerically for a given system. This point can be illustrated by the simple example of a Fokker-Planck-type equation with *constant* coefficients. Consider, for instance, a particle subject to a (white noise) stochastic acceleration in one dimension. The usual 'markovian' approximation leads to the equation[9]:

$$\partial_t f \;+\; v\,\partial_x f \;=\; d\,\partial_v^2 f \;+\; c\,\partial_v\,\partial_x f \tag{4.14}$$

where all coefficients are constant ($c, d \in \Re_+$). At a first step, let us set $c = 0$:

$$\partial_t f \;+\; v\,\partial_x f \;=\; d\,\partial_v^2 f \tag{4.15}$$

Considering the Fourier transform of $f(x, v; t)$, one may solve the associated *PDE* in Fourier space and then integrate to obtain a solution in terms of a Green function $G(x, x', v, v'; t)$, for a given initial distribution $f(x, v; 0)$ [42].

[8] Rigorously speaking, of course, this should result from the velocity-diffusion matrix $\mathbf{A}$ being a positive definite matrix itself. Whether this is the case, should be checked in the particular problem considered. The relevant confirmation for magnetized plasma will be given where appropriate.

[9] See also [47], where an equation of 4.14 describing stochastic acceleration is obtained and discussed, in the context of electrostatic turbulence (coefficients therein are not constant, though).

For a $\delta-$function initial distribution in phase space, one thus obtains an explicit solution for $f(x,v;t)$, which is easily shown to be positive at all times $t \geq 0$.

Now let us set $c \neq 0$, recovering the cross-V-X diffusion term $c\,\partial_v\,\partial_x\,f$ in the *rhs* of the above equation, which now becomes:

$$rhs \,=\, \begin{pmatrix} \partial_x \\ \partial_v \end{pmatrix} \begin{pmatrix} 0 & c/2 \\ c/2 & d \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_v \end{pmatrix} f$$

One thus introduces a 2d-'diffusion' matrix which yields two eigenvalues of opposite signs. Contrary to the *'correct' FPE* 4.15, equation (4.14) in general has no *probability* solution, since the second-order coefficient matrix is not positive. Indeed, as one may check analytically, the corresponding Green function develops a singularity at some instant of time[10]. The details of this calculation will be omitted here, yet the method is discussed in a forthcoming chapter in detail (see ch. 10).

[10] What happens, exactly, is that one comes up with integrals of the form $\int_{-\infty}^{\infty} dk\, e^{iA(t)x}\, e^{-B(t)x^2}$ where $B(t)$ becomes negative for certain values of t.

Chapter 5

A Markovian kinetic operator - construction of the $\Phi-$operator

Summary

In search for a *correct* Markovian approximation, we consider a modified collision operator (referred to as *the* 'Φ*-operator'*). A *modified kinetic equation* is obtained, taking into account the external field. The result of this calculation is valid for any specific dynamical problem (assuming that an exact solution is known).

I accept no principles of physics
which are not also accepted in mathematics ...

René Déscartes
in *Principles of Philosophy*

5.1 Towards a Markovian kinetic equation

We are now in search of a well-defined kinetic evolution operator, possessing the expected mathematical properties (see discussion in the previous chapter). We will consider a markovian operator (defined as the 'Φ *operator*') which was introduced in the so-called *theory of subdynamics* by I. Prigogine and his collaborators [66], [67], [97]. It was extensively studied in the context of weakly-coupled open quantum systems by Davies [13], [62], who pointed out the importance of the positivity (non-)preservation problem mentioned above[1].

For *classical* systems, the operator was later considered for systems possessing a *discrete* spectrum of eigenvalues (of the free Liouville operator) [46]. Its explicit construction (to second order in λ) was formally carried out therein; its mathematical properties were thoroughly investigated in [68] and properly established. The Φ operator has also been considered for a special class of classical systems possessing a *continuous* spectrum (see ch. 8 in [46]; also in [112]).

In this chapter, we will try to obtain a less abstract formulation of this evolution operator. Its construction, to λ^2, for the system described before, will lead to a *correct* new kinetic equation for a test-particle problem (in the weak-coupling approximation).

5.2 Time-averaging operator

The Φ operator essentially amounts to:

$$\Phi \;=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt'\;U^{(0)}(t')\,\Theta\,U^{(0)}(-t') \tag{5.1}$$

(cf. the above references). As we see, the action of this operation consists in shifting the system successively backward and forward in time, by an interval t', and then averaging over a time step which is finally extended to the whole of the system's history in time. This operation has the desired effect i.e. 'loss of memory' on behalf of the system.

This averaging operator will now be applied to the kinetic evolution operator derived previously.

5.3 The modified equation

Let us now apply the above operation (5.1) to the evolution equations obtained in subsequent orders in λ. The 0th- and 1st-order terms coincide. However, the 2nd-order (collision) term (see (3.27)) is dramatically modified. It now reads:

$$\Phi_2\,f=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt'\int_{0}^{\infty}d\tau\int d\mathbf{X_1}\,L_I(t')\,L_I(t'+\tau)\,\phi_{eq}^{\alpha'}(\mathbf{X_1})\,f(\mathbf{X};t) \tag{5.2}$$

[1] In fact, the $\Phi-$operator is reported as '*Davies' device*' in Van Kampen's book [49] (nevertheless, curiously enough, the classical case is not addressed therein).

i.e.

$$\Phi_2\, f = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \int_0^{\infty} d\tau\, \mathcal{K}_\sigma(\tau, t')\, f \tag{5.2-bis}$$

The definition of the *kernel* $\mathcal{K}_\sigma$ is evident. L_I denotes the interaction Liouville operator $L_{\sigma 1}$ (3.16). Note that, by setting $t' = 0$, we readily recover the Θ-operator:

$$\Theta_2\, f = \int_0^{\infty} d\tau \int d\mathbf{X_1}\, L_I\, L_I(\tau)\, \phi_{eq}^{\alpha'}(\mathbf{X_1})\, f(\mathbf{X};t)$$

$$\equiv \int_0^{\infty} d\tau\, \mathcal{K}_\sigma(\tau, t'=0)\, f \tag{5.3}$$

In the following we shall use the convenient notation

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \cdot \; \equiv \; \mathcal{A}_{t'} \cdot \tag{5.4}$$

5.4 Evaluation of the kernel

We may now evaluate the kernel in equation (5.2) by explicitly substituting from definitions (3.15), (3.16), in combination with (3.24) and (4.3) (just as we did in §4.4).

The calculation is rather lengthy, yet quite straightforward. It consists in evaluating the action of the propagator $U^{(0)}(\pm t')$ on quantities appearing in the collision term. In specific, $L_I(t)$ ($t \in \Re$) appearing in the above equation is equal to:

$$L_I(t) = U(t)\, L_{\sigma 1}\, U(-t)$$

so, recalling definition (3.16):

$$L_I(t) = \mathbf{F} \left[m_\sigma^{-1} \frac{\partial}{\partial \mathbf{v}_\sigma} \,-\, (\sigma \to 1) \right]$$

we have:

$$L_I(t) = \mathbf{F}(t) \left[m_\sigma^{-1}\, U(t) \frac{\partial}{\partial \mathbf{v}} U(-t) - (\sigma \to 1) \right] \equiv \mathbf{F}(t) \left[m_\sigma^{-1}\, \mathbf{D}_\mathbf{v}(t) - (\sigma \to 1) \right]$$

(the generalized gradients $\mathbf{D}_\mathbf{v}(t)$, $\mathbf{D}_\mathbf{x}(t)$ were defined in §4.3.1; also see App. B).

To be more specific, let us consider, for instance, the quantity:

$$U^{(0)}(t') \left[\frac{\partial}{\partial v_i} C_{ij}(\tau)\, D_{v_j}(\tau) \right] U^{(0)}(-t') =$$

$$\left[U^{(0)}(t') \frac{\partial}{\partial v_i} U^{(0)}(-t') \right] \left[U^{(0)}(t')\, C_{ij}(\tau)\, U^{(0)}(-t') \right] \left[U^{(0)}(t')\, D_{v_j}(\tau)\, U^{(0)}(-t') \right]$$

Now recall the definition (4.3) for the generalized phase space gradients:

$$D_{v_j}(\tau) \equiv U^{(0)}(\tau)\partial_{v_j} U^{(0)}(-\tau)$$

The first factor within brackets is equal to $D_{v_i}(t')$, while the third one is $D_{v_j}(t'+\tau)$ (since $U(t')\,U(\tau) = U(t'+\tau)$). The second factor is exactly equal to $C_{ij}(\tau)$ [2]. Therefore, we obtain:

$$U^{(0)}(t')\,\big[\frac{\partial}{\partial v_i}\,C_{ij}(\tau)\,D_{v_j}(\tau)\big]\,U^{(0)}(-t') = D_{v_i}(t')\,C_{ij}(\tau)\,D_{v_j}(t'+\tau) \qquad (5.5)$$

The rest of the calculation follows the same methodology.

The modified kernel $\mathcal{K}_\sigma(\tau, t')$ finally becomes:

$$\begin{aligned}\mathcal{K}_\sigma(\tau, t') = n_{\alpha'}\ \mathbf{D_V}(t')\,\mathbf{C}(\mathbf{x}, \mathbf{v}; t-t', t-t'-\tau)\ \mathbf{D_V}(t'+\tau) \\ - n_{\alpha'}\ \mathbf{D_V}(t')\ \mathbf{d}(\mathbf{x}, \mathbf{v}; t-t', t-t'-\tau) \qquad (5.6)\end{aligned}$$

where all differential operators are assumed to act on everything on their right; $\mathbf{D}_V(t')$ was defined in (4.3); $\mathbf{C}(\mathbf{x}, \mathbf{v}; t_1, t_2) = \langle \mathbf{F_{int}}(t_1)\ \mathbf{F_{int}}(t_2)\rangle_R$ was evaluateded in §3.4) (remember, once more: $\mathbf{C}(\mathbf{x}, \mathbf{v}; t-t', t-t'-\tau) = \mathbf{C}(\mathbf{x}, \mathbf{v}; \tau)$, for a stationary process)
and:

$$\begin{aligned}\mathbf{d}(\mathbf{x}, \mathbf{v}; t_1, t_2) \quad = \quad \int_{\Gamma_1} d\mathbf{x_1} d\mathbf{v_1}\,\mathbf{F_{int}}\big(|\mathbf{x}(t_1) - \mathbf{x_1}(t_1)|\big)\,\mathbf{F_{int}}\big(|\mathbf{x}(t_2) - \mathbf{x_1}(t_2)|\big) \\ \mathbf{D_{V_1}}(\tau)\,\phi^{\alpha'}_{eq}(\mathbf{v_1}) \qquad (5.7)\end{aligned}$$

For comparison, the kernel in the QM-ME (4.1), say $\mathcal{K}(\tau)$, can be expressed as:

$$\mathcal{K}(\tau) = n_{\alpha'}\ \frac{\partial}{\partial \mathbf{v}}\ \mathbf{C}(\mathbf{x}, \mathbf{v}; t, t-\tau)\ \mathbf{D_V}(\tau)\ \ -\ \ n_{\alpha'}\ \frac{\partial}{\partial \mathbf{v}}\ \mathbf{d}(\mathbf{x}, \mathbf{v}; t, t-\tau)$$

so

$$\mathcal{K}^{(\Theta)}(\tau) = \mathcal{K}^{(\Phi)}(\tau, t=0) \qquad .$$

One is now left with the task of substituting from (5.6) into equation (5.2) and then applying the averaging operator $\mathcal{A}_{t'}$ defined above, for the specific physical problem considered. For a stationary process, for instance, one has to evaluate the action of $\mathcal{A}_{t'}$ on the *rhs* of (5.5), upon substitution with:

$$D_{v_i}(t') = N_{il}(t')\ \frac{\partial}{\partial x_i} + N'_{il}(t')\ \frac{\partial}{\partial v_i}$$

(cf. (4.3)) and so forth. One thus needs, basically, to evaluate the action of $\mathcal{A}_{t'}$ quantities like:

$$\mathcal{A}_{t'}\ \mathcal{M}_{il}(t')\,C_{ij}(\tau)\,\mathcal{M}_{jk}(t'+\tau)$$

[2]To see this, one must apply the property (3.24) in expressions in §3.4 for $\mathbf{C}$; formally, for a stationary process: $U(t')\,C(\tau)\,U(-t') \equiv U(t')\,C(t, t+\tau)\,U(-t') = C(t+t', t+t'+\tau)\,U(t')\,U(-t') = C(\tau)$.

where $\mathcal{M}_{il}$ can be either of the matrices: N_{il}, N'_{il} defined above. Assuming that the correlation matrix $C_{ij}(\tau)$ is diagonal in an appropriate reference frame: $C_{ij}(\tau) = C_{ii}(\tau)\,\delta_{ij}$, one finally obtains, for the above quantity:

$$C_{ii}(\tau)\,\mathcal{A}_{t'}\,\mathcal{M}_{il}(t')\,\mathcal{M}_{ik}(t'+\tau)$$

which can be evaluated once the matrices are specified [3].

5.5 The markovian Fokker-Planck equation (M-FPE)

Proceeding as described above, the kernel in the modified ('Φ'-) kinetic equation can be evaluated explicitly. We thus obtain a new (markovian) Fokker-Planck-type PDE of the form:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} &= \frac{\partial}{\partial \mathbf{q}}\left[\mathbf{D}^{(\Phi)}(\mathbf{v})\,\frac{\partial}{\partial \mathbf{q}} + \mu\,\mathbf{a}^{(\Phi)}(\mathbf{v})\right] f(\mathbf{q};t) \\ &\equiv \Phi_2 f \end{aligned} \tag{5.8}$$

After an algebraic manipulation, this second-order PDE may take the form of a Fokker-Planck-type diffusion equation in the 6-dimensional space $\{\mathbf{q} \equiv (\mathbf{x},\mathbf{v}) \in \Re^6\}$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{q}}(\mathcal{F}^{(\Phi)}\,f) + \frac{\partial}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{q}} : (\mathbf{D}^{(\Phi)}\,f) \tag{5.8-bis}$$

cf. (4.5).

The new 6x6 *diffusion matrix* $\mathbf{D}^{(\Phi)}$ is:

$$\begin{aligned} \mathbf{D}^{(\Phi)} &= \mathbf{D}^{(\Phi)}(\mathbf{q}; field) \\ &= \frac{n}{m^2}\int_0^\infty d\tau\,\mathcal{A}_{t'}\begin{pmatrix} \mathbf{N}(t') \\ \mathbf{N}'(t') \end{pmatrix}\mathbf{C}(t-t', t-t'-\tau) \\ &\qquad \begin{pmatrix} \mathbf{N}^{\mathbf{T}}(t'+\tau) & \mathbf{N}'^{\mathbf{T}}(t'+\tau) \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{D_{11}}^{(\Phi)} & \mathbf{D_{12}}^{(\Phi)} \\ \mathbf{D_{21}}^{(\Phi)} & \mathbf{D_{22}}^{(\Phi)} \end{pmatrix} \end{aligned} \tag{5.9}$$

and the 6d vector $\mathbf{a}$ reads:

$$\begin{aligned} \mathbf{a}^{(\Phi)} &= \mathbf{a}^{(\Phi)}(\mathbf{q}; field) \\ &= \frac{n}{m^2}\int_0^\infty d\tau\,\mathcal{A}_{t'}\begin{pmatrix} \mathbf{N}^{\mathbf{T}}(t')\,\mathbf{d}(t-t', t-t'-\tau) \\ \mathbf{N}'^{\mathbf{T}}(t')\mathbf{d}(t-t', t-t'-\tau) \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{a_1}^{(\Phi)} \\ \mathbf{a_2}^{(\Phi)} \end{pmatrix} \end{aligned} \tag{5.10}$$

[3] If the process is *not* stationary, C_{ii} depends on t', so the integration in $\mathcal{A}_{t'}$ should be done very carefully (remember however that interactions in our system of interest *are* stationary).

As we saw in §3.4.2, $\mathbf{C}(t-t', t-t'-\tau)$ in (5.9) (as well as $\mathbf{C}(t, t-\tau)$ in (4.10)) is equal to $\mathbf{C}(\tau)$ (i.e. $= \mathbf{C}(t, t+\tau) = \mathbf{C}(0, \tau)$, cf. (3.37) in §3.4) in a *stationary* process.

In a single species system all quantities are functions of $\mathbf{v} - \mathbf{v_1} \equiv \mathbf{g}$; one may then prove that

$$(a_1^{(\Phi)})_i = -\partial (D_{12})_{ij} / \partial v_j \quad , \qquad (a_2^{(\Phi)})_i = -\partial (D_{22})_{ij} / \partial v_j \tag{5.11}$$

The 6-d *dynamical friction vector* $\mathcal{F}$ is related to $\mathbf{a}^{(\Phi)}$ by:

$$\begin{aligned} \mathcal{F}_i &= \begin{pmatrix} -\mu\, a_{1,i} + \frac{\partial D_{11,ij}}{\partial x_j} + \frac{\partial D_{12,ij}}{\partial v_j} \\ -\mu\, a_{2,i} + \frac{\partial D_{21,ij}}{\partial x_j} + \frac{\partial D_{22,ij}}{\partial v_j} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{F}^{(\Phi)}_{(X),i} \\ \mathcal{F}^{(\Phi)}_{(V),i} \end{pmatrix} \end{aligned} \tag{5.12}$$

Note that only the *symmetric* part of $\mathbf{D}^{(\Phi)}$ contributes to the diffusive part of (5.8) i.e.

$$\begin{aligned} \mathbf{D}^{SYM} &= \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^T\right) = \begin{pmatrix} \mathbf{D_{11}}^{SYM} & \frac{1}{2}\left(\mathbf{D_{21}} + \mathbf{D_{12}}^T\right)^T \\ \frac{1}{2}\left(\mathbf{D_{21}} + \mathbf{D_{12}}^T\right) & \mathbf{D_{22}}^{SYM} \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{D_{XX}} & \mathbf{D_{VX}}^T \\ \mathbf{D_{VX}} & \mathbf{D_{VV}} \end{pmatrix} \end{aligned}$$

5.5.1 Expanded form of the M-FPE

The new kinetic equation (5.8) obtained above can be expanded to the form:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \\ &\quad \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}} : (\mathbf{D}^{(VV)} f) + 2\frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{x}} : (\mathbf{D}^{(VX)} f) + \frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}} : (\mathbf{D}^{(XX)} f) \\ &\quad\quad - \frac{\partial}{\partial \mathbf{v}}(\mathcal{F}^{(\Phi)}_{(V)} f) - \frac{\partial}{\partial \mathbf{x}}(\mathcal{F}^{(\Phi)}_{(X)} f) \end{aligned} \tag{5.13}$$

We note the appearance of the third term in the *rhs*, accounting for diffusion in real (position) space. This element was absent from known previous collision operators, derived rigorously from a *GME*. Remember that the *rhs* of the *GME* is equal to the *velocity* gradient of a *'probability current'* in Γ-space; this excludes the appearance of a *position* diffusion operator in the collision term. The appearance of such a term is nevertheless expected from an *intuitive* point of view and, not surprisingly, certain authors have chosen to add this term by hand [73]. In this respect, our study is more systematic and may claim the ambition to serve as a link between phenomenology and microscopic theories.

5.5.2 M-FPE in the homogeneous case

In the *homogeneous case* i.e. $f = f(\mathbf{v}; t)$, $\mathbf{D} = \mathbf{D}(\mathbf{v})$ and eq.(5.8) becomes:

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\big[\mathbf{D_{22}}^{(\Phi)}\frac{\partial}{\partial \mathbf{v}} + \mu\,\mathbf{a_2}^{(\Phi)}\big]\, f \qquad (5.8 - HOM)$$

It is interesting to see that the modified $\Phi-$ operator coincides with the old one ($\Theta-$) in this case[4], i.e. (5.8-HOM) is identical to (4.8). To see this, follow the argument in the last paragraph in §5.4. For the velocity diffusion part, $\mathcal{M}_{il}$ is N'_{il}, so we have to deal with the quantity:

$$\begin{aligned} C_{ii}(\tau)\,\mathcal{A}_{t'}\,N'_{il}(t')\,N'_{ik}(t'+\tau) &= C_{ii}(\tau)\,\mathcal{A}_{t'}\,N'_{li}(-t')\,N'_{ik}(t'+\tau) \\ &= C_{ii}(\tau)\,\mathcal{A}_{t'}\,N'_{lk}(\tau) \\ &= C_{ii}(\tau)\,N'_{lk}(\tau) \end{aligned}$$

due to the group properties (3.11). One thus immediately sees that:

$$\mathbf{D_{22}}^{\Phi} = \mathbf{D_{22}}^{\Theta} = \mathbf{A}$$

(see (4.10). We draw the conclusion that the two kinetic operators defined above *coincide* in the homogeneous case.

5.6 Mathematical properties

The mathematical properties stated in §4.5 were thoroughly established, one by one, in [46], [68] with respect to the operator defined in (5.1). In particular, having constructed the operator by using *canonical variables*, it was shown therein that in order for the probability distribution to be positive at any instant t under the action of the evolution operator Φ_2 constructed above, the diffusion matrix $\mathbf{D}^{\Phi}$ should be *positive definite*, i.e. one should have, for any $\mathbf{a} \in \Re^6$:

$$\begin{aligned} (\mathbf{a}, \mathbf{D}^{\Phi}\,\mathbf{a}) &= \mathbf{a}^T\mathbf{D}^{\Phi}\,\mathbf{a} = \mathbf{a}^T\mathbf{D}^{\Phi\,SYM}\,\mathbf{a} \\ &= (D_{XX})_{ij}x_i x_j + (D_{VV})_{ij}v_i v_j + 2(D_{VX})_{ij}v_i x_j \geq 0 \end{aligned} \qquad (5.14)$$

Remember, however, that our $\Phi-$ *FP* equation (5.8) has been derived by making use of *non-canonical variables* (see discussion in ch. 3), so the results in [46], [68] do not *automatically* hold, just as they stand. Nevertheless, in every particular problem of interest, one may show that the associated diffusion matrix $\mathbf{D}$ is non-negative definite, and that an equilibrium solution exists. Both positivity conservation and existence of an H-theorem then follow directly from the results of [113].

In conclusion, the Markovian *F.P.* equation (5.8) *does* guarantee the preservation of the positivity of the probability d.f. f; this statement should however be confirmed in each specific system considered, and cannot generally be asserted at this (formal) stage.

[4]Literally speaking, this is shown here for a reference frame where $\mathbf{C}$ is a diagonal matrix. Nevertheless, in *any* frame, the expression obtained for the time evolution of angle-independent observables (e.g. $\langle\rho\rangle = \langle(x^2+y^2)^2\rangle$, $\langle v_\perp\rangle = \langle(v_x^2+v_y^2)^2\rangle$) *is* indeed the same between the two operators, once the phase angle has been averaged out; see discussion in §6.5.4.

5.7 Conclusions - discussion

The kinetic equation (5.8), together with definitions (5.9) to (5.13), are to be retained as the explicit result of this chapter. The proposed *'recipe'* for the kinetic study of a test-particle problem is now complete. Given a specific problem, one has to evaluate the force correlations, as described previously, and then substitute into the above expressions for the coefficients. The generic form of the kinetic equation (5.13) will then be valid as it stands. Remember that the velocity- part of the collision operator remains unchanged, the cross-velocity-space diffusion part is a priori modified and, finally, a new term arises, allowing for the description of space diffusion phenomena.

Part A: Concluding remarks

The previous section concludes the first part of this thesis. *Part A* was devoted to the kinetic formalism applied in the description of a test-particle problem, when an external field is present. So far, we have presented a general formalism, from first microscopic principles, which leads to a non-Markovian master equation. This method, which is actually valid for any particular dynamical problem (provided that an explicit solution of the 'free' problem of motion is given), takes into account the existence of long-range interactions *and* the field.

We have shown that the 'standard' *'pseudo-Markovian'* assumption leads to a linear kinetic equation which does not preserve the positivity of the distribution function $f(\mathbf{x}, \mathbf{v}; t)$. By considering a different Markovian operator, a *correct Fokker-Planck-type equation* was obtained.

A set of exact general expressions for all coefficients were derived; their form explicitly depends on the external field and also on the interaction law $V(r)$ and the reservoir configuration $\phi_{eq}(\mathbf{v})$.

Part B
Application to Magnetized Plasma

Because in the beginning,
was the plasma ...

Hannes Alfvén
(The concluding words of H. Alfvén's
Nobel lecture, in 12-11-1970.)

Chapter 6

Kinetic description of magnetized plasma: construction of the plasma Θ-operator

Summary

The formalism presented in the first part is now applied in the case of an electrostatic plasma inside a uniform and stationary magnetic field. Considering the Θ-operator defined above, the corresponding kinetic equation is derived. Explicit expressions for the coefficients are obtained, in the form of an infinite series of functions involving Bessel functions of the first kind. Their form depends explicitly on the magnitude of the magnetic field and also on the interaction potential considered $V(r)$ and the reservoir equilibrium configuration $\phi_{eq}(\mathbf{v})$.

All models are wrong but some are useful.

G. E. P. Box
in
Robustness in the strategy
of scientific model building.

6.1 Introduction

The test-particle problem was formulated previously and two distinct kinetic evolution operators were discussed with respect to it. In this chapter, we proceed by constructing the *pseudo-'Markovian'* kinetic evolution operator presented in chapter 4 (the Θ-operator), in the case of an electrostatic plasma inside a magnetic field. The mathematical properties of the plasma kinetic equation thus obtained will be questioned in the end of this Chapter.

6.2 Kinetic description

Magnetized plasma as a statistical mechanical system, as well as the formulation of the related test-particle problem, were defined previously. The formalism which is necessary for its study follows exactly the guidelines set in the first part of the thesis. According to our previous considerations, the basis for a kinetic description is exactly provided by the set of equations derived in the previous chapters. They will be summarized here for clarity and reference.

(i) The Fokker-Planck-type equation (4.8) (or, better, (4.9)):

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i}(F_i\, f) + \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}(A_{ij}\, f) \tag{6.1}$$

for a *homogeneous* distribution $f(\mathbf{v};\, t)$.

(ii) The Markovian kinetic equation (5.8) (our *'MFP equation'*):

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} =$$

$$= \frac{\partial^2}{\partial v_i \partial v_j}(D^{(\Phi)}_{(VV),ij}\, f) + 2\,\frac{\partial^2}{\partial v_i \partial x_j}(D^{(\Phi)}_{(VX),ij}\, f) + \frac{\partial^2}{\partial x_i \partial x_j}(D^{(\Phi)}_{(XX),ij}\, f)$$
$$- \frac{\partial}{\partial v_i}(F^{(\Phi)}_{(V),i}\, f) - \frac{\partial}{\partial x_i}(F^{(\Phi)}_{(X),i}\, f) \tag{6.2}$$

in the general case $f(\mathbf{x}, \mathbf{v};\, t)$. This is the *correct* generalization of the former for *inhomogeneous* systems - in contrast, that is, with:

(iii) the *'pseudo-Markovian'* equation (4.4):

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\mathbf{F_{ext}}\frac{\partial f}{\partial \mathbf{v}} =$$

$$= \frac{\partial^2}{\partial v_i \partial v_j}(D^{(\Theta)}_{(VV),ij}\, f) + 2\,\frac{\partial^2}{\partial v_i \partial x_j}(D^{(\Theta)}_{(VX),ij}\, f) - \frac{\partial}{\partial v_i}(F^{(\Theta)}_{(V),i}\, f) \tag{6.3}$$

Note that equations (6.2), (6.3), both 2nd order *PDE*s, are different in *structure*[1] but also (presumably) in the *form* of their coefficients. However, as ar-

[1] The last term in the second and third lines of the former is absent in the latter.

gued in the previous chapter (and confirmed later in this one) all coefficients are functions of particle velocity $\mathbf{v}$ only, and the velocity-diffusion matrices will be proved to coincide in all three equations:

$$D^{(\Theta)}_{(VV),ij} = D^{(\Phi)}_{(VV),ij} = A_{ij}$$

(the same is valid for the friction vector $F_{(V)}$). Therefore, by integrating either of these two equations over position $\{\mathbf{x}\}$, one recovers precisely the FPE (6.1), describing the evolution of, say

$$f_{local}(\mathbf{v};t) = \int d\mathbf{x} f(\mathbf{x}, \mathbf{v}; t)$$

We thus see that the two operators Θ and Φ coincide in the homogeneous case, where they are both represented by equation (6.1)[2]. As we mentioned previously, this equation can be viewed as a *'linearized'* version of a kinetic equation which has appeared in earlier works e.g. [16], [64], [85], [103]. We will come back on this point later on.

6.3 Method of calculation

The general expressions for the coefficients in the kinetic equation, presented in Chapter 4, are valid in this case. Therefore, one just has to evaluate those expressions by appropriately substituting with the dynamic matrices $\mathbf{M}(t)$, $\mathbf{N}(t)$ for helicoidal motion, i.e. as defined in §3.1.1 (briefly recalled in the next paragraph, for convenience). This is a rather lengthy yet quite straightforward task. In general, we have proceeded by
(i) substituting the definition of the dynamic matrices $M_{ij}(t)$, $N_{ij}(t)$ (appearing in the solution of the problem of motion, see below) into the formulae derived in Chapter 4,
(ii) expressing the integrals (in $\mathbf{k}$, $\mathbf{v_1}$) in cylindrical coordinates, as suggested above - this is quite convenient since, due to the symmetry of the problem, the angle variable is absent in the integrand quantities[3] and can then be integrated out immediately -
and then
(iii) evaluating the integrals in an appropriate reference frame.

In the *'conventional'* approach[4], one evaluates the time integral first (actually obtaining a $\delta-$function relating $\mathbf{k}$ and $\mathbf{v_1}$) and then discusses the form of the potential *F.T.* $\tilde{V}_k$ or the reservoir equilibrium df ϕ_{eq}. This is exactly what we will do in the following. However, an alternative method of evaluation, allowing for a more tractable analytic treatment of the coefficients, will be presented in chapter 8.

---

[2] Literally speaking, this is true as a result of phase-averaging; see discussion in the previous chapter.

[3] Remember that both the *F.T.* of $V(r)$ i.e. $\tilde{V}_k$ and the reservoir *df* $\phi_{eq}(v_1)$ depend on the modulus of their argument *only*, and *not* on the phase.

[4] See e.g. Appendix 2.A.1 in [5]; cf. the unmagnetized case in Appendix J.

In order for this presentation to remain concise and self-contained (yet rigorous) we will only briefly expose the procedure in the following, omitting lengthy details. However, a few sample evaluations incorporating all the necessary details are provided in the Appendix.

The different expressions for *equal-* and *different-species* collisions (i.e. whether $\alpha = \alpha'$ or not) will be presented in separate sections below.

6.3.1 Particle motion in a magnetic field

Let us consider a *uniform* field along the $z-$axis:

$$\mathbf{B} = B\,\hat{z}$$

The exact (spiral) solution of the problem of motion was given by (3.8) above; let us recall that it can be expressed in the following form:

$$\begin{pmatrix} \mathbf{x}^{(0)}(t) \\ \mathbf{v}^{(0)}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{N}(t) \\ \mathbf{0} & \mathbf{R}(t) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} \equiv \mathbf{E}(t) \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} \tag{6.4}$$

where we have defined the matrices:

$$\mathbf{M}(t) = \mathbf{I}, \qquad \mathbf{M}'(t) = \mathbf{0}$$

$$\mathbf{N}'^{\alpha}(t) = \mathbf{R}^{\alpha}(t) = \begin{pmatrix} \cos\Omega_\alpha t & s\,\sin\Omega_\alpha t & 0 \\ -s\,\sin\Omega_\alpha t & \cos\Omega_\alpha t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{N}^{\alpha}(t) = \int_0^t dt'\ \mathbf{R}^{\alpha}(t) = \Omega_\alpha^{-1} \begin{pmatrix} \sin\Omega_\alpha t & s\,(1-\cos\Omega_\alpha t) & 0 \\ s\,(\cos\Omega_\alpha t - 1) & \sin\Omega_\alpha t & 0 \\ 0 & 0 & \Omega_\alpha t \end{pmatrix} \tag{6.5}$$

Ω is the gyro- (cyclotron) frequency:

$$\Omega = \Omega_\alpha = \frac{|e_\alpha| B}{m_\alpha c} \tag{6.6}$$

($|e_{\alpha'}| = +Z_{\alpha'}\,|e|$ for an '$Z-$ply' charged positive ion) and s is the sign of the particle:

$$s = s_\alpha = \frac{e_\alpha}{|e_\alpha|} = \pm 1$$

Notice that the Ω is defined as a positive quantity here. The influence (if any) of the *sense* of gyration (i.e. *clockwise* or *anti-clockwise*, depending on the *sign* of the particle charge) will be reflected in the appearance of s in certain expressions.

Notice that the perpendicular and parallel components of particle velocity, $v_\perp$ and $v_\parallel$ respectively:

$$v_\perp \equiv \left(v_x^2 + v_y^2\right)^{1/2}, \qquad v_\parallel \equiv v_z$$

are conserved. Particle energy therefore remains constant under the action of a magnetic field[5]; as a consequence, the equilibrium df of the reservoir will be $\phi_{eq} = \phi_{eq}(v_\perp, v_\|)$ [6].

We now have to substitute this solution into the formulae derived in Chapter 4 and then evaluate the resulting integrals in a convenient coordinate frame.

6.3.2 Reference frame

As the problem is characterized by *cylindrical symmetry* (compare to the unmagnetized problem, which is spherical-symmetric), all vectors appearing in the formulae, say $\mathbf{a} = (a_x, a_y, a_z)$ (namely, $\mathbf{a} \in \{\mathbf{v}, \mathbf{v_1}, \mathbf{k}\}$) will be naturally expressed in cylindrical coordinates $\{a_\perp, \theta, v_\|\}$ for convenience, where

$$a_\perp \equiv \left(a_x^2 + a_y^2\right)^{1/2} \quad \geq 0\,, \qquad a_\| \equiv a_z \qquad\qquad \forall \mathbf{a} \in \Re^3$$

and θ is an appropriate angle, ie. $arctan(a_y/a_x)$, whose value depends on the reference frame considered. Therefore, any vector $\mathbf{a}$ may be expressed as:

$$\mathbf{a} = (a_\perp \cos\theta, a_\perp \sin\theta, a_\|)$$

and integration in $d^3\mathbf{a}$ becomes:

$$\int_0^\infty da_\perp\, a_\perp \int_{-\infty}^\infty da_\| \int_0^{2\pi} d\theta\, \cdots \equiv \int\!\!\int d^c\mathbf{a} \int_0^{2\pi} d\theta\, \cdots$$

('c'='cylindrical').

As the $z-$ axis is defined by the magnetic field, once and for all, one is left with the freedom of choice of the $x-$ and $y-$ axes. One choise of frame is the following, in view of making the evaluation of the Fourier integral more convenient.

Frame 1: The x-axis is taken along $\mathbf{g}_\perp$:

$$\{\hat{e}_1,\, \hat{e}_2,\, \hat{e}_3\} \;=\; \{\hat{g}_\perp,\, \hat{b} \times \hat{g}_\perp,\, \hat{b}\}$$

where

$$\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$$

(see figure 6.1); obviously

$$\hat{a} \equiv \frac{1}{|\mathbf{a}|}\mathbf{a} \qquad\qquad \forall \mathbf{a} \in \Re^3$$

is the unit vector in the direction of $\mathbf{a}$. Remember that $\mathbf{v}$ is the test-particle velocity - actually the final variable in the diffusion coefficients - and $\mathbf{v_1}$ is the velocity of the (*any*) reservoir particle, i.e. the integration variable (which "disappears" once the integrations are carried out).

[5]Lorentz forces are perpendicular to the (helicoidal) displacement and thus provide no mechanical work.

[6]For instance, a Maxwellian *df* $\phi_{Max}(v)$ is a function of the *norm* of the velocity $v \equiv \left(v_\perp^2 + v_\|^2\right)^{1/2}$.

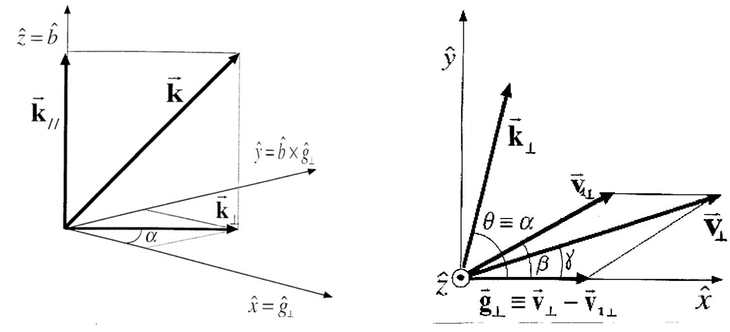

Figure 6.1: Frame 1: the x-axis is taken along the velocity difference $\mathbf{g} = \mathbf{v} - \mathbf{v_1}$.

This frame is chosen for convenience in manipulating the integrals in our expressions. A slightly different frame (*'frame 2'*) will be considered later on (see figure 6.2).

6.4 Evaluation of correlations

The results of §3.4.2 can be applied in order to calculate the (interaction-) force autocorrelation matrix. Details of the calculation can be found in Appendix C.

We will first evaluate the exponential factor $e^{i\mathbf{k\Delta r}}$ appearing in (3.36) - (3.35) in the *single species* case[7]. Let α (θ) be the angles between the x axis and $\mathbf{k}_\perp$ ($\mathbf{g}_\perp$, respectively) in any frame [8]. The exponential $e^{i\mathbf{k\Delta r}(\tau)}$ is equal to:

$$e^{i\mathbf{k\Delta r}^\alpha} = e^{iZ\sin(\theta-\alpha)}\, e^{-iZ\sin(\theta-\alpha-\Omega\tau)}\, e^{ik_\| g_\| \tau} \tag{6.7}$$

where we have defined:

$$Z \equiv \frac{k_\perp g_\perp}{\Omega}, \qquad \mathbf{g} \equiv \mathbf{v} - \mathbf{v_1} \tag{6.8}$$

Notice that, taking the limit $\Omega \to 0$, one readily recovers $\mathbf{k\Delta r}(\tau) \to \mathbf{kg}\tau$, just as expected from the free-of-field case [9].

Substituting into the general relation (3.39) for the 2-time force-correlation matrix and using the Bessel function identity:

$$e^{i\,x\,\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\,\phi} \qquad \forall x, \phi \in \Re \tag{6.9}$$

[7] Assuming $\alpha = \alpha'$ that is, i.e. *either* considering a single component system, e.g. electron plasma *or* the $\alpha = \alpha'$ term (in the general case), where the *t.p.* and R particles colliding with it obey the same dynamics $(\mathbf{N_1}(t) = \mathbf{N_\Sigma}(t))$.

[8] As obvious, the angle variable is distinguished in meaning from α, denoting particle species.

[9] Indeed, as $\Omega \to 0$, $\mathbf{N}(\tau) \to \tau\mathbf{I}$ and

$$\mathbf{k\,\Delta r} = \mathbf{k\,N}(\tau)\,\mathbf{g} \to \mathbf{k\,g}\,\tau$$

we obtain the symmetric form:

$$\underline{\underline{\mathbf{C}}}(t_1, t_2) = n \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1})\, (2\pi)^3 \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^{\infty} dk_\parallel\, \tilde{V}_k^2 \sum_{n=-\infty}^{\infty} e^{i\omega_n (t_1 - t_2)}$$

$$J_n \begin{pmatrix} \frac{1}{4} k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & i\, s\, \frac{1}{4} k_\perp^2 (J_{n+2} - J_{n-2}) & \frac{1}{2} k_\perp k_\parallel (J_{n-1} + J_{n+1}) \\ i\, s\, \frac{1}{4} k_\perp^2 (J_{n+2} - J_{n-2}) & \frac{1}{4} k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & -s\frac{1}{2} k_\perp k_\parallel (J_{n-1} - J_{n+1}) \\ \frac{1}{2} k_\perp k_\parallel (J_{n-1} + J_{n+1}) & -s\frac{1}{2} k_\perp k_\parallel (J_{n-1} - J_{n+1}) & k_\parallel^2 J_n \end{pmatrix} \tag{6.10}$$

(in frame 1). Note the definitions:

$$Z \equiv \frac{k_\perp g_\perp}{\Omega}, \qquad \mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}, \qquad \omega_n \equiv n\Omega + k_\parallel g_\parallel \tag{6.11}$$

$J_n \equiv J_n(Z)$ are Bessel functions of the first kind; their argument is Z everywhere.

Notice that the integration in $\alpha-$ (the angle variable in $\mathbf{k}$-Fourier space) has been carried out, since the integrand does *not* depend on α (the Fourier transform $\tilde{V}_k$ of the interaction potential depends on $k \equiv (k_\perp^2 + k_\parallel^2)^{1/2}$ only, as $V = V(r)$).

The (random) interaction between the test-particle and the heat bath comes out to be a *stationary* process, as $\mathbf{C}(t_1, t_2) = \mathbf{C}(t_1 - t_2 \equiv \tau)$. Notice that, as expected, C_{ij} come out to be *even* in the time argument[10] τ:

$$C_{ij}(-\tau) = C_{ij}(\tau) \qquad .$$

6.5 Diffusion coefficients - same-species

We may now substitute relation (6.10) for the force-correlation as well as the zeroth-order solution (6.4) of the problem of motion into the general expression (4.10) for the coefficients, in order to obtain the analytic expressions for the coefficients in the magnetized plasma case. For clarity, a sample evaluation of the matrix elements of $\mathbf{A}$ in detail, is provided in the Appendix.

(4.10-a) now reads:

$$\begin{aligned} (\mathbf{A}(\mathbf{v}),\, \mathbf{G}(\mathbf{v})) &= \frac{1}{m^2} n_\alpha (2\pi)^3 \int_0^\infty d\tau \int d\mathbf{v_1}\, \phi_{eq}^\alpha(\mathbf{v_1}) \\ &\quad \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^\alpha(\tau)(v_m - v_{1,m})}\, \mathbf{k} \otimes \mathbf{k}\, ({\mathbf{R}_\alpha}^T(\tau),\, {\mathbf{N}_\alpha}^T(\tau)) \\ &\equiv \frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{C}^{(\alpha,\alpha)}(\tau)\, ({\mathbf{R}_\alpha}^T(\tau),\, {\mathbf{N}_\alpha}^T(\tau)) \end{aligned} \tag{6.12}$$

[10] $\mathbf{C}(\tau)$ is by definition a *real* matrix, so only the *real* part ($\cos \omega_n \tau$) of the exponential is relevant.

One may now substitute from (6.10) and successively evaluate integrals in $\mathbf{k}$, $\mathbf{v_1}$, τ.

The time integral can be evaluated by using the *Plemelj* formula:

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty d\tau\, e^{i\omega\tau} &= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\frac{\epsilon}{\omega^2+\epsilon^2} + i\,\frac{1}{\pi}\lim_{\epsilon\to 0^+}\frac{\omega}{\omega^2+\epsilon^2} \\ &\equiv \delta(\omega) + i\,\frac{1}{\pi}\,\mathcal{P}(\frac{1}{\omega}) \\ &\equiv \delta_+(\omega) \qquad (6.13)\end{aligned}$$

where $\mathcal{P}(\frac{1}{x})$ denotes the principal value of the Cauchy integral [3], [7]. The final result has the form:

$$\left\{\begin{array}{c}\mathbf{A}(\mathbf{v})\\ \mathbf{G}(\mathbf{v})\end{array}\right\} = \frac{n_\alpha}{m^2}(2\pi)^5\pi\int d^c\mathbf{v_1}\,\phi_{Max}(v_{1\perp}, v_{1\parallel})\int d^c\mathbf{k}\,\tilde{V}_k^2 \sum_{n=-\infty}^{\infty}\left\{\begin{array}{c}\mathbf{M_1}^{(n)}\\ \mathbf{M_2}^{(n)}\end{array}\right\}\delta_+(\omega_n) \qquad (6.14)$$

where $\tilde{V}_k$ denotes the Fourier transform of the interaction potential (defined in §3.4.2). Keep in mind that $\int d^c\mathbf{a}$ ('c'='cylindrical') denotes the integration $\int_0^\infty da_\perp\, a_\perp \int_{-\infty}^\infty da_\parallel$ $(\mathbf{a} \in \Re^3)$ (the angle integration has already been carried out in (6.14), since neither ϕ_{Max} nor $\tilde{V}_k$ depend on the angle variable); once more, $J_n = J_n(Z)$ are Bessel functions of the first kind (remember the definitions (6.11)).

The form matrices appearing in the right-hand-side depend on the choise of frame[11]. In frame 1, they are presented in the following.

6.5.1 Diffusion matrix in frame 1

In frame 1, the matrices $\mathbf{M_{1,2}}^{(n)}$ in (6.14) are given by:

$$\mathbf{M_1}^{(n)} = \mathbf{M_1^\alpha}(k_\perp, k_\parallel; g_\perp, g_\parallel; \Omega)$$

$$= \begin{pmatrix} k_\perp^2\,\frac{n^2}{Z^2}\,J_n^2(Z) & i\,s\,k_\perp^2\,\frac{n}{Z}\,J_n(Z)\,J_n'(Z) & k_\perp\,k_\parallel\,\frac{n}{Z}\,J_n^2(Z) \\ -i\,s\,k_\perp^2\,\frac{n}{Z}\,J_n(Z)\,J_n'(Z) & k_\perp^2\,J'^2_n(Z) & -i\,s\,k_\perp\,k_\parallel\,J_n(Z)\,J_n'(Z) \\ k_\perp\,k_\parallel\,\frac{n}{Z}\,J_n^2(Z) & i\,s\,k_\perp\,k_\parallel\,J_n(Z)\,J_n'(Z) & k_\parallel^2\,J_n^2(Z) \end{pmatrix}$$

$$\mathbf{M_2}^{(n)} = \mathbf{M_2^\alpha}(k_\perp, k_\parallel; g_\perp, g_\parallel; \Omega)$$

[11] The form of matrices $\mathbf{M1, 2}$ presented here depends on the frame used for the Fourier integration, indeed. However, this is rather 'fictitious'. Once the final integration in velocity $\mathbf{v_1}$ has been carried out, the result should be valid in *any* frame; the only final argument in the coefficients should be the *t.p.* velocity $\mathbf{v}$.

$$= \Omega^{-1} \begin{pmatrix} i\, k_\perp^2 \left[\frac{n}{Z^2} J_n^2 - \frac{2n}{Z} J_n J_n'\right] & s\, k_\perp^2\, \frac{1}{Z}\, J_n\, J_n' & -i\Omega k_\perp\, k_\parallel\, \frac{n}{Z}\, J_n^2\, \frac{d}{d\omega_n} \\ -s\, k_\perp^2 \left[\left(\frac{n^2}{Z^2} - 1\right) J_n^2 - \frac{1}{Z} J_n J_n' + {J_n'}^2\right] & -i k_\perp^2\, \frac{n}{Z^2} J_n^2 & -s\Omega\, k_\perp\, k_\parallel\, J_n\, J_n' \frac{d}{d\omega_n} \\ -2i\, k_\perp\, k_\parallel\, J_n\, J_n' & 0 & -i\Omega k_\parallel^2\, J_n^2 \frac{d}{d\omega_n} \end{pmatrix} \tag{6.15}$$

where

$$J_n'(Z) = \frac{dJ_n(Z)}{dZ}$$

The argument of Bessel functions is $Z \equiv \frac{k_\perp g_\perp}{\Omega}$ everywhere; cf. (6.11). The δ_+ function was defined above[12].

6.5.2 Reduction of the coefficients

We may carry out the $k_\parallel$-integration in the above formulae, using:

$$\int_{-\infty}^{\infty} dk_\parallel f(k_\parallel) \delta(\omega_n) = \frac{1}{|g_\parallel|} \int_{-\infty}^{\infty} d\omega_n\, f(\frac{\omega_n - n\Omega}{g_\parallel})\, \delta(\omega_n) = \frac{1}{|g_\parallel|} f(-\frac{n\Omega}{g_\parallel})$$

where we took into account the definition (6.11c) (the contribution of the principal part $\mathcal{P}$ in $\delta_+(\cdot)$ cancels for reasons of symmetry). In this way, the tensor, say $\mathbf{Q}$ (cf. (6.39), (6.40))[13], in the velocity-diffusion matrix $\mathbf{A}(\mathbf{v})$ comes out to be of the form:

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} & -\frac{g_\perp}{g_\parallel} Q_{11} \\ -Q_{12} & Q_{22} & Q_{23} \\ -\frac{g_\perp}{g_\parallel} Q_{11} & -Q_{23} & \left(\frac{g_\perp}{g_\parallel}\right)^2 Q_{11} \end{pmatrix} \tag{6.16}$$

for a given form of potential $V(r)$. The matrix elements are:

$$Q_{11} = \frac{n}{m^2} (2\pi)^4\, \pi\, \frac{1}{|g_\parallel|}\, \frac{\Omega^4}{{g_\perp}^4} \sum_{m=-\infty}^{+\infty} m^2 \int_0^\infty dZ\, Z\, J_m^2(Z)\, \hat{V}_k^2(Z)$$

$$Q_{22} = \frac{n}{m^2} (2\pi)^4\, \pi\, \frac{1}{|g_\parallel|}\, \frac{\Omega^4}{{g_\perp}^4} \sum_{m=-\infty}^{+\infty} \int_0^\infty dZ\, Z^3\, {J_m'}^2(Z)\, \hat{V}_k^2(Z)$$

---

12 In the last column of the second matrix, i.e. $M^{(n)}_{2,\ i3}$, we have used:

$$\frac{1}{\pi} \int_0^\infty d\tau\, \tau\, e^{i\omega\tau} = \frac{1}{\pi} \int_0^\infty d\tau\, \frac{1}{i}\, \frac{\partial}{\partial\omega} e^{i\omega\tau} = (-i)\, \frac{d}{d\omega} \left(\frac{1}{\pi} \int_0^\infty d\tau\, e^{i\omega\tau} \right) = \cdots = -i\, \delta_+'(\omega_n)$$

13 Remember that $\mathbf{A}$ has the form:

$$A_{ij} = \int d\mathbf{v_1}\, Q_{ij}\, \phi_{eq}(\mathbf{v_1})\, f(\mathbf{x}, \mathbf{v}; t) \qquad .$$

$$
\begin{aligned}
Q_{12} &= -s\frac{n}{m^2}(2\pi)^4\,\pi\,\frac{\Omega^4}{{g_\perp}^4}\sum_{m=-\infty}^{+\infty} m\int_0^\infty dZ\,Z^2 \\
&\qquad \mathcal{P}\int_{-\infty}^{\infty} dk_\parallel\,\tilde{V}_k^2\,\frac{1}{k_\parallel g_\parallel + m\Omega}\,J_m(Z)\,J'_m(Z) \\
Q_{23} &= s\frac{n}{m^2}(2\pi)^4\,\pi\,\frac{\Omega^3}{{g_\perp}^3}\sum_{m=-\infty}^{+\infty}\int_0^\infty dZ\,Z^2 \\
&\qquad \mathcal{P}\int_{-\infty}^{\infty} dk_\parallel\,k_\parallel\,\tilde{V}_k^2\,\frac{1}{k_\parallel g_\parallel + m\Omega}\,J_m(Z)\,J'_m(Z)
\end{aligned}
\tag{6.17}
$$

(the $k_\perp$ integration variable was shifted to Z defined in (6.11)) where:

$$
\hat{V}_k(Z) \equiv \tilde{V}_{k\equiv\sqrt{k_\perp^2+k_\parallel^2}}(Z)\Big|_{k_\parallel=-\frac{m\Omega}{g_\parallel}}
\tag{6.18}
$$

Note that A_{11}, A_{22}, A_{33} are positive quantities.

For the Coulomb potential:

$$
V(\mathbf{r}) = V(r) = \frac{e_\Sigma e_1^\alpha}{r}\,, \qquad \tilde{V}_{\mathbf{k}} = \tilde{V}_k = \frac{e_\Sigma e_1^\alpha}{2\pi^2}\frac{1}{k^2} \equiv \frac{e_\Sigma e_1^\alpha}{2\pi^2}\frac{1}{k_\perp^2+k_\parallel^2}
\tag{6.19}
$$

we have:

$$
\hat{V}_k(Z) = \frac{e_\Sigma e_1^\alpha}{2\pi^2}\Big(\frac{g_\perp}{\Omega}\Big)^2\frac{1}{Z^2+\big(m\frac{g_\perp}{g_\parallel}\big)^2} \equiv \frac{e_\Sigma e_1^\alpha}{2\pi^2}\Big(\frac{g_\perp}{\Omega}\Big)^2\frac{1}{Z^2+a_m^2}
\tag{6.20}
$$

A similar reduction can be carried out for the $\mathbf{G}$ matrix; the analytic expression for the result would be of no interest here and will be omitted. Note that upon substitution from (6.20) into (6.17) certain series may be shown to diverge. A more sophisticated form of potential therefore has to be considered, eventually taking into account dynamical screening of charge. This well-known problem actually points out the limits of the weak-coupling approximation, since one only considers *binary* interactions up to 2nd order, neglecting all other contributions.

The reduction carried out here was actually suggested by Montgomery et al. in [85] (for the velocity diffusion matrix $\mathbf{A}$ only, as a matter of fact); it was explicitly reported here in order to point out the very structure of the matrix $\mathbf{A}$: see that (6.16) immediately implies:

$$
\sum_1^3 Q_{ij}\,v_i\,v_j \;=\; \left(v_1 - \frac{g_\perp}{g_\parallel}\,v_3\right)^2 + v_2^2 \qquad \geq 0
$$

for any vector $\mathbf{v} \in \Re^3$, so $\mathbf{Q}$ is readily seen to be positive definite.

Nevertheless, the result of this section is quite complicated to manipulate[14]. It was only provided for reference, here, and will be abandoned in the following,

[14] Notice that the integration variable $\mathbf{v_1}$ is included in Z; also, an infinite series of integrals is obtained, and its convergence is rather questionable.

in favor of a different (analytically far more tractable) approach (see Chapter 8).

6.5.3 Friction terms

The vector **a** in the kinetic equation(s) (4.8), (4.4) is given by:

$$a_i = -\partial A_{ij}/\partial v_j \tag{6.21}$$

that is:

$$\begin{aligned}
a_x &= -\frac{\partial A_{11}}{\partial v_x} - \frac{\partial A_{12}}{\partial v_y} - \frac{\partial A_{13}}{\partial v_z} \\
a_y &= -\frac{\partial A_{21}}{\partial v_x} - \frac{\partial A_{22}}{\partial v_y} - \frac{\partial A_{23}}{\partial v_z} \\
a_z &= -\frac{\partial A_{31}}{\partial v_x} - \frac{\partial A_{32}}{\partial v_y} - \frac{\partial A_{33}}{\partial v_z}
\end{aligned} \tag{6.22}$$

(in frame 1). Thus, the 3d dynamical friction vector (drift term) $\mathcal{F}_{(V)}$ in the kinetic equation(s) in the beginning of this chapter, is given by:

$$\mathcal{F}_i = -\,\mu\, a_i + \frac{\partial A_{ij}}{\partial v_j} \;=\; (1+\mu)\,\partial A_{ij}/\partial v_j \tag{6.23}$$

where μ is the t.p./ion mass ratio:

$$\mu = \frac{m}{m_1} = \frac{m_\alpha}{m_{\alpha'}} \qquad .$$

6.5.4 An alternative expression for the diffusion coefficients

When deriving a kinetic operator, say $\mathcal{K}$, one is interested in studying the evolution of a macroscopic magnitude, say A, via an associated microscopic function, say a, evolving under the action of the kinetic operator:

$$A(t) = \int d\mathbf{x}\, d\mathbf{v}\; a\, f(\mathbf{x}, \mathbf{v};\, t), \qquad \left.\frac{\partial A}{\partial t}\right|_{coll} = \ldots = \int d\mathbf{x}\, d\mathbf{v}\; a\mathcal{K}\, f$$

(see the discussion in the Introduction). In fact, we are mostly interested in studying phenomena related to the plane perpendicular to the magnetic field, and actually to phase-independent quantities a e.g. $v_\perp = (v_x^2 + v_y^2)^{1/2}$, $\rho = (x^2+y^2)^{1/2}$ etc. Therefore, the angle variable(s) involved in the above integral(s) (expressed in polar coordinates) can be averaged out immediately, provided that the quantity $a = (\mathbf{x}, \mathbf{v};\, t)$ in the integrand is independent of the angle (phase-) variable. This fact can be exploited by:

1st) defining a new reference frame $x'y'z'$, say *'frame 2'*: once more, the z-axis lies along the field $\mathbf{B}$, yet the x-axis is now taken along $v_\perp$ [15]:

$$\{\hat{e}_1, \, \hat{e}_2, \, \hat{e}_3\} \; = \; \{\hat{v}_\perp, \, \hat{b} \times \hat{v}_\perp, \, \hat{b}\} \, .$$

2nd) expressing the velocity derivatives (in the collision term) in the new frame. Basically, this amounts to a rotation by a specific angle, say γ with respect to the $x-$ (and around the $z-$) axis.
3rd) carrying out the integration in γ [16].
We thus obtain a kinetic equation which essentially describes the evolution of gyro-tropic (or, better, gyro-averaged) distribution functions: $f(\mathbf{x}; \, v_\perp, v_\parallel) = \int_0^{2\pi} d\gamma \, f(\mathbf{x}; \, v_\perp, \gamma, v_\parallel)$.

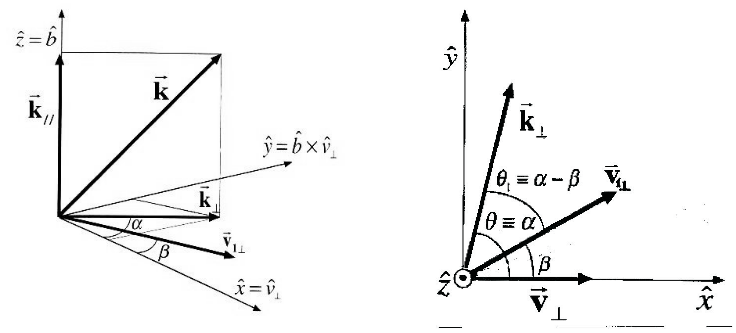

Figure 6.2: Frame 2: the x-axis is taken along the test-particle velocity $\mathbf{v}_\perp$. Compared to frame 1, this is essentially a rotation with respect to the z-axis.

The expressions presented in the previous section, were derived in frame 1. In frame 2, (6.14) still holds, yet with a different form for $\mathbf{M_1}^{(n)}$:

$$\mathbf{M_1}^{(n)} = \mathbf{M}^{\alpha}_{\mathbf{1}}(k_\perp, k_\parallel; g_\perp, g_\parallel; \, \Omega) =$$

[15]Since the $x-$ axis is now along the t.p. velocity component $\mathbf{v}_\perp$, this frame is expected to yield a form of the operator describing evolution in *two* directions i.e. a variation in the $\parallel -$ (i.e. $z-$) and $\perp = x-$ (*only*) directions, as often considered, phenomenologically. We therefore expect, and will indeed see, that $x-$ and $y-$ directions will be equivalent in this frame.

[16]One may easily check that the above amount to the gyro-averaging operator:

$$\frac{1}{2\pi} \int_0^{2\pi} d\gamma \, \mathbf{R}^{-1}(\gamma) \, \mathbf{D} \, \mathbf{R}(\gamma) = ... = \frac{1}{2\pi} \begin{pmatrix} \frac{1}{2}(D_{11} + D_{22}) & \frac{1}{2}(D_{12} - D_{21}) & 0 \\ -\frac{1}{2}(D_{12} - D_{21}) & \frac{1}{2}(D_{11} + D_{22}) & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \tag{6.24}$$

where $\mathbf{R}(\gamma)$ is the usual matrix of rotation around the $z-$axis; we have used the fact that: $\int_0^{2\pi} d\gamma \, \sin^2\gamma = \int_0^{2\pi} d\gamma \, \frac{1}{2}(1 - \cos 2\gamma) = \frac{1}{2}$ and $\int_0^{2\pi} d\gamma \, \sin\gamma = \int_0^{2\pi} d\gamma \, \sin 2\gamma = 0$ (the same results hold for a *cosine* instead of the *sine*). $\mathbf{D}$ is e.g. the diffusion matrix $\mathbf{A}$ here. The factor $\int_0^{2\pi}$ will be omitted in the following.

$$= \begin{pmatrix} k_\perp^2 \, \frac{1}{4} \left[J_{n-1}^2(Z) \, + \, J_{n+1}^2(Z) \right] & i\, s\, k_\perp^2 \, \frac{1}{4} \left[J_{n-1}^2(Z) \, - \, J_{n+1}^2(Z) \right] & 0 \\ -i\, s\, k_\perp^2 \, \frac{1}{4} \left[J_{n-1}^2(Z) \, - \, J_{n+1}^2(Z) \right] & k_\perp^2 \, \frac{1}{4} \left[J_{n-1}^2(Z) \, + \, J_{n+1}^2(Z) \right] & 0 \\ 0 & 0 & k_\parallel^2 \, J_n^2(Z) \end{pmatrix} \tag{6.25}$$

The diffusion matrix **A** now takes the form:

$$\mathbf{A}(v_\perp, v_\parallel) = \begin{pmatrix} D_\perp & D_\angle & 0 \\ -D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \equiv D_\perp \, \mathbf{1}_\perp \, + \, D_\angle \, \mathbf{1}_\angle \, + D_\parallel \, \mathbf{1}_\parallel$$

(definitions are obvious). The same operation actually annihilates the contribution of $\mathbf{M_2}$ (i.e. the cross-V-X term **G**)[17] (i.e. for a gyrotropic *df* f).

Remember that *all* of the coefficients are functions of $\{(v_x^2 + v_y^2)^{1/2}, v_z\} \equiv \{v_\perp, v_\parallel\}$ only.

Friction terms

In frame 2, the vector **a** in the kinetic equation(s) (4.8), (4.4) is given by:

$$a_i = -\partial A_{ij}/\partial v_j \tag{6.26}$$

where A_{ij} is now expressed in frame 2, that is:

$$\begin{aligned} a_x &= -\frac{\partial A_{11}}{\partial v_x} - \frac{\partial A_{12}}{\partial v_y} = -\frac{\partial D_\perp}{\partial v_x} - \frac{\partial D_\angle}{\partial v_y} \\ a_y &= -\frac{\partial A_{21}}{\partial v_x} - \frac{\partial A_{22}}{\partial v_y} = +\frac{\partial D_\angle}{\partial v_x} - \frac{\partial D_\perp}{\partial v_y} \\ a_z &= -\frac{\partial A_{33}}{\partial v_z} = -\frac{\partial D_\parallel}{\partial v_z} \end{aligned} \tag{6.27}$$

Thus, the 3d dynamical friction vector (drift term) $\mathcal{F}_{(V)}$ in the kinetic equation(s) in the beginning of this chapter, is given by:

$$\mathcal{F}_i = -\,\mu\, a_i + \frac{\partial A_{ij}}{\partial v_j} \; = \; (1+\mu)\, \partial A_{ij}/\partial v_j \tag{6.28}$$

or

$$\mathcal{F}_x \;\; = \;\; (1+\mu)\, \big(\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y} \big)$$

[17] Check that the calculation suggested in the previous footnote simply cancels the cross-V-X term **G** (i.e. $\mathbf{M_2}$), if one assumes that $\partial f/\partial z = 0$, as we will (see next chapter); indeed, if $\mathbf{x} = (\rho, \gamma', z)$, $\mathbf{v} = (v_\perp, \gamma, v_\parallel)$, we have:

$$\int_0^{2\pi} \frac{\partial}{\partial \mathbf{v}} \, \mathbf{G} \, \frac{\partial}{\partial \mathbf{x}} = \int_0^{2\pi} \frac{\partial}{\partial \mathbf{v}'} \, \mathbf{R}^{-1}(\gamma) \, \mathbf{G} \, \mathbf{R}(\gamma') \, \frac{\partial}{\partial \mathbf{x}'} \; = \; ... \; = (...) \, \frac{\partial f}{\partial z} \, , 0 \qquad .$$

$$\begin{aligned}
\mathcal{F}_y &= (1+\mu)\,\big(-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y}\big) \\
\mathcal{F}_z &= (1+\mu)\,\frac{\partial D_\parallel}{\partial v_z}
\end{aligned} \tag{6.29}$$

Remember that μ is the t.p./ion mass ratio:

$$\mu = \frac{m}{m_1} = \frac{m_\alpha}{m_{\alpha'}}$$

Correlations in frame 2

It is interesting to see that the force auto-correlation matrix entering the formulae can be considered to be *diagonal* in this frame:

$$\mathbf{C}'(t_1, t_2) = n \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1})\, (2\pi)^3 \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^{\infty} dk_\parallel\, \tilde{V}_k^2 \sum_{n=-\infty}^{\infty} e^{i\omega_n (t_1 - t_2)}$$

$$J_n \begin{pmatrix} \frac{1}{4}\, k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & 0 & 0 \\ 0 & \frac{1}{4}\, k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & 0 \\ 0 & 0 & k_\parallel^2 J_n \end{pmatrix}$$

$$\equiv \begin{pmatrix} C_\perp & 0 & 0 \\ 0 & C_\perp & 0 \\ 0 & 0 & C_\parallel \end{pmatrix} \tag{6.30}$$

(compare to (6.10)); see in App. C for details[18].

6.6 Coefficients - the multiple-species case

In the case of *t.p.* and *R*- particles of *different species* ($\alpha \neq \alpha'$) a different set of expressions replaces the above ones. Once more, only the final results are presented in this section; see in the Appendix for details.

Expression (4.10-a) is now equivalent to:

$(\mathbf{A}(\mathbf{v}),\, \mathbf{G}(\mathbf{v})) =$

---

[18] Recall the form of the diffusion coefficients:

$$\mathbf{A} \sim \int d\tau\, \mathbf{C}(\tau)\, \mathbf{R}^T(\Omega\tau)$$

where $\mathbf{R}(\theta)$ is the matrix of (anti-clockwise) rotation around the $z-$axis. Now, shifting to frame 2 (cf. previous footnote):

$$\mathbf{R}^{-1}(\gamma)\, \mathbf{C}(\tau)\, \mathbf{R}^T(\Omega\tau)\, \mathbf{R}(\gamma) \;=\; ... = [\mathbf{R}^{-1}(\gamma)\, \mathbf{C}(\tau)\, \mathbf{R}(\gamma)]\, \mathbf{R}^T(\Omega\tau)$$

so:

$$\mathbf{A}' \sim \int d\tau\, \mathbf{C}'(\tau)\, \mathbf{R}^T(\Omega\tau)$$

in the new frame.

$$
\begin{aligned}
&= \frac{1}{m^2} \sum_{\alpha'} n_{\alpha'} (2\pi)^3 \int_0^\infty d\tau \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \\
&\quad \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\, \mathbf{k} \otimes \mathbf{k}\, (\mathbf{R}_\alpha{}^T(\tau),\, \mathbf{N}_\alpha{}^T(\tau)) \\
&\equiv \frac{1}{m^2} \sum_{\alpha'} \int_0^\infty d\tau\, \mathbf{C}^{(\alpha,\alpha')}(\tau)\, (\mathbf{R}_\alpha{}^T(\tau),\, \mathbf{N}_\alpha{}^T(\tau))
\end{aligned}
\tag{6.31}
$$

Once more, one may now substitute all quantities as defined above and then successively evaluate the integrals in $\mathbf{k}$, $\mathbf{v_1}$, τ. This procedure goes as previously.

The exponential $e^{i\mathbf{k}\Delta\mathbf{r}}$ is now equal to:

$$
e^{i\mathbf{k}\Delta\mathbf{r}^{\alpha,\alpha'}(\tau)} = e^{i\, Z^\alpha\, [\sin(\Omega^\alpha \tau - s\,\theta) + \sin(s\,\theta)]}\, e^{i\, Z_1^{\alpha'}\, [\sin(\Omega^{\alpha'} \tau - s^{\alpha'}\,\theta_1) + \sin(s_1\,\theta_1)]}\, e^{i\, k_\parallel\, g_\parallel\, \tau}
\tag{6.32}
$$

where the subscript $1^{\alpha'}$ denotes the R-particle (and σ^α - or '*no subscript*' - denotes the test-particle); θ, θ_1 denote the angle between $\mathbf{k}_\perp$ and $\mathbf{v}_\perp$, $\mathbf{v_1}_\perp$, respectively (see figure 6.1); $\mathbf{g} = \mathbf{v} - \mathbf{v_1}$. We have defined:

$$
Z^\alpha \equiv \frac{k_\perp v_\perp}{\Omega_\alpha}, \qquad Z_1^{\alpha'} \equiv \frac{k_\perp v_{1,\perp}}{\Omega_{\alpha'}}
\tag{6.33}
$$

Notice the difference from the above definitions for Z (in the single-species case).

The long calculation is carried out just like previously (see in Appendix C). It finally yields an expression surprisingly similar to the previous one (6.15):

$$
\begin{aligned}
(\mathbf{A}(\mathbf{v}),\, \mathbf{G}(\mathbf{v})) &= \sum_{\alpha'} \frac{n_{\alpha'}}{m_\alpha^2} (2\pi)^5 \pi \int d^c\mathbf{v_1}\, \phi_{Max}^{\alpha'}(v_{1\perp}, v_{1\parallel}) \\
&\quad \int d^c\mathbf{k} \sum_{m=-\infty}^{\infty} J_m^2(Z_1)\, \tilde{V}_k^2 \sum_{n=-\infty}^{\infty} (\mathbf{M_1}^{\alpha(n)},\, \mathbf{M_2}^{\alpha(n)})\, \delta_+(\omega_{nm})
\end{aligned}
\tag{6.34}
$$

where $\tilde{V}_k$ denotes the Fourier transform of the interaction potential (defined in §3.4.2). The matrices appearing in the right-hand-side are exactly the same as in (6.15) (with Z as argument, still evaluated for the *t.p.* species α). The argument in the δ_+−function is:

$$
\omega_{nm} = n\,\Omega^\alpha \;-\; m\,\Omega^{\alpha'} \;+\; k_\parallel\, v_\parallel
\tag{6.35}
$$

6.7 Relation between drift and diffusion coefficients

In addition to the general result, relating the drift coefficients to the velocity derivatives of the diffusion ones:

$$
a_i = -\partial A_{ij} / \partial v_j
$$

i.e.

$$\mathcal{F}_i = (1+\mu)\,\frac{\partial A_{ij}}{\partial v_j}$$

($\mu = \frac{m_\alpha}{m_{\alpha'}}$; see in chapter 4), it can be proved that, for a Maxwellian reservoir state:

$$\phi_{eq}(\mathbf{v_1}) = \phi_{eq}(0)\, e^{-\beta\, v_1^2}$$

the vector a_i is related to D_{ij} by[19]:

$$\mu\, a_i = 2\beta\, A_{ij}\, v_j \tag{6.36}$$

In consequence,

$$\mathcal{F}_i = -2\beta\,\frac{1+\mu}{\mu} A_{ij}\, v_j \tag{6.37}$$

implying that the diffusion coefficients are directly related to the dynamical friction coefficient η, defined via the drift vector: $\mathcal{F}(\mathbf{v}) = -\eta(\mathbf{v})\,\mathbf{v}$ (see in [3]).

In order to prove expression (6.36), one first needs to recall the original form of the collision term:

$$\mathcal{K}\{f\} = \frac{\partial \mathbf{J}}{\partial \mathbf{v}} \tag{6.38}$$

where the *probability current* J_i has the structure:

$$\begin{aligned}
\mathbf{J} &= \int d\mathbf{v_1}\,\mathbf{Q}\left(\frac{\partial}{\partial \mathbf{v}} - \mu\frac{\partial}{\partial \mathbf{v_1}}\right)\phi_{eq}(\mathbf{v_1})\, f(\mathbf{x},\mathbf{v};\,t) \\
&= \ldots \\
&= D_{ij}\,\frac{\partial f}{\partial v_j} + a_i\, f
\end{aligned} \tag{6.39}$$

where the definition of $\mathbf{Q}$ is obvious, upon inspection from preceding formulae:

$$\mathbf{Q} = \frac{n}{m^2}\,(2\pi)^3 \int_0^\infty d\tau \int d\mathbf{k}\,\mathbf{k}\otimes\mathbf{k}\,\tilde{V}_{\mathbf{k}}^2\, e^{i\mathbf{k}\mathbf{\Delta r}(\tau)}\,\mathbf{N'}^T(\tau) \tag{6.40}$$

Now, one may prove the important relation:

$$\mathbf{Q}\,(\mathbf{v} - \mu\mathbf{v_1}) = \mathbf{0} \tag{6.41}$$

(see in the Appendix); (6.36) follows immediately, as:

$$\begin{aligned}
\mu\, a_i &= -\mu \int d\mathbf{v_1}\, Q_{ij}\,\frac{\partial \phi_{Max}(\mathbf{v_1})}{\partial v_{1,j}} = -\mu \int d\mathbf{v_1}\, Q_{ij}\,(-2\beta\,\phi_{Max} v_{1,j}) \\
&= 2\beta \int d\mathbf{v_1}\, Q_{ij}\,\mu v_{1,j}\,\phi_{Max} = 2\beta \int d\mathbf{v_1}\, Q_{ij}\, v_j\,\phi_{Max} \\
&= 2\beta\, v_j \int d\mathbf{v_1}\, Q_{ij}\,\phi_{Max} \\
&\equiv 2\beta\, v_j\, A_{ij}
\end{aligned} \tag{6.42}$$

As can be easily verified, expression (6.41) implies that $\mathcal{K}\{f_{Max}(\mathbf{v})\} = 0$ i.e. the Maxwellian function is an equilibrium state[20].

---

[19] Notice that this is a special case of a more general result: see in [46] (eq. (6.32) in p. 6-7 therein), also in [111] (p. 98).

[20] Check by directly substituting into (G.2) that $\mathbf{J} = \mathbf{0}$ in this case .

6.8 Conclusions - discussion

This chapter was devoted to the construction of the *pseudo-Markovian* kinetic operator defined in Part A, as applied in magnetized plasma. A Fokker-Planck-type kinetic equation was derived and exact analytic expressions were obtained, their form depending explicitly on the magnitude of the magnetic field as well as the interaction potential considered $V(r)$ and the reservoir equilibrium configuration $\phi_{eq}(\mathbf{v})$. As we have seen, this is an *ill-defined* evolution operator, since it does not respect the properties of the probability distribution function it acts upon. An alternative operator will be constructed in the next chapter.

Relation to previous work. It is worth peeking into previous work concerning the kinetic description of magnetized plasma. As a matter of fact, the calculation presented in this chapter is quite similar to the one carried out, independently, by V. P. Silin and co-workers [64], [105], P. Schram [103], D. Montgomery et al. [85] (*in chronological order*) in the *homogeneous* case, and later by P. Ghendrih [65] and A. Øien [88], [90], [94] in the *inhomogenenous* case; expressions (6.15a,b) correspond exactly to the formulae appearing therein (upon substitution with a Maxwellian for the reservoir). It should be noticed that the latter two authors address the mathematical properties of the operator only *after* the cross-velocity-space diffusion term (responsible for the mathematical *'misbehaviour'* of the collision term) has been neglected[21].

Let us end with a remark concerning the methodology of this Chapter (as compared to subsequent ones). In seek of an *asymptotic* form for the kernel (i.e. for $t \to \infty$) we chose to straightforward carry out the time-integration (in τ) first and thus obtained a set of final expressions in terms of an infinite series of Bessel functions of the first kind (as an experienced reader might have expected from the cylindrical symmetry of the problem). That result is exact, yet rather delicate to manipulate, since convergence of the series is not guaranteed at this level. However, the test-particle formulation allows for rather more analytical tractability. The reservoir equilibrium function may be assumed to be e.g. Maxwellian, allowing for an exact analytic computation of coefficients, in our case; this would not be possible in the complete nonlinear problem (considered by studies mentioned above, for instance). This calculation will be the object of the following chapter.

Note added in proof. The final expressions (6.14) (also (6.34)) for the FPE coefficients (along with related definitions) were presented just as they stand, for the sake of comparison with previous work. Nevertheless, the existence of the infinite series in these expressions is *not* necessary! Indeed, shifting back to the time-integration, i.e. substituting $\delta_+(\omega)$ with $\frac{1}{\pi}\int_0^\infty d\tau\, e^{i\omega\tau}$, the series in all

[21] Physical intuitive arguments are employed therein, in order for the term in question (**G** in our *QM-FPE*) to be plainly discarded. These arguments are rather questionable, though; as a simple dimensional analysis shows (see in App. A), the cross-V-X term is not *necessarily* small in magnitude.

matrix elements can be evaluated as follows:

$$\sum_{n=-\infty}^{\infty} J_n(Z)\, J_{n+p}(Z)\, e^{i\,n\,\Omega\,\tau} \;=\; J_p(2\,Z\,\sin\frac{\Omega\,\tau}{2})\, e^{i\,p\,\Omega\,\tau/2} \tag{6.43}$$

according to Gegenbauer's formula (see (F.11), F.12) in the Appendix); the convergence of the infinite series above is thus ensured. Then, one may proceed by evaluating velocity $\mathbf{v_1}$ integrals and so forth. We do not go into further detail, since the calculation exactly results in the expressions that will be presented (in detail) in the following chapter.

Chapter 7

New Markovian kinetic equation for magnetized plasma: construction of the plasma Φ-operator

Summary

We construct the Φ-operator, defined in Part A, in the case of a magnetized plasma. We obtain a *new* kinetic equation, in the form of a multivariate *Fokker - Planck equation* in the complete (i.e. velocity *and* position) phase-space. Expressions for all coefficients are obtained. The mathematical properties of the new collision operator are established.

... a reasoned theory is preferable to blind extrapolation.

`Arthur Stanley Eddington`
in *The Expanding Universe*

7.1 Introduction

In order to construct the Markovian evolution operator $\Phi_2(t)$ for magnetized plasma, we have to evaluate the expressions derived in §5.5 for the coefficients in the M-FP equation (5.13).

A word of caution seems to be necessary here. The action of the Φ-operator seems to be well established analytically in cases of classical systems with a *discrete* spectrum of eigenvalues of the associated Liouville operator (see [46], [68]). In our system, even though particle motion in the $xy-$plane (perpendicular to the magnetic field) is confined by the field (and thus bounded), motion *along* the field remains essentially *free* [1]. It was thus expected, and indeed verified, that certain *ill-defined* (or, rather, *indefinite*) quantities might appear in the calculation, actually related to motion in the z-axis. Since we are mostly interested in phenomena related to the $xy-$plane, in the rest of this text, we will assume that our system is translationally invariant along z i.e. $\frac{\partial f}{\partial z} = 0$. This hypothesis is physically plausible, given the uniformity of the magnetic field considered.

As in the previous section, only the final result will be explicitly given in the following, while the detailed calculation will be described in the Appendix.

7.2 Markovian kinetic equation for magnetized plasma

7.2.1 M-FP equation: the general (inhomogeneous) case

The Markovian ($\Phi-$) kinetic equation derived in Chapter 5 can now be constructed and, finally, compared to its ($\Theta-$) analogue presented in the previous Chapter. Details are reported in the Appendix: the matrix elements are evaluated in Appendix D (also see App. C); the action of the averaging operator $\mathcal{A}_{t'}$ on certain functions of time is evaluated in Appendix E.

The kinetic equation thus obtained is of the expected 2nd-order parabolic (linear) form (cf. (6.2)):

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\frac{\partial f}{\partial \mathbf{v}} \;\equiv\; \Phi_2\, f(\mathbf{x},\mathbf{v};t)$$

$$\begin{aligned}
= \quad & \left[\left(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\right)\left[D_\perp(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)\right] \;+\; \frac{\partial^2}{\partial v_z^2}\left[D_\parallel(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)\right]\right. \\
& +2\, s\, \Omega^{-1}\left[\frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x}\right]\left[D_\perp(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)\right] \\
& \qquad\qquad\qquad + \frac{\partial^2}{\partial z \partial v_z}\left[D_\parallel^{(VX)}(\mathbf{v}) f(\mathbf{x},\mathbf{v};t)\right] \\
& + \Omega^{-2}\left[Q(\mathbf{v}) + D_\perp(\mathbf{v})\right]\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(\mathbf{x},\mathbf{v};t)
\end{aligned}$$

[1] The Lorentz force: $\mathbf{F} = \frac{e}{c}(\mathbf{v}\times\mathbf{B})$ yields no component along the magnetic field $\mathbf{B}$.

$$+ \frac{\partial^2}{\partial z^2} \big[D_\parallel^{(XX)}(\mathbf{v}) f(\mathbf{x}, \mathbf{v}; t) \big]$$

$$- \frac{\partial}{\partial v_x} \Big[\mathcal{F}_x(\mathbf{v})\, f(\mathbf{x}, \mathbf{v}; t) \Big] - \frac{\partial}{\partial v_y} \Big[\mathcal{F}_y(\mathbf{v})\, f(\mathbf{x}, \mathbf{v}; t) \Big] - \frac{\partial}{\partial v_z} \Big[\mathcal{F}_z(\mathbf{v})\, f(\mathbf{x}, \mathbf{v}; t) \Big]$$

$$+ s\, \Omega^{-1}\, \mathcal{F}_y(\mathbf{v})\, \frac{\partial}{\partial x}\, f(\mathbf{x}, \mathbf{v}; t) - s\, \Omega^{-1}\, \mathcal{F}_x(\mathbf{v})\, \frac{\partial}{\partial y}\, f(\mathbf{x}, \mathbf{v}; t) \tag{7.1}$$

All coefficients are defined in the following paragraph. Notice that the parts corresponding to the directions $\parallel$ and $\perp$ to the magnetic field, are completely decoupled.

A few comments need to be made, concerning the difference in structure with respect to the 'old' $\Theta-$FPE (6.3).

1. Notice, in particular, the appearance of the real *space-* $(XX-)$ *diffusion term* (4th line in the *rhs*). This term, suggesting diffusion in real (position) space, was completely absent from previous equations[2]. The dependence on the inverse square of the field magnitude ($\sim 1/\Omega^2$), actually rather imposed by dimensional arguments (see in App. A), is in agreement with results in [80][3].

2. See that the *cross-velocity-position* $(VX-)$ diffusion term[4] is strongly modified. Still, this term varies as the inverse magnetic field ($\sim 1/\Omega$).

3. The *velocity-diffusion* $(VV-)$ term is *not* modified. Recall that *all* coefficients are functions of particle velocity $\mathbf{v}$ *only*; therefore, by integrating the full FPE (7.1) over position $\mathbf{x}$, we readily recover the equation presented in the previous paragraph.

4. The terms related to the $z-$ spatial direction (parallel to the field) disappear since both f [5] and $D_\parallel^{(VX)}$, $D_\parallel^{(XX)}$ [6] are explicitly independent of z. Therefore, the 3rd and 5th lines in the *rhs*, here provided for completeness, will be omitted in the following.

[2]This remark refers to rigorously derived kinetic equations, i.e. excluding artificially injected space diffusion terms e.g. [73].

[3]In reference [80], the authors actually used the same (dynamic) arguments as in the original *Rosenbluth - McDonald - Judd* paper [99] (in the unmagnetized limit) in order to discuss binary interactions in the presence of a uniform magnetic field and derive an appropriate diffusion equation.

[4]i.e. 2nd and 3rd line in the *rhs* of (7.1).

[5]due to previous assumption, see in §7.1.

[6]according to the exact calculation, see comment in the beginning of the next paragraph; also in App. D.

Reduced form of the collision term. The new *collision term* $\mathcal{C}^{(\Phi)} \equiv rhs(7.1)$ can be cast into the (cylindrical-symmetric) form[7]:

$$\begin{aligned}
\Phi_2 f \; = \; & \left[\left(\frac{\partial}{\partial v_x} + s\Omega^{-1} \frac{\partial}{\partial y} \right)^2 + \left(\frac{\partial}{\partial v_y} - s\Omega^{-1} \frac{\partial}{\partial x} \right)^2 \right] \Big[D_\perp(\mathbf{v}) f(\mathbf{x}, \mathbf{v}; t) \Big] \\
& + \frac{\partial^2}{\partial v_z^2} \big[D_\parallel(\mathbf{v}) f(\mathbf{x}, \mathbf{v}; t) \big] \\
& + \Omega^{-2} \, Q(\mathbf{v}) (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f(\mathbf{x}, \mathbf{v}; t) \\
- & \left(\frac{\partial}{\partial v_x} + s\Omega^{-1} \frac{\partial}{\partial y} \right) \Big[\mathcal{F}_x(\mathbf{v}) \, f(\mathbf{x}, \mathbf{v}; t) \Big] - \left(\frac{\partial}{\partial v_y} - s\Omega^{-1} \frac{\partial}{\partial x} \right) \Big[\mathcal{F}_y(\mathbf{v}) \, f(\mathbf{x}, \mathbf{v}; t) \Big] \\
& - \frac{\partial}{\partial v_z} \Big[\mathcal{F}_z(\mathbf{v}) \, f(\mathbf{x}, \mathbf{v}; t) \Big]
\end{aligned} \qquad (7.1 - bis)$$

Equation (7.1) is the strong result of this chapter. It will be referred in the rest of the thesis as '*the Fokker-Planck-type kinetic equation*' (*FPE*) (or the *M-FPE*) and will be used as the basis of the study that follows.

7.2.2 Coefficients

Let us define the coefficients appearing in eq. (7.1-2) above.

The coefficients $D_\perp$, $D_\parallel$, $\mathcal{F}_{x,y,z}$ are just as defined in the previous chapter (see §6.5.4). The cross-v_z-z diffusion coefficient $D_\parallel^{(VX)}(\mathbf{v})$ (omitted in the following, see previous comment) is computed in the Appendix (see §D.2-3 in combination with App. E) and comes out to be precisely equal to $\frac{1}{2} G_{33}$ (as defined e.g. in (6.14-15) above). The $zz-$ diffusion coefficient $D_\parallel^{(XX)}(\mathbf{v})$ (also omitted in the following, see previous comment) has been computed in the Appendix (see §D.4 in combination with App. E)[8].

The new quantity $Q(\mathbf{v})$ (see the third line in (7.1-bis)) has been derived in the Appendix (see §D.4). Following the method of calculation suggested in Appendix C (§C.1 - 3), it is found to be:

$$\begin{aligned}
Q(\mathbf{v}) = & \frac{n_{\alpha'}}{m^2} (2\pi)^5 \pi \int d^c\mathbf{v_1} \, \phi_{Max}(v_{1\perp}, v_{1\parallel}) \\
& \int_0^\infty dk_\perp \, k_\perp \int_{-\infty}^\infty dk_\parallel \, \tilde{V}_k^2 \, J_n^2(Z) \, \delta_+(\omega_n)
\end{aligned} \qquad (7.2)$$

[7]We henceforth omit the $\parallel -$ components in the $V - X-$ and $X - X-$ parts (i.e. the 3rd and 5th lines in the right-hand-side) in (7.1), here, as we said above.

[8]It is precisely given by $D_\parallel^{(XX)} = D_\parallel \, \mathcal{A}_{t'} \, t'^2$, the latter quantity giving infinity; see App. D.4, App. E.

For the sake of future reference, let us explicitly present the form of the 6×6 diffusion matrix $\mathbf{D}$ and the $(6d)$ friction (drift) vector $\vec{\mathcal{F}}$ [9] (defined in §5.5 in general) as they apply in equation (7.1) above (see in the Appendix for details). $\mathbf{D}$ is given by:

$$\mathbf{D} = \begin{pmatrix} \Omega^{-2}(D_\perp + Q) & 0 & 0 & s\,\Omega^{-1}D_\angle & -s\,\Omega^{-1}D_\perp & 0 \\ 0 & \Omega^{-2}(D_\perp + Q) & 0 & s\,\Omega^{-1}D_\perp & s\,\Omega^{-1}D_\angle & 0 \\ 0 & 0 & D_\parallel^{(XX)} & 0 & 0 & D_\parallel^{(VX)}/2 \\ -s\,\Omega^{-1}D_\angle & s\,\Omega^{-1}D_\perp & 0 & D_\perp & D_\angle & 0 \\ -s\,\Omega^{-1}D_\perp & -s\,\Omega^{-1}D_\angle & & -D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel^{(VX)}/2 & 0 & 0 & D_\parallel \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{D_{11}} & \mathbf{D_{12}} \\ \mathbf{D_{21}} & \mathbf{D_{22}} \end{pmatrix} \qquad (7.3)$$

where the definition of the 3×3 matrices $\mathbf{D_{ij}}$ is obvious[10 11]. The general relations:

$$(a_1^{(\Phi)})_i = -\partial (D_{12})_{ij}/\partial v_j \quad , \qquad (a_2^{(\Phi)})_i = -\partial (D_{22})_{ij}/\partial v_j \qquad (7.4)$$

clearly hold here (see (5.11)), so the drift (friction) vector $\mathcal{F}_{(V),i}$ is defined by:

$$\mathcal{F}_{(V),i} = \left(1 + \frac{m}{m_{\alpha'}}\right) \frac{\partial (D_{22})_{ij}}{\partial v_j} \qquad (7.5)$$

i.e.

$$\begin{aligned} \mathcal{F}_{(V),x} &= \left(1 + \frac{m}{m_{\alpha'}}\right) \left(\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y}\right) \\ \mathcal{F}_{(V),y} &= \left(1 + \frac{m}{m_{\alpha'}}\right) \left(-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y}\right) \\ \mathcal{F}_{(V),z} &= \left(1 + \frac{m}{m_{\alpha'}}\right) \frac{\partial D_\parallel}{\partial v_\parallel} \end{aligned} \qquad (7.6)$$

---

[9] Remember that $\vec{\mathcal{F}}$ is related to the vector $\mathbf{a}$; see in §5.5.

[10] Remember that the third line and the third column in matrix $\mathbf{D}$ have been cancelled previously in the text, since the involved elements, related to space direction $\parallel$ to the field, were ill-defined. We will therefore straightforward set $D_\parallel^{(VX)} = D_\parallel^{(XX)} = 0$ in the subsequent formulae.

[11] Only the symmetric part of $\mathbf{D}$ appears in the diffusive part of the FPE; see definitions in §5.5.

and the associated space-drift vector is, obviously:

$$\mathcal{F}_{(X),i} = \left(1 + \frac{m}{m_{\alpha'}}\right) \frac{\partial (D_{12})_{ij}}{\partial v_j} \tag{7.7}$$

so

$$\mathcal{F}_{(X),x} = -s\,\Omega^{-1}\,\mathcal{F}_{(V),y}\,, \qquad \mathcal{F}_{(X),y} = s\,\Omega^{-1}\,\mathcal{F}_{(V),x}\,, \qquad \mathcal{F}_{(X),z} = 0 \tag{7.8}$$

(cf. (7.1), in comparison with the general form (5.13)).

7.2.3 M-FP equation: the homogeneous case

Let us consider a uniform *df* $f = f(\mathbf{v};\, t)$. The "new" Φ-operator coincides with the "old" Θ- in this case[12]: both give exactly the Fokker-Planck-type equation (4.9) (or (6.1), along with definitions provided in the previous chapter). This is essentially a 'linearized' version of a kinetic equation which has appeared in earlier works e.g. [101], [103].

Let us express this equation in velocity space cylindrical coordinates:

$$v_x = v_\perp\, \cos\theta\,, \qquad v_y = v_\perp\, \sin\theta\,, \qquad v_z = v_\parallel$$

The FPE becomes:

$$\begin{aligned} \frac{\partial f}{\partial t} - s\,\Omega\, \frac{\partial f}{\partial \theta} = & \left(\frac{\partial^2}{\partial v_\perp^2} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp^2}\frac{\partial^2}{\partial \theta^2}\right)\Big[D_\perp(\mathbf{v}) f\Big] + \frac{\partial^2}{\partial v_\parallel^2}\big[D_\parallel(\mathbf{v}) f\big] \\ & -\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\Big[\mathcal{F}_\perp(\mathbf{v})\, f\Big] + \frac{1}{v_\perp}\frac{\partial}{\partial \theta}\Big[\mathcal{F}_\theta(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_\parallel}\Big[\mathcal{F}_\parallel(\mathbf{v})\, f\Big] \end{aligned} \tag{7.9}$$

where:

$$\begin{aligned} \mathcal{F}_\perp &= \left(1 + \frac{m}{m_{\alpha'}}\right)\frac{\partial D_\perp}{\partial v_\perp} \qquad \mathcal{F}_\theta = \left(1 + \frac{m}{m_{\alpha'}}\right)\frac{\partial D_\angle}{\partial v_\perp} \\ \mathcal{F}_\parallel &= \left(1 + \frac{m}{m_{\alpha'}}\right)\frac{\partial D_\parallel}{\partial v_\parallel} \end{aligned} \tag{7.10}$$

It is interesting to see that, for a gyrotropic *d.f.* $f = f(v_\perp, v_\parallel)$, the field appears only *inside* the coefficients, since $\partial f/\partial\theta$ cancels in both sides:

$$\begin{aligned} \frac{\partial f}{\partial t} = & \left(\frac{\partial^2}{\partial v_\perp^2} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\right)\Big[D_\perp(\mathbf{v}) f\Big] + \frac{\partial^2}{\partial v_\parallel^2}\big[D_\parallel(\mathbf{v}) f\big] \\ & -\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\Big[\mathcal{F}_\perp(\mathbf{v})\, f\Big] - \frac{\partial}{\partial v_\parallel}\Big[\mathcal{F}_\parallel(\mathbf{v})\, f\Big] \end{aligned} \tag{7.11}$$

This reduced equation is, rigorously speaking, *the* kinetic equation one should use in a test-particle description of magnetized plasma, if inhomogeneities in the (gyrotropic) distribution function are *not* to be taken into account.

[12] See discussion in Part A.

7.3 Properties of the kinetic equation

7.3.1 Reality of the distribution function

The distribution function f is a *real* function of the dynamical variables $\{\mathbf{x}\}$. It should remain such for all values of time t. This criterion is indeed satisfied by the new kinetic operator defined here; to see this, check that *all* quantities in the FPE are *real*, so *no* imaginary part may arise for a given real initial condition $f(t=0)$.

7.3.2 Norm of the distribution function

The norm of the probability should be preserved under the action of a kinetic evolution operator. This means that the time derivative of the quantity: $\int_\Gamma d\mathbf{q}\, f(\mathbf{q},t)$ should be zero at all values of time t. Indeed, the time-derivative of the norm is given by[13]:

$$\frac{d}{dt}\int_\Gamma d\mathbf{q}\, f(\mathbf{q},t) = \int_\Gamma d\mathbf{q}\,\frac{df}{dt} = \int_\Gamma d\mathbf{q}\,\mathcal{K}\{f\} = \int_\Gamma d\mathbf{q}\,\frac{\partial}{\partial \mathbf{q}}\cdots = 0$$

As a consequence, the (test-) particle number (or density) $\langle n\rangle = \int_\Gamma d\mathbf{q}\, n f(\mathbf{q},t)$ will be conserved as well.

Notice that our model is expected to allow for either momentum or energy transfer between the test- and field- ($R-$) particles, so no other conservation laws are to be examined (i.e. of the first or second velocity moments, 'as usual').

7.3.3 Diffusion matrix - positivity preservation

The preservation of the positivity of $f(\mathbf{x},\mathbf{v};\,t)$ by eq.(7.1) is now ensured. Indeed, as we have seen, the above equation may be seen as an equation describing diffusion in the *total* configuration space considered, say $\Gamma \equiv \{x, y, v_x, v_y, v_z\}$ (see in Chapters 4, 5). The symmetric part of the new (Φ-) diffusion matrix thus obtained is[14]:

$$\mathbf{D}^{(\Phi)\,(SYM)} =$$

$$\begin{pmatrix} \Omega^{-2}(D_\perp + Q) & 0 & 0 & -s\,\Omega^{-1}D_\perp & 0 \\ 0 & \Omega^{-2}(D_\perp + Q) & s\,\Omega^{-1}D_\perp & 0 & 0 \\ 0 & s\,\Omega^{-1}D_\perp & D_\perp & 0 & 0 \\ -s\,\Omega^{-1}D_\perp & 0 & 0 & D_\perp & 0 \\ 0 & 0 & 0 & 0 & D_\parallel \end{pmatrix} \tag{7.12}$$

[13] The zeroth order Liouville operator does not contribute to the integral, due to its form: $L_0 = \frac{d}{d\mathbf{q}}\,(\mathbf{v},\mathbf{a})$.

[14] Only the *symmetric* part of $\mathbf{D}$ (i.e. $D_{ij}^{(SYM)} = \frac{1}{2}\big(D_{ij} + D_{ji}\big)$) is of relevance in the diffusion matrix: see the form of the symmetric operator $\partial^2(D_{ij}\,\cdot)/\partial v_i \partial v_j$; nevertheless, the elements of the *anti-symmetric* part are of importance as they enter the definitions of the friction vectors $\mathcal{F}_i$ (see in the text).

Notice the separation between $\perp$ − and $\|$ − parts in the block-diagonal form of this matrix. We can readily see that **D** *is* positive definite, indeed as:
$\forall \quad \mathbf{V} \equiv (x, y, v_x, v_y, v_z) \in \Re^5$,

$$(\mathbf{V}, \mathbf{D}^{(\Phi)}\, \mathbf{V}) = (\mathbf{V}, \mathbf{D}^{(\Phi)}\, \mathbf{V}) = \mathbf{V}^T (\mathbf{D}^{(\Phi)})^{SYM}\, \mathbf{V} = (D^{(\Phi)})^{SYM}_{ij}\, V_i\, V_j \ =$$
$$= D_\perp \left[\left(v_x + s\Omega^{-1} y\right)^2 \,+\, \left(v_y - s\Omega^{-1} x\right)^2 \right] \,+\, D_\| \, v_z^2 \,+\, \Omega^{-2} Q \,(x^2 + y^2) \quad \geq 0$$

($D_{\perp,\|}$ and Q are positive quantities by construction). In other words, the characteristic polynomial $p(\lambda) = Det(\mathbf{D} - \lambda \mathbf{I})$ of the diffusion matrix **D** possesses only *positive* zeroes[15], as one may readily verify.

For the sake of comparison, note that the corresponding diffusion matrix for the old (Θ-) kinetic operator reads:

$$\mathbf{D}^{\Theta}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{0} & \frac{1}{2}\mathbf{G}^{\mathbf{T}} \\ \frac{1}{2}\mathbf{G} & \mathbf{A} \end{pmatrix}$$

(see (4.6)) or:

$$\mathbf{D}^{(\Theta)\,(SYM)} \quad = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} D_\perp^{(VX)} & 0 \\ 0 & 0 & \frac{1}{2} D_\perp^{(VX)} & 0 & 0 \\ 0 & \frac{1}{2} D_\perp^{(VX)} & D_\perp & 0 & 0 \\ -\frac{1}{2} D_\perp^{(VX)} & 0 & 0 & D_\perp & 0 \\ 0 & 0 & 0 & 0 & D_\| \end{pmatrix} \qquad (7.13)$$

(all quantities where defined in the previous Chapter). Proceeding as described above, we may straightforward see that the characteristic polynomial $p(\lambda) = Det(\mathbf{D}^{(\Theta)} - \lambda \mathbf{I})$ of the diffusion matrix $\mathbf{D}^{(\Theta)}$ possesses a *negative* (double) solution. As the eigenvalues of the matrix are therefore *not* all positive, the action of the operator of the distribution function f may result in a negative value of the latter.

7.3.4 Equilibrium state and H - Theorem

Consider a Maxwellian reservoir:

$$\phi_{eq}(v_1) = \phi_{eq}(0)\, e^{-\beta\, v_1^2}\,.$$

One may easily verify that the function:

$$f_0(v) = \phi_{eq}(v) = \ \phi_{eq}(0)\, e^{-\beta\, v^2} \qquad (7.14)$$

is an equilibrium function of the kinetic operator[16], i.e. $\mathcal{K}\{f_{eq}\}$. Collisions therefore result in the test-particle(s) monotonously relaxing to the exact temperature of the bath.

[15] In fact two double roots and one simple, all positive.
[16] Check by substituting into (7.1); it is necessary to recall relations in (6.36), (6.37).

The evolution of the system state in time is characterized by the monotonous growth of a quantity defined as its *entropy* S (which reaches its maximum at equilibrium). Following the standard Boltzmann picture, one may define a function H and then prove that $dH/dt \leq 0$ as the system evolves in time under the influence of collisions (i.e. essentially $H = -S$); the system is then said to obey an *H-Theorem.*

In *open systems* theory, a (sub)system (state: $f(\Gamma)$, phase space: Γ) is free to exchange energy with its environment. The formal kinetic framework for open systems is displayed and discussed in [113]. An H-theorem holds, presumably corresponding to relaxation towards an equilibrium state f_0.

Let us define the function:

$$H = \int f \, ln \frac{f}{f_0} \, d\Gamma \tag{7.15}$$

We shall show that H monotonously decreases in time under the action of a kinetic operator $\frac{df}{dt}|_{coll} = \mathcal{K}$.

For clarity, let us first consider a *homogeneous* system state $f(\mathbf{v}; t)$, evolving under the kinetic operator defined here. We have:

$$\begin{aligned}
\frac{dH}{dt} &= \int \left(1 + ln \frac{f}{f_0}\right) \frac{\partial f}{\partial t} \, d\mathbf{v} \\
&= \int \left(1 + ln \frac{f}{f_0}\right) \frac{\partial}{\partial v_i} \left(D_{ij} \frac{\partial f}{\partial v_j} + \mu a_i \, f\right) d\mathbf{v} \\
&= -\int \left(\frac{1}{f} \frac{\partial f}{\partial v_i} - \frac{1}{f_0} \frac{\partial f_0}{\partial v_i}\right) \left(D_{ij} \frac{\partial f}{\partial v_j} + \mu a_i \, f\right) d\mathbf{v} \\
&= -\int \left[\frac{D_{ij}}{f} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} + \mu a_i \frac{\partial f}{\partial v_i} - \frac{D_{ij}}{f_0} \frac{\partial f_0}{\partial v_i} \frac{\partial f}{\partial v_j} - \mu a_i \frac{f}{f_0} \frac{\partial f_0}{\partial v_i}\right] \\
&= -\int \frac{D_{ij}}{f} \left(\frac{\partial f}{\partial v_i} - \frac{f}{f_0} \frac{\partial f_0}{\partial v_i}\right) \left(\frac{\partial f}{\partial v_j} + \frac{\mu a_i}{D_{ij}} \, f\right)
\end{aligned} \tag{7.16}$$

where a summation $\sum_{i,j=1}^{3}$ is understood in the beginning of every line (except the first). An integration by parts was carried out in the third step, assuming that boundary terms vanish (at infinity). Remember that $\mathbf{D}$ here is the (3×3) velocity diffusion matrix $\mathbf{D_{22}}$ defined previously. Now, recall that:

$$\frac{\partial f_0}{\partial v_i} = -2\beta \, f_0 v_i$$

(see (7.14) above); also:

$$\mu \, a_i = 2\beta \, D_{ij} \, v_j$$

as was proved in §6.7. We thus end up with:

$$\begin{aligned}
\frac{dH}{dt} &= -\int \frac{D_{ij}}{f} \left(\frac{\partial f}{\partial v_i} + 2\beta \, v_i\right) \left(\frac{\partial f}{\partial v_j} + 2\beta \, v_j\right) \\
&= -\int \frac{D_{ij}}{f} A_i \, A_j \qquad \leq 0
\end{aligned} \tag{7.17}$$

where the definition of the vector A_i is obvious. The conclusion drawn about the sign of the quantity is a consequence of the analytic form of the diffusion matrix (cf. discussion in the previous paragraph).

The proof in the *general* case $f(\mathbf{x}, \mathbf{v};\, t) = f(\mathbf{q};\, t)$ is quite straightforward, given the structure of the kinetic operator; one now has to deal with a 6−dimensional phase space equipped with a 2nd-order (differential) operator which bears the same form as in the previous calculation: see the general expression (5.8), together with definitions in §7.2.2 above. Along the same lines, one finds:

$$\begin{aligned}\frac{dH}{dt} &= \int \left(1 + ln\frac{f}{f_0}\right)\frac{\partial f}{\partial t}\, d\mathbf{q} \\ &= \int \left(1 + ln\frac{f}{f_0}\right)\frac{\partial}{\partial q_i}\left(D_{ij}\frac{\partial f}{\partial q_j} + \mu a_i\, f\right) d\mathbf{q} \\ &= \ldots \\ &= -\int \frac{D_{ij}}{f}\left(\frac{\partial f}{\partial q_i} - \frac{f}{f_0}\frac{\partial f_0}{\partial q_i}\right)\left(\frac{\partial f}{\partial q_j} + \frac{\mu a_i}{D_{ij}}\, f\right) \end{aligned} \tag{7.18}$$

where the 6×6 matrix $\mathbf{D}$ and the vector $\mathbf{a} \in \Re^6$ were defined previously (also see in ch. 5). At this stage, one might *"wish"* that relation:

$$\mu\, a_i = 2\beta\, D_{ij}\, q_j \tag{7.19}$$

be fulfilled by the (6d) quantities in it. Definitely, nothing could have been more satisfying than verifying that, yes, this relation *is* indeed satisfied[17] by our coefficients in the $\Phi-$ operator (only). Therefore, we may define the vector:

$$A'_i = \frac{\partial f}{\partial q_i} + 2\beta_i\, q_i\, f$$

where: $\overrightarrow{\beta} = (0, 0;\, \beta, \beta, \beta)$[18]. We thus obtain:

$$\frac{dH}{dt} = -\int \frac{D_{ij}}{f}\, A'_i\, A'_j \qquad \leq 0 \tag{7.20}$$

where, once more, we have used the fact that the diffusion matrix is positive definite.

Notice that, once equilibrium is reached, $f = f_0$ and $A'_i = 0$, so the derivative dH/dt cancels and the function H reaches its minimum value ($S = max$).

[17] For instance, from (7.3), (7.4) we have:

$$a_1^{(\Phi)} \equiv (a_1^{(\Phi)})_1 = -\frac{\partial (D_{12})_{1j}}{\partial v_j} = \ldots = s\,\Omega^{-1}\,\frac{\partial (D_{22})_{2j}}{\partial v_j} = -s\,\Omega^{-1}\, 2\,\beta\,(D_{22})_{2j}\, v_j = +2\,\beta\,(D_{12})_{1j}\, v_j$$
$$= +2\,\beta\, D_{1j}^{(\Phi)}\, v_j$$

as can be seen by careful inspection; in the same way, one may prove that (7.19) is met by all 5 (having excluded the $z-$direction; see in §7.1, 2) components of $\mathbf{a}$.

[18] i.e.

$$A' = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y};\, \frac{\partial f}{\partial v_x} + 2\beta\, v_x\, f,\; \frac{\partial f}{\partial v_y} + 2\beta\, v_y\, f,\; \frac{\partial f}{\partial v_z} + 2\beta\, v_z\, f\right).$$

7.4 Dynamical friction

A comment regarding the vector $\vec{\mathcal{F}}$ is appropriate. The existence of $\vec{\mathcal{F}}$ results in a modification of the force exerted on the test-particle, which may now be expressed as:

$$\vec{F}_{total} = \vec{F}_{Lorentz} + \vec{\mathcal{F}}$$

(in order to see this, take the term containing $\mathcal{F}_j$ to the left part in the kinetic equation and re-arrange[19]. Taking into account definitions (6.29), we can seek an analytic form for the rate of *energy transfer*, say $\mathcal{R} \equiv \vec{\mathcal{F}} \cdot \vec{v}$, via the action of this vector. Expressing the velocity in cylindrical coordinates $\{v_\perp, \theta, v_\parallel\}$, we find that:

$$\mathcal{R} \equiv \mathbf{F} \cdot \mathbf{v} = \cdots \sim v_\perp \frac{\partial D_\perp}{\partial v_\perp} + v_\parallel \frac{\partial D_\parallel}{\partial v_\parallel} \tag{7.21}$$

$$\equiv v_\perp \mathcal{F}_\perp + v_\parallel \mathcal{F}_\parallel \quad < 0 \tag{7.22}$$

The *rhs* will later be proved to be negative, due to the very construction of the coefficients; in fact, we shall later see that $\mathcal{F}_\perp \sim \frac{\partial D_\perp}{\partial v_\perp}$ and $\mathcal{F}_\parallel \sim \frac{\partial D_\parallel}{\partial v_\parallel}$ are of opposite sign to $v_\perp$, $v_\parallel$ respectively[20]. Therefore, the vector $\vec{\mathcal{F}}$ represents a continuous slowing down of the particle (and energy loss through heat produsction), via a mechanism often called the *'dynamical friction'* [3] (the strength of the acceleration felt by the particle depends on its velocity).

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[19] This modification actually appears in the order λ^2 (2nd order in the interaction).

[20] Coefficients $D_\perp$, $D_\parallel$ will be seen to be monotonous functions of $v_\perp, v_\parallel$ respectively: they actually take *lower* values as velocity increases, hence present a negative slope - see figures in Chapter 8 - and thus a negative derivative, with respect to the absolute value of $v_\perp, v_\parallel$ (don't forget that $v_\perp \in \Re_+$ but $v_\parallel \in \Re$); furthermore, we will see that $\mathcal{F}_\parallel \sim \frac{\partial D_\parallel}{\partial v_\parallel}$ is an *odd* function of $v_\parallel$).

Chapter 8

Derivation of an exact form for the coefficients in a Maxwellian background

Summary

Explicitly considering a Maxwellian bath and Debye-type interactions, a set of exact expressions are derived for the force auto-correlations $C_{ij}(\tau)$ and the running diffusion coefficients $D_{ij}(t)$, by following an alternative analytical method (different from the one presented previously). This procedure leads to a new analytically tractable non-dimensional form of the coefficients in the kinetic equation, which are now expressed as double integrals of exact expressions (so *no* infinite series are involved). The calculation for both single- and multiple species plasma is presented.

The simplicities of natural laws arise through the complexities of the language we use for their expression.

`Eugene Wigner`

8.1 Introduction

We saw that the coefficients in the plasma kinetic equation are given by a set of rather complicated expressions (actually 7 coupled integrals!) containing the exact solution of the problem of (gyrating) motion. The appearance of the latter (involving trigonometric functions) in the exponential (inside the integrand) resulted in the use of infinite series of Bessel functions[1].

This difficulty seems rather inevitable, in general. Remember that a kinetic equation is generally *intrinsically* nonlinear in f: it involves the *d.f.* f evaluated simultaneously in two different points in phase space, hence the analytical difficulty in finding a closed concise exact form for the kinetic operator. However, in a *test-particle problem*, the background (*field*-) particle *d.f.* is prescribed (e.g. a Maxwellian, or another equilibrium *d.f.*, if one is known) so, for instance, the velocity integrals may be calculated *first*[2,3].

In this chapter, we will try to develop this idea, in order to advance the analytical computation as far as possible and derive exact tractable expressions for the coefficients.

8.2 Coefficients: summary of expressions

We have seen that the $D_\perp$, $D_\angle$, $D_\parallel$ and Q coefficients appearing in the plasma kinetic equation derived in the previous chapter are given by the expressions:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} \frac{n_{\alpha'}}{m_\alpha^2} (2\pi)^3 \int_0^t d\tau \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1})$$
$$\int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^\alpha(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}$$
$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2 + k_y^2)\, \cos \Omega^\alpha \tau \\ (-s^\alpha)\, \frac{1}{2}\,(k_x^2 + k_y^2)\, \sin \Omega^\alpha \tau \\ (k_x^2 + k_y^2)(1 + \frac{1}{2} \cos \Omega^\alpha \tau) \\ k_z^2 \end{array} \right\} \tag{8.1}$$

where v_i ($v_{1,i}$), $i = 1, 2, 3$ denote the velocity coordinates of the test-particle (reservoir-particle) of species $\alpha_\sigma \equiv \alpha$ ($\alpha_1 = \alpha' \in \{e, i, ...\}$) respectively and $\tilde{V}_k$ stands for the Fourier transform of the (Coulomb-type) interaction potential (see in Chapter 1); remember that $V = V(|\mathbf{r}|) = V(r) \in \Re$ implies $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k \in \Re$. By taking the limit $t \to \infty$, the asymptotic form of the coefficients

[1] Such a series is rather hard to manipulate, either analytically or numerically (convergence? unknown exact form of integrals? lack of physical transparency).

[2] i.e. *before* the time integration, which is traditionally a first step, leading to the appearance of $\delta-$ functions (see e.g. the derivation of the Landau collision term in [5]).

[3] Of course, this is no more than a different sequence in the series of integrations, so the result found in this way should be equal to the one in chapter 6, further advanced by evaluating the infinite sum of integrals for a Maxwellian ϕ_{eq}. We will comment on this point elsewhere

is obtained. Recall that $D_{\perp}^{(XX)} = D_{\perp} + Q$ (see in the previous chapter), so cancelling the *cosine* in the third line in the above formulae (and all subsequent formulae of the same structure) we readily obtain the expression for Q alone. The obvious convention:

$$\left\{ \begin{array}{c} A \\ B \end{array} \right\} = \left\{ \begin{array}{c} C \\ D \end{array} \right\} E \qquad \Rightarrow \qquad \left\{ \begin{array}{c} A = CE \\ B = DE \end{array} \right\}$$

will be extensively used throughout this chapter.

The above relations can be cast in the form:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_{\perp} \\ D_{\angle} \\ D_{\perp}^{(XX)} \end{array} \right\} \\ D_{\parallel} \end{array} \right\} =$$

$$= \sum_{\alpha'} \frac{1}{m_{\alpha'}^2} \int_0^{t\to\infty} d\tau \left\{ \begin{array}{c} \left\{ C_{\perp}^{\alpha,\alpha'} \right\} \\ C_{\parallel}^{\alpha,\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_{\alpha}\tau \\ (-s^{\alpha}) \frac{1}{2} \sin \Omega_{\alpha}\tau \\ 1 + \frac{1}{2} \cos \Omega_{\alpha}\tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.2}$$

where $C_{\{\perp,\parallel\}}^{\alpha,\alpha'}(v_{\perp}, v_{\parallel}; \Omega\,\tau)$ are elements of the force-correlation matrix $\mathbf{C}(\tau) = \langle \mathbf{F_{int}}(t)\ \mathbf{F_{int}}(t-\tau) \rangle_R$; they are given by:

$$C_{\{\perp,\parallel\}} = n_{\alpha'} (2\pi)^3 \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}}\, k_{\{\perp,\parallel\}}^2 \tag{8.3}$$

(a summation over n, m is understood). Obviously, i (j) in $\{i, j\}$ correspond to the upper (lower) i.e. $\perp$ ($\parallel$) parts respectively.

Remember that the dynamical friction vector $\mathcal{F}$ is defined through the above coefficients; cf (6.29).

The α index, as well as the summation over particle species α', will be dropped where obvious.

8.3 Reduction of the formulae for Maxwellian plasma

The v_1- integration in (8.1) can be carried out, once one assumes an analytic form for the equilibrium reservoir distribution function (df) $\phi_{eq}(\mathbf{v_1})$ [4]. Here, it will be explicitly taken to be a Maxwellian of the form:

$$\phi_{Max}^{\alpha'}(v_1) = \prod_{i=1,2,3} \phi_0^{(i,\alpha')}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \tag{8.4}$$

[4] Remember that the homogeneous equilibrium distribution function is a function of $\{v_{\perp}, v_{\parallel}\}$ (i.e. the motion invariants - here, the *conserved* velocity components).

where $\phi_0^{(i)}$ is the normalization constant:

$$\phi_0^{(i)} = \Big(\frac{m_{\alpha'}}{2\pi T_{\alpha'}^{(i)}}\Big)^{1/2} \equiv \frac{1}{(2\pi)^{1/2}\, v_{i,th}^{\alpha'}} \equiv \frac{1}{\sqrt{\pi\sigma_i^{\alpha'}}}$$

and σ is related to the plasma temperature:

$$\sigma_i^{\alpha'} \equiv 2\, {v_{i,th}^{\alpha'}}^2 \equiv \frac{2T_i^{\alpha'}}{m_{\alpha'}} \qquad \forall i \in \{1,2,3\} \equiv \{x,y,z\}$$

We will assume here that

$$\sigma_1^{\alpha'} = \sigma_2^{\alpha'} = \sigma_\perp \,, \qquad \sigma_3^{\alpha'} = \sigma_\parallel$$

8.3.1 Elimination of the velocity integral $\int d^3\mathbf{v_1}\cdots$

By substituting from (8.4) into (8.1) we obtain:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \\ D_\parallel \end{array} \right\} =$$

$$= \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \int d\mathbf{v_1} \prod_i \Big[\phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \Big] \tilde{V}_k^2$$
$$e^{ik_n N_{nm}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}} \left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega_\alpha\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega_\alpha\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega_\alpha\tau) \\ k_z^2 \end{array} \right\}$$

$$= \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \prod_i \Big[\int dv_{1,i}\, \phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}}\, e^{-ik_n N_{ni}^{\alpha'}(\tau) v_{1,i}} \Big] \tilde{V}_k^2$$
$$e^{ik_n N_{nm}(\tau) v_m} \left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega_\alpha\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega_\alpha\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega_\alpha\tau) \\ k_z^2 \end{array} \right\}$$
$$= \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \prod_i \Big[\int dv_{1,i}\, \phi_0^{(i)}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}}\, e^{-ip_i^{\alpha'} v_{1,i}} \Big] \tilde{V}_k^2$$
$$e^{ip_m v_m} \left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega_\alpha\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega_\alpha\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega_\alpha\tau) \\ k_z^2 \end{array} \right\} \qquad (8.5)$$

where we defined: $k_n N^\beta_{nm} \equiv p^\beta_m$ (β is either α or α'; a summation over m is understood where appropriate) [5]. Note that:

$$p^{\alpha'}_m(\mathbf{k};\tau) = \sum_{n=1}^{3} k_n N^{\alpha'}_{nm}(\tau) = \sum_{n=1}^{3} k_n \int_0^\tau R^{\alpha'}_{nm}(t')dt' \tag{8.6}$$

The definite integrals in brackets are of a well-known general form, which can be shown to yield:

$$\int_{-\infty}^{\infty} e^{-iAx}\, e^{-Bx^2}\, dx \;=\; e^{-\frac{A^2}{4B}} \int_{-\infty}^{\infty} e^{-B(x+i\frac{A}{2B})^2}\, dx \;=\; e^{-A^2/4B}\,\frac{\sqrt{\pi}}{\sqrt{B}}$$

i.e.

$$\int_{-\infty}^{\infty} e^{-ip_i^{\alpha'}\, v_{1,i}}\, e^{-\frac{v_{1,i}^2}{\sigma_i^{\alpha'}}}\, dv_{1,i} \;=\; \sqrt{\pi\sigma_i^{\alpha'}}\; e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4}$$

Therefore, expressions (8.5) above become:

$$\begin{aligned}
\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \\ D_\parallel \end{array} \right\} \;&=\; \frac{n_{\alpha'}}{m^2}(2\pi)^3 \int_0^{t\to\infty} d\tau \int d\mathbf{k} \prod_i \left[\phi_0^{(i)}\, \sqrt{\pi\sigma_i^{\alpha'}}\; e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4} \right] \tilde{V}_k^2 \\
&\qquad e^{ip_m^\alpha v_m} \left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega^\alpha\tau) \\ k_z^2 \end{array} \right\} \\
&=\; \frac{n_\alpha}{m^2}(2\pi)^3 \int_0^{t\to\infty} d\tau \int d\mathbf{k} \prod_i \left[e^{-\sigma_i^{\alpha'}\, p_i^{\alpha'\,2}/4}\, e^{ip_i^\alpha v_i} \right] \tilde{V}_k^2 \\
&\qquad \left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega_\alpha\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega_\alpha\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega_\alpha\tau) \\ k_z^2 \end{array} \right\}
\end{aligned} \tag{8.7}$$

(remember the definition of $\phi_0^{(i)}$). A summation over α' is understood; once more, let us remind that α (α') denotes the test- (1^R-) particle species respectively.

Note: of particle species - remarks on the model.

A clarifying remark needs to be made here. Remember the way our physical system is described: a *charged* test - particle σ_α, of species α (charge e_α, mass m_α), is moving against a thermalized background of particles of (several) species α' (charge $e_{\alpha'}$, mass $m_{\alpha'}$, density $n_{\alpha'}$, temperature $T_{\alpha'}$). Type α' may denote *electrons*, *ions* (maybe several types of), etc.; the t.p. may be either e.g. a

[5] In the limit $\Omega \to 0$ we have: $p_m \to k_m\tau$.

'foreign' particle injected in the plasma, in general, or an *'astray'* particle of the inside, which found itself off equilibrium, for some reason. Therefore α may, or may *not*, be the same type of particle as α', depending of the problem posed[6] [7]. Of course, in the former case ($\alpha = \alpha'$, considered in the preceding part of this chapter) the t.p. obeys the same dynamic laws (e.g. same characteristic frequency $\Omega_\alpha = \Omega_{\alpha'}$) as particles α' and *relaxes to* the same temperature (i.e. exactly the temperature of the bath); nevertheless, needless to say, should one invoke density n in the calculations, this would refer to the density $n_{\alpha'}$ of particles in the bath, and *not* that of the t.p.[8]; nevertheless, this point having been made clear, the 'erroneous' notation n_α may be used when $\alpha' = \alpha$.

Now notice, in the above formulae, that the two exponentials correspond to the two different species, α and α', of the particles entering in collisions (t.p. σ^α and field particle 1_R^α). Therefore, attention should be paid to the evaluation of the above expression, depending on whether the term concerns particles of equal or distinct species. The next two sections will be devoted to the case of equal- ($\alpha = \alpha'$) and different ($\alpha \neq \alpha'$) species respectively.

8.4 Collisions between equal-species: $\alpha = \alpha'$

Let us assume that $\alpha = \alpha'$ (indices denoting species are now dropped for simplicity).

For $\alpha_\sigma = \alpha_1$ ($\alpha = \alpha'$) one may write:

$$-\sigma_j\, p_j^2/4 + i p_j v_j \;=\; -\frac{1}{4}\,\sigma_j \left[\big(p_j^2 - i\frac{2v_j}{\sigma_j}\big)^2 + \frac{4v_j^2}{\sigma_j^2} \right] \;=\; -\frac{1}{4}\,\sigma_j\,\big(p_j^2 - i\frac{2v_j}{\sigma_j}\big)^2 - \frac{v_j^2}{\sigma_j}$$

(we henceforth drop the subscript α in this section). Thus, setting:

$$q_m(\mathbf{k};\tau;v) \;=\; p_m - i\frac{2v_m}{\sigma_m} \;\equiv\; \sum_{n=1}^{3} k_n N_{nm}(\tau) - i\frac{2v_m}{\sigma_m} \tag{8.8}$$

(cf. (8.6)[9]) we have:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \\ D_\parallel \end{array} \right\} \;=\; \frac{n}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k} \prod_i \left(e^{-\sigma_i\, q_i^2/4}\, e^{-v_i^2/\sigma_i} \right) \tilde{V}_k^2$$

---

[6]As we saw, the diffusion coefficient D^α, regarding the t.p. σ_α, will be a sum over α' of terms i.e. $D^{\alpha,\alpha'}$; therefore, both cases may be of interest.

[7]The problem may refer to a variety of situations e.g. (a) electrons moving against a background of active ions (so $m_{\alpha'} \gg m_\alpha$), (b) energetic ions relaxing against a thermalized background of electrons ($m_{\alpha'} \ll m_\alpha$), also referring to situations like energy pumping via energetic $\alpha-$particle (^4_2He nuclei) injection into plasma, mentioned previously, etc.

[8]Literally speaking, the density of the t.p. is $n_\alpha = \frac{1}{V} \ll n_{\alpha'}$.

[9]$[q_m] = [p_m] = 1/[v] = L^{-1}T^1$; in the limit $\Omega \to 0$ we have: $q_m \to k_m\tau - i\frac{2u_m}{\sigma}$.

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega\tau) \\ k_z^2 \end{array} \right\}$$

$$= \frac{n}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k}\; e^{-\sum_i \sigma_i\, q_i^2/4}\, e^{-\sum_i v_i^2/\sigma_i}\, \tilde{V}_k^2$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega\tau) \\ k_z^2 \end{array} \right\}$$

$$= \frac{n}{m^2}(2\pi)^3\; e^{-v_i^2/\sigma_i} \int_0^t d\tau \int d\mathbf{k}\; e^{-\sigma_i\, q_i^2/4}\tilde{V}_k^2$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,(k_x^2+k_y^2)\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,(k_x^2+k_y^2)\,\sin\Omega\tau \\ (k_x^2+k_y^2)(1+\frac{1}{2}\cos\Omega\tau) \\ k_z^2 \end{array} \right\}$$

i.e.

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} = \frac{n}{m^2}(2\pi)^3\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\; e^{-\sigma_m\, q_m^2/4}\, k_\perp^2\, \tilde{V}_k^2 \right)$$

$$\left\{ \begin{array}{c} \frac{1}{2}\,\cos\Omega\tau \\ (-s)\,\frac{1}{2}\,\sin\Omega\tau \\ 1+\frac{1}{2}\,\cos\Omega\tau \end{array} \right\}$$

$$D_\parallel = \frac{n}{m^2}(2\pi)^3\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\; e^{-\sigma_m\, q_m^2/4}\, k_\parallel^2\, \tilde{V}_k^2 \right) \quad (8.9)$$

(once more, a summation over m is understood where appropriate; thus $e^{-v_m^2/\sigma_m}$ $= e^{-v_\perp^2/\sigma_\perp}\, e^{-v_\parallel^2/\sigma_\parallel}$). Note that the integrals within parenthesis in the last relations present a cylindrical symmetry due to the existence of the $\mathbf{N}$ matrix in it (cf. (8.8)); in the absence of an external magnetic (or *any*) field, it reduces to a *spherically* symmetric form, as $\mathbf{N}(\tau) \to \tau\mathbf{I}$ [10].

For convenience we shall define the quantities in parenthesis in the latter

---

[10] If $\Omega \to 0$, the $\mathbf{k}-$ integral (cf. (8.9)) in $D_{11} = D_{22} = D_{33}$ will be:

$$D_{jj} = \cdots \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\, \tilde{V}_k^2\, k_j^2 = \cdots =$$

$$= \int_{-\infty}^{\infty} dk_i\; e^{-\sigma\, q_i^2/4} \int_{-\infty}^{\infty} dk_l\; e^{-\sigma\, q_l^2/4} \int_{-\infty}^{\infty} dk_j\; e^{-\sigma\, q_j^2/4}\, \tilde{V}_k^2\, k_j^2$$

$(i \neq l \neq j = 1, 2, 3)$

relation as[11]:

$$I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}} \equiv \int d\mathbf{k}\; e^{-\sigma\, q^2/4}\; \tilde{V}_k^2 \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\} \tag{8.10}$$

As a matter of fact, relation (8.9) holds as it stands for *any* particular form of $V(r)$. However, this is not the most general form in terms of $V(r)$ (still not specified, that is). Remember that, in principle, $V = V(|\mathbf{r}|) = V(r)$ implies that $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$ ($= V(k_\perp^2 + k_\|^2)$ in a cylindrical-symmetric problem). Therefore the Fourier transform of the interaction potential does *not* depend on the angle α ($\equiv (\hat{x}, \mathbf{k}_\perp)$) and the corresponding angle integration inside the triple integral $I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}}$ can be performed straightaway, giving a final double integration in $k_\perp$, $k_\|$. This is precisely what we will do in the next paragraph.

8.4.1 The Fourier-integral(s) $\int d^3\mathbf{k} ... = I_{\mathbf{k}^{(3)}}^{\{\perp,\|\}}$

Let us remark that q^2 can be expressed in an elegant manner as [12]:

$$\begin{aligned} q^2 &= \sum_m q_m^2 = \sum_m \left(p_m - i\frac{2v_m}{\sigma_m}\right)^2 = \sum_m \left(p_m^2 - 4\frac{v_m^2}{\sigma_m^2} - 4i\frac{p_m v_m}{\sigma_m}\right) \\ &\equiv p^2 - 4\frac{v^2}{\sigma^2} - 4i\frac{\mathbf{p}\cdot\mathbf{v}}{\sigma} \end{aligned}$$

and can be decomposed into $\{x, y\} \equiv \perp$ and $\{z\} \equiv \|$- parts:

$$q^2 = \left(p_x - i\frac{2v_x}{\sigma_x}\right)^2 + \left(p_y - i\frac{2v_y}{\sigma_y}\right)^2 + \left(p_z - i\frac{2v_z}{\sigma_z}\right)^2 \equiv q_\perp^2 + q_\|^2$$

By making use of the explicit definition of the $\mathbf{p}$ vector above (see (8.6)), as well as that of the $\mathbf{N}$ matrix in it, we find

$$\begin{aligned} q_\|^2 &= \left(k_\|\tau - i\frac{2v_\|}{\sigma_\|}\right)^2 \\ q_\perp^2 &= p_\perp^2 - 4\frac{v_\perp^2}{\sigma_\perp^2} - 4i\frac{\mathbf{p}_\perp\cdot\mathbf{v}_\perp}{\sigma_\perp} \\ &= \Omega^{-2}\left(k_x^2 + k_y^2\right) 2\,(1-\cos\Omega\tau) - 4\frac{v_x^2 + v_y^2}{\sigma_\perp^2} \end{aligned} \tag{8.11}$$

[11] e.g. for a Coulomb potential $V(r) = e^2/r$, (8.10) reduces to:

$$I_{Coulomb}{}_{\mathbf{k}^{(3)}}^{\{\perp,\|\}} \equiv \int d\mathbf{k}\; e^{-\sigma\, q^2/4} \left(\frac{e^2}{2\pi^2}\right)^2 \frac{1}{(\sum_{i=1}^3 k_i^2)^2} \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\}$$

[12] In the limit $\Omega \to 0$ we have:

$$\begin{aligned} q^2 \to \sum_m q_m^2 &= \sum_m \left(k_m\tau - i\frac{2v_m}{\sigma_m}\right)^2 = \sum_m \left(k_m^2\tau^2 - 4\frac{v_m^2}{\sigma_m^2} - 4i\frac{k_m v_m \tau}{\sigma_m}\right) \\ &\equiv k^2\tau^2 - 4\frac{v^2}{\sigma_m^2} - 4i\frac{\tau(\mathbf{k}\cdot\mathbf{v})}{\sigma_m} \end{aligned}$$

$$-i\frac{4}{\sigma_\perp}\Omega^{-1}\Big[\big(k_x v_x + k_y v_y\big)\sin\Omega\tau - s(1-\cos\Omega\tau)\big(k_y v_x - k_x v_y\big)\Big] \tag{8.12}$$

in cartesian coordinates. In cylindrical coordinates, one may set:

$$\mathbf{k} = \begin{pmatrix} k_\perp \cos\alpha \\ k_\perp \sin\alpha \\ k_\parallel \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_\perp \cos\theta \\ v_\perp \sin\theta \\ v_\parallel \end{pmatrix}$$

so relation (8.12) takes the form:

$$\begin{aligned} q_\perp^2 &= 2\,\frac{k_\perp^2}{\Omega^2}\,(1-\cos\Omega\tau) - 4\frac{v_\perp^2}{\sigma_\perp^2} \\ &\qquad -i\frac{4}{\sigma_\perp}\frac{k_\perp v_\perp}{\Omega}\,s\,\Big[\sin(\theta-\alpha) - \sin(\theta-\alpha-s\Omega\tau)\Big] \\ &= 4\,\frac{k_\perp^2}{\Omega^2}\,\sin^2\frac{\Omega\tau}{2} - 4\frac{v_\perp^2}{\sigma_\perp^2} \\ &\qquad -i\frac{4}{\sigma_\perp}\frac{k_\perp v_\perp}{\Omega}\,2\,s\,\sin(s\frac{\Omega\tau}{2})\cos(\theta-\alpha-s\frac{\Omega\tau}{2}) \end{aligned} \tag{8.13}$$

where we used the trigonometric identities:

$$\sin\alpha - \sin\beta \;=\; 2\,\sin(\alpha-\beta)\,\cos(\alpha+\beta)$$

$$1-\cos\alpha \;=\; 2\,\sin^2\frac{\alpha}{2}$$

The **k**- integral(s) in parenthesis in eq. (8.9) become:

$$\begin{aligned} &\int d\mathbf{k}\; e^{-\sigma\, q^2/4}\,\tilde{V}_k^2\,k_i\,k_l \;= \\ &\qquad \int_0^\infty dk_\perp\,k_\perp \int_{-\infty}^\infty dk_\parallel \int_0^{2\pi} d\alpha\;\tilde{V}_k^2\,e^{-\sigma_\perp\,q_\perp^2/4}\,e^{-\sigma_\parallel\,q_\parallel^2/4} \\ &\qquad\qquad \begin{pmatrix} k_\perp\cos\alpha \\ k_\perp\sin\alpha \\ k_\parallel \end{pmatrix}_i \begin{pmatrix} k_\perp\cos\alpha \\ k_\perp\sin\alpha \\ k_\parallel \end{pmatrix}_j \end{aligned}$$

that is

$$\begin{aligned} I_{\mathbf{k}^{(3)}}^{(\perp,\parallel)} &= \int d\mathbf{k}\; e^{-\sigma\,q^2/4}\,\tilde{V}_k^2 \left\{ \begin{matrix} k_x^2 + k_y^2 \\ k_z^2 \end{matrix} \right\} = \\ &= \int_0^\infty dk_\perp\,k_\perp \int_{-\infty}^\infty dk_\parallel \int_0^{2\pi} d\alpha\,e^{-\sigma_\perp\,q_\perp^2/4}\,e^{-\sigma_\parallel\,q_\parallel^2/4}\,\tilde{V}_k^2 \left\{ \begin{matrix} k_\perp^2 \\ k_\parallel^2 \end{matrix} \right\} \\ &= \int_0^\infty dk_\perp \left\{ \begin{matrix} k_\perp^3 \\ k_\perp \end{matrix} \right\} \Big(\int_0^{2\pi} d\alpha\;e^{-\sigma_\perp\,q_\perp^2/4}\Big) \end{aligned}$$

$$\left[\int_{-\infty}^{\infty} dk_\parallel \; e^{-\sigma_\parallel \, q_\parallel^2/4} \, \tilde{V}^2_{k\equiv(k_\perp^2+k_\parallel^2)^{1/2}} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\}\right]$$

$$\equiv \int_0^\infty dk_\perp \left\{ \begin{array}{c} k_\perp^3 \\ k_\perp \end{array} \right\} I_\alpha \; I_{k_\parallel}^{\{\perp,\parallel\}} \tag{8.14}$$

Note the appearance of the magnetic field *only* in the first (angle-) integration I_α and not in the quantity in brackets. On the contrary, the exact form of the interaction potential *only* enters the latter (and not at all the former).

The integral I_α will be analytically evaluated in the next paragraph. The remaining part of the $\mathbf{k}-$ integral has to be evaluated once a specific form of interaction potential is chosen.

8.4.2 The α- integration

The α- integral in parenthesis in (8.14), say I_α, can now be evaluated analytically. For convenience, we can use expression (8.13) which, once substituted into I_α, yields:

$$\begin{aligned} I_\alpha &= \int_0^{2\pi} d\alpha \, e^{-\sigma_\perp \, q_\perp^2/4} \\ &= \int_0^{2\pi} d\alpha \, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \, e^{i \frac{k_\perp v_\perp}{\Omega} 2 \sin \frac{\Omega\tau}{2} \cos(\theta - \alpha - s\frac{\Omega\tau}{2})} \\ &= e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \int_0^{2\pi} d\alpha \, e^{i \, 2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2} \sin(\frac{\pi}{2} - \theta + \alpha + s\frac{\Omega\tau}{2})} \end{aligned}$$

In order to perform the angle integration, we may now use the Bessel function identity:

$$e^{i \, x \, \sin \phi} = \sum_{n=-\infty}^{\infty} J_n(x) \, e^{in\,\phi} \qquad \forall x, \phi \in \Re \tag{8.15}$$

(J_n are Bessel functions of the first kind) to express the integrand as:

$$e^{i\left(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}\right) \sin(\frac{\pi}{2} - \theta + \alpha + s\frac{\Omega\tau}{2})} = \sum_{n=-\infty}^{\infty} J_n(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}) \, e^{in\,\alpha} \, e^{in\,(\frac{\pi}{2} - \theta + s\frac{\Omega\tau}{2})}$$

Now, using:

$$\int_0^{2\pi} e^{in\alpha} d\alpha = 2\pi \, \delta_{n,0}^{Kr}$$

(where $\delta_{n,0}^{Kr}$ is the Kronecker-delta symbol)[13] we obtain:

$$I_\alpha = (2\,\pi) \, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}} \, e^{\frac{v_\perp^2}{\sigma_\perp}} \, J_O(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}) \tag{8.16}$$

$J_O(x)$ is the zeroth-order Bessel function of the first kind.

It is very interesting to remark that the angle-integral contains *all* the information about the magnetic field (through Ω) (see the formulae in the previous paragraph).

[13] The dummy summation index n is thus set to zero.

8.4.3 Form of the coefficients for an arbitrary potential

By substituting from (8.16) into (8.14) and then back into (8.9) we obtain:

$$\begin{aligned}
\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} &= \frac{n}{m^2}(2\pi)^4\, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp^3 \left[\int_{-\infty}^\infty dk_\parallel\; e^{-\sigma_\parallel\, q_\parallel^2/4}\, \tilde{V}_k^2 \right] \\
&\quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2}) \left\{ \begin{array}{c} \frac{1}{2} \cos\Omega\tau \\ (-s)\, \frac{1}{2} \sin\Omega\tau \\ 1 + \frac{1}{2} \cos\Omega\tau \end{array} \right\} \\
D_\parallel &= \frac{n}{m^2}(2\pi)^4\, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp \\
&\quad \left[\int_{-\infty}^\infty dk_\parallel\, k_\parallel^2\; e^{-\sigma_\parallel\, q_\parallel^2/4}\, \tilde{V}_k^2 \right] \\
&\quad e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega\tau}{2})
\end{aligned} \tag{8.17}$$

See that $k_\parallel$ only appears inside the brackets (so the $k_\parallel-$ integration may be easier to carry out first).

Expression (8.17) is the most general form of the diffucion coefficients for a test-particle in a maxwellian background, interacting through a central potential $V(r)$ (which remains to be specified).

8.4.4 Explicit calculation for screened electrostatic interactions

Debye potential

Let us now explicitly assume that the (long-range) interaction potential $V(r)$ is a Debye-type potential:

$$V(r) = V(|\mathbf{x}_{\sigma^\alpha} - \mathbf{x}_{1^{\alpha'}}|) = \frac{e_\alpha\, e_{\alpha'}}{r}\, e^{-r/\lambda_D} \equiv V_0\, \frac{e^{-k_D\, r}}{r} \tag{8.18}$$

i.e.

$$\tilde{V}_k = \frac{e_\alpha\, e_{\alpha'}}{2\,\pi^2} \frac{1}{k^2 + k_D^2} \tag{8.19}$$

where λ_D is the Debye length [5], [22] [14]:

$$\lambda_D \equiv k_D^{-1} = \left(\frac{4\pi e^2 n}{k_B T} \right)^{-1/2} \tag{8.20}$$

[14] In general:

$$\lambda_D \equiv k_D^{-1} = min\left\{ \left(\frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha} \right)^{-1/2} \right\}.$$

and $V_0 = e_\alpha \, e_{\alpha'}$, $\tilde{V}_0 = \frac{e_\alpha \, e_{\alpha'}}{2\pi^2}$ are constant quantities [15]. Obviously

$$k^2 = \sum k_i^2 = k_x^2 + k_y^2 + k_z^2 \equiv k_\perp^2 + k_\parallel^2$$

Eq. (8.17) directly becomes:

$$\begin{aligned}
\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} &= \frac{4\, n\, e^4}{m^2} \, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp \, k_\perp^3 \\
& \qquad \left[\int_{-\infty}^{\infty} dk_\parallel \; e^{-\sigma_\parallel \, q_\parallel^2/4} \, \frac{1}{(k^2 + k_D^2)^2} \right] \\
& \qquad e^{-\sigma_\perp \, \frac{k_\perp^2}{\Omega^2} \, \sin^2 \frac{\Omega\tau}{2}} \, J_O(2 \frac{k_\perp v_\perp}{\Omega} \, \sin \frac{\Omega\tau}{2}) \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega\tau \\ (-s)\, \frac{1}{2} \sin \Omega\tau \\ 1 + \frac{1}{2} \cos \Omega\tau \end{array} \right\} \\
D_\parallel &= \frac{4\, n\, e^4}{m^2} \, e^{-v_\parallel^2/\sigma_\parallel} \int_0^t d\tau \int_0^\infty dk_\perp \, k_\perp \\
& \qquad \left[\int_{-\infty}^{\infty} dk_\parallel \, k_\parallel^2 \; e^{-\sigma_\parallel \, q_\parallel^2/4} \, \frac{1}{(k^2 + k_D^2)^2} \right] \\
& \qquad e^{-\sigma_\perp \, \frac{k_\perp^2}{\Omega^2} \, \sin^2 \frac{\Omega\tau}{2}} \, J_O(2 \frac{k_\perp v_\perp}{\Omega} \, \sin \frac{\Omega\tau}{2}) \qquad (8.21)
\end{aligned}$$

The $k_\parallel$- integration

We may now attempt to evaluate the $k_\parallel$- integral (in brackets in (8.21)), say $I_{k_\parallel}^{(\perp,\parallel)}$. Note the absence of the field i.e. Ω in the $\parallel$ $-$ part of the formulae; as a consequence, the results of this paragraph are valid as they stand in the free-of-field (free motion) case, as well.

Substituting from expression (8.11) we obtain:

$$\begin{aligned}
I_{k_\parallel}^{\{\perp,\parallel\}} &= \int_{-\infty}^{\infty} dk_\parallel \; e^{-\sigma_\parallel \, q_\parallel^2/4} \, \frac{1}{(k^2 + k_D^2)^2} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \\
&= \int_{-\infty}^{\infty} dk_\parallel \; e^{-\sigma_\parallel \, \left(k_\parallel \tau - i \frac{2 v_\parallel}{\sigma_\parallel}\right)^2/4} \, \frac{1}{(k_\perp^2 + k_\parallel^2 + \tilde{k}_\perp^2)^2} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \\
&= e^{v_\parallel^2/\sigma_\parallel} \int_{-\infty}^{\infty} dk_\parallel \, e^{-\sigma_\parallel \, k_\parallel^2 \, \tau^2/4} \, \frac{\cos(k_\parallel \, v_\parallel \, \tau)}{(k_\parallel^2 + \tilde{k}_\perp^2)^2} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \qquad (8.22)
\end{aligned}$$

where we have set:

$$\tilde{k}_\perp = (k_\perp^2 + k_D^2)^{1/2} \qquad (8.23)$$

Notice that the imaginary part of the integral cancels for reasons of symmetry, as the integrand in it is an *odd* function of $k_\parallel$ [16]. After a tedious calculation

[15] Remember that $\alpha = \alpha'$ throughout this section, so $(2\pi)^4 \, \tilde{V}_0^2$ appearing in the formulae is here replaced by $4\, e_\alpha^4$.

[16] This was expected: the coefficients in the kinetic equation *have to* be real, in order the reality of the distribution function to be preserved in time.

(provided in the Appendix) the integral(s) in (8.22) are found to be:

$$I_{k_\parallel}^{\{\perp,\parallel\}} = \frac{1}{\left\{ \begin{array}{c} \tilde{k}_\perp^3 \\ \tilde{k}_\perp \end{array} \right\}} \Big\{ \pm \frac{\sqrt{\pi}}{2} \sqrt{\sigma_\parallel}\, \tilde{k}_\perp \tau + \frac{\pi}{4} e^{v_\parallel^2/\sigma_\parallel}\, e^{\sigma_\parallel \tilde{k}_\perp^2 \tau^2/4}$$

$$\sum_{s=+1,-1} \Big[e^{s\tilde{k}_\perp v_\parallel \tau} \left(1 \mp \sigma_\parallel \tilde{k}_\perp^2 \tau^2/2 \mp s\tilde{k}_\perp v_\parallel \tau\right) Erfc\big(\frac{1}{2}\sqrt{\sigma_\parallel}\tilde{k}_\perp \tau + s\frac{v_\parallel}{\sqrt{\sigma_\parallel}}\big)\Big]\Big\} \tag{8.24}$$

where the upper (lower) index in the above formulae holds for $I^{(\perp)}$ ($I^{(\parallel)}$). $Erfc(x)$ denotes the complementary error function (see in Appendix F).

Note that the above results for the integrals $I_{k_\parallel}^{(\perp,\parallel)}$:

(i) converge to zero at both $k_\perp \to \infty$ and $\tau \to \infty$, as intuitively expected.
(ii) give a *finite* limit at $\tau \to 0$ [17] :

$$\lim_{\tau \to 0} I_{k_\parallel}^{\{\perp,\parallel\}} = \frac{\pi}{2} e^{v_\parallel^2/\sigma_\parallel}$$

(ii) do *not* diverge at $k_\perp \to 0$.

The quantities defined in (8.24), re-scaled over $\frac{\pi}{2} e^{v_\parallel^2/\sigma_\parallel}$, are depicted in figures 8.1, 8.2 (actually in a reduced form, see further below). Notice that they converge very rapidly: the numerical value of this expression is practically below 10^{-5} everywhere above $v \approx 4\sqrt{\sigma}$ and $\sqrt{\sigma} k_\perp \tau/2 \approx 2$.

Diffusion coefficients

The coefficients in (8.9) are functions of $\{v_\perp, v_\parallel, t;\, \sigma_\perp, \sigma_\parallel, \Omega\}$ given by the expressions:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{n}{m^2} 4e^4 \int_0^t d\tau \int_0^\infty dk_\perp\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O(2\frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega \tau}{2})$$

$$\left(1 - \frac{k_D^2}{k_D^2 + k_\perp^2}\right)^{\{3/2,1/2\}} \left\{ \begin{array}{c} \{\ F_\perp\ \} \\ F_\parallel \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega\tau \\ \frac{-s}{2} \sin \Omega\tau \\ 1 + \frac{1}{2} \cos \Omega\tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.25}$$

where the functions $F = F_{\{\perp,\parallel\}}(k_\perp, v_\parallel, \tau;\, \sigma_\parallel)$ are given by:

[17] Note that $Erfc(x) + Erfc(-x) = 2$.

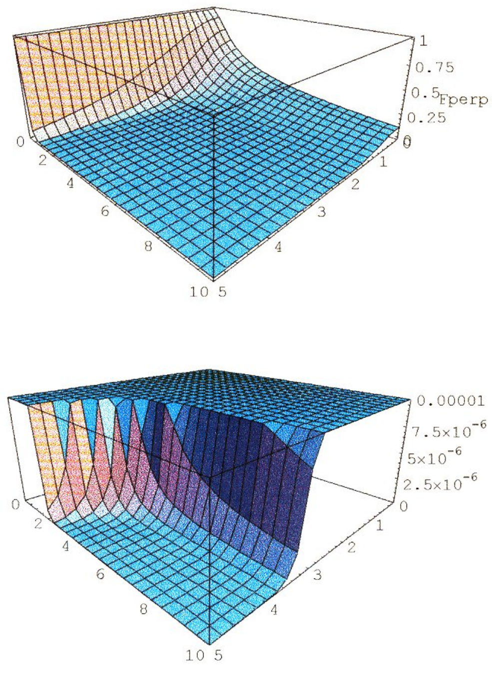


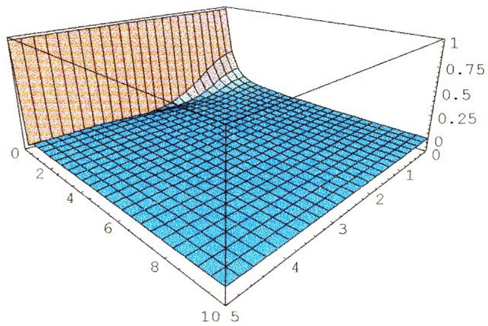
Figure 8.1: The dimensionless quantity $F_\perp(\phi, \tilde{v})$ defined in (8.29), scaled over $F_\perp(0,0) = \frac{\pi}{2}\, e^{\tilde{v}^2}$ (so its value at the origin is unity). The dimensionless velocity $\tilde{v}_\parallel$ in the argument appears in the region between 0 and 5, while ϕ is taken between 0 and 10. The second plot is focused in the region where $F_\perp(\phi, \tilde{v})$ is below 10^{-5}; notice that it converges to zero very rapidly.

$$F_{\{\perp,\parallel\}} = \pm \frac{\sqrt{\pi}}{2} \sqrt{\sigma_\parallel}\, \tilde{k}_\perp \, \tau \, e^{-v_\parallel^2/\sigma_\parallel} \; + \frac{\pi}{4}\, e^{\sigma_\parallel \tilde{k}_\perp^2 \tau^2/4} \times$$
$$\sum_{s=+1,-1} \left[e^{s \tilde{k}_\perp v_\parallel \tau} \left(1 \mp \sigma_\parallel \tilde{k}_\perp^2 \tau^2/2 \mp s\tilde{k}_\perp v_\parallel \tau\right) Erfc\big(\frac{1}{2}\sqrt{\sigma_\parallel}\tilde{k}_\perp \tau + s\frac{v_\parallel}{\sqrt{\sigma_\parallel}}\big) \right] \tag{8.26}$$

the upper (lower) signs corresponding to the $\perp$ ($\parallel$)- parts respectively.

Limit cases - discussion

It is very interesting to remark that, as we mentioned before, the angle-integral in (8.16) contains *all* the information about the magnetic field (through Ω) in the above formulae. Even though a more thorough investigation of limit cases will follow, it may be appropriate to point out a few facts here.



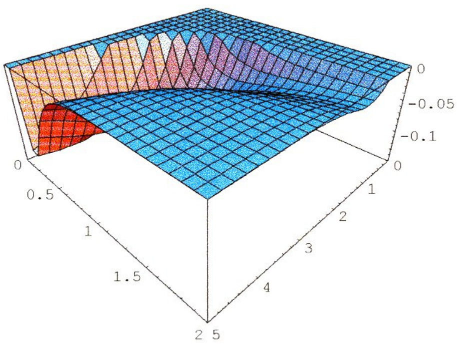


Figure 8.2: The dimensionless quantity $F_{\|}(\phi, \tilde{v})$ defined in (8.29), scaled over $F_{\|}(0,0) = \frac{\pi}{2}\, e^{\tilde{v}^2}$. For the sake of comparison with the previous figure, the region of plot (a) is exactly the same: $\tilde{v}$ appears in the region between 0 and 5, while ϕ is taken between 0 and 10. Notice that $F_{\|}(\phi, \tilde{v})$ converges even more rapidly than $F_{\perp}(\phi, \tilde{v})$. See the second plot, zoomed in the region where ϕ is between 0 and 2 ($\tilde{v}_{\|}$ is again taken between 0 and 5) and *below* zero: $F_{\|}$ has a negative tail, which is nevertheless smoothed out as $\phi \approx k_{\perp}\, \tau$ grows above 1.

In the infinite magnetic field limit ($\Omega \to \infty$) we have [18]:

$$I_{\alpha} \to 2\, \pi\, e^{v_{\perp}^2/\sigma_{\perp}}$$

since $J_0(0) = 1$, so the dependence on the field *and* wavenumber $k_{\perp}$ disappears.

[18]Express $\frac{1}{\Omega} \sin \frac{\Omega\tau}{2}$ as $\frac{2}{\tau} \frac{\sin \frac{\Omega\tau}{2}}{\frac{\Omega\tau}{2}} \equiv \frac{2}{\tau} \frac{\sin x}{x}$ and then:

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0\,.$$

On the other hand, in the limit $\Omega \to 0$ we have [19]:

$$I_\alpha \to (2\pi)\, e^{-\sigma_\perp k_\perp^2 \tau^2/4}\, e^{v_\perp^2/\sigma_\perp}\; J_0(k_\perp v_\perp \tau)$$

For large times $\tau \to \infty$, I_α remains a bounded oscillating function of time τ, so the dependence on the field Ω *only* survives in $\cos\Omega\tau$, $\sin\Omega\tau$.

At $k_\perp \to 0$

$$e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\; J_O(2\frac{k_\perp v_\perp}{\Omega}\,\sin\frac{\Omega\tau}{2}) \to 1$$

so the dependence on the field Ω only survives in $\cos\Omega\tau$, $\sin\Omega\tau$.

8.4.5 Final non-dimensional formulae

In the following we shall set $\sigma_\perp = \sigma_\parallel = \sigma$ for simplicity.

Force correlations $C^\alpha_{\{\perp,\parallel\}}(\tau)$

Notice that the above relations imply a set of expressions for the force correlation functions $C^\alpha_{\{\perp,\parallel\}}(\tau)$, readily obtained by comparing (8.25) to (8.2). The integration variable: $k_\perp$ therein can be rescaled to the non-dimensional variable:

$$x = \frac{\tilde{k}_\perp}{k_D} = (1 + \frac{k_\perp^2}{k_D^2})^{1/2} \tag{8.27}$$

The relations for the correlations can thus be expressed as:

$$\begin{aligned} C^\alpha_{\{\perp,\parallel\}}(\tau) \;&=\; 4\,n\,e^4\,k_D \int_1^\infty dx\; e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\Omega\tau}{2}} \left(1 - \frac{1}{x^2}\right)^{\{1,0\}} \\ &\qquad e^{-\tilde{v}_\parallel^2}\; J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\Omega\tau}{2})\;\tilde{F}_{\{\perp,\parallel\}} \end{aligned} \tag{8.28}$$

$\tilde{F}_{\{\perp,\parallel\}} = \tilde{F}_{\{\perp,\parallel\}}(\phi(x,\tau), \tilde{v}_\parallel)$ is given by:

$$\tilde{F}^{\alpha'}_{\{\perp,\parallel\}} = \pm\sqrt{\pi}\,\phi\; + \frac{\pi}{4} \sum_{s=+1,-1} \left[e^{(\phi + s\tilde{v}_\parallel)^2} \left(1 \mp 2\,\phi^2 \mp s2\,\phi\,\tilde{v}_\parallel\right) Erfc\big(\phi + s\,\tilde{v}_\parallel\big)\right] \tag{8.29}$$

where ϕ in the argument is given by:

$$\phi = \frac{\sqrt{\sigma}}{2}\,k_D\,x\,\tau \;=\; \frac{1}{\sqrt{2}}\,\omega_{p,\alpha}\,\tau\,x \;\equiv\; \frac{\lambda}{2}(\Omega\tau)\,x$$

the re-scaled velocity components are:

$$\tilde{v}_\perp = v_\perp/\sqrt{\sigma}\,, \qquad \tilde{v}_\parallel = v_\parallel/\sqrt{\sigma}$$

[19] As in the previous footnote, and then:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

Remember that

$$\sigma_\alpha = 2\, k_B\, T_\alpha / m_\alpha = 2\, v_{th,\alpha}^2$$

is related to the thermal velocity (i.e. to the temperature), Ω is the cyclotron (gyroscopic) frequency:

$$\Omega = \Omega_\alpha = e_\alpha\, B / m_\alpha\, c$$

k_D is the Debye wave-number:

$$k_D = \left(\frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha} \right)^{1/2}$$

and the field is 'hidden' in the dimensionless parameter λ:

$$\lambda = \sqrt{\sigma}\, \frac{k_D}{\Omega} = \cdots = \sqrt{2}\, \frac{\omega_p}{\Omega}$$

where ω_p is the plasma (Langmuir) frequency:

$$\omega_p = \omega_{p,\alpha} = \left(\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha} \right)^{1/2}$$

(so $\omega_p = \sqrt{\sigma\, k_D / 2}$). Notice the interplay of collision and magnetic field scales through $\lambda \approx \frac{T_{gyro}}{T_{coll}} \equiv \frac{v_{thermal}}{v_{Alfven}}$.

The correlations $C(\tau)$ are now expressed as a single definite integral in x from 1 to ∞. All physical parameters enter through ω_p and Ω. Therefore, for a given set of parameter values (namely temperature T, particle density n, mass m_α, particle charge e_α and field magnitude B), one only has to determine the values of ω_p, Ω and then λ; the above formulae for $C(\tau)$ can then be evaluated as functions of τ (or, rather, $\Omega\tau$), by carrying out the integration in x numerically.

For convenience, we shall define the quantity:

$$C_0 = 4\, n\, e^4\, k_D$$

(which has the dimensions[20] of $[force^2]$.).

Diffusion coefficients D_{ij}

The diffusion coefficients $D_{\perp,\angle,\parallel,\cdots}(t)$ are defined as a definite integral in τ (from 0 to $t \to \infty$, for the asymptotic value) (cf. (8.2)); they are functions of the velocity components $v_\perp, v_\parallel$ and physical parameters, including the magnitude of the magnetic field (through the cyclotron frequency Ω).

Let us define a non-dimensional time variable, say $\tilde{\tau} = \Omega\tau$, so $d\tau \to \Omega^{-1}\, d\tilde{\tau}$ (assuming $\Omega \neq 0$). Our final expressions for the diffusion coefficients now read:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{1}{m^2} 4\, n\, e^4\, k_D\, \Omega^{-1} \int_0^{\Omega t \to \infty} d\tilde{\tau} \int_0^\infty dx\, e^{\lambda^2\, (1-x^2)\, \sin^2 \frac{\tilde{\tau}}{2}}$$

[20]Remember that: $[n] = [length^{-3}]$ and $[k_D] = [length^{-1}]$

$$\left(1-\frac{1}{x^2}\right)^{\{1,0\}} e^{-\tilde{v}_\parallel^2}\, J_O(2\lambda\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})$$
$$\left\{ \begin{array}{c} \left\{ \tilde{F}_\perp \right\} \\ \tilde{F}_\parallel \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\tilde{\tau} \\ \frac{-s}{2}\sin\tilde{\tau} \\ 1+\frac{1}{2}\cos\tilde{\tau} \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.30}$$

(obviously 1, 0 in $\{1,0\}$ hold for the $\perp$ (first three terms on the *lhs*), $\parallel$ (last terms on the *lhs*) parts, respectively) or, finally[21]:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{1}{m^2}\,\frac{\lambda}{k_D\sqrt{\sigma}} \int_0^{\Omega t\to\infty} d\tilde{\tau}\, C_{\{\perp,\parallel\}}(\tilde{\tau};\,\tilde{v}_\perp,\tilde{v}_\parallel)$$
$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\tilde{\tau} \\ \frac{-s}{2}\sin\tilde{\tau} \\ 1+\frac{1}{2}\cos\tilde{\tau} \end{array} \right\} \\ 1 \end{array} \right\}$$
$$\equiv D_0\,\lambda \int_0^{\Omega t\to\infty} d\tilde{\tau} \int_0^{\infty\to\infty} dx\, e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\tilde{\tau}}{2}} \left(1-\frac{1}{x^2}\right)^{\{1,0\}}$$
$$e^{-\tilde{v}_\parallel^2}\, J_O(2\lambda\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2}) \left\{ \begin{array}{c} \left\{ \tilde{F}_\perp \right\} \\ \tilde{F}_\parallel \end{array} \right\}$$
$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\tilde{\tau} \\ \frac{-s}{2}\sin\tilde{\tau} \\ 1+\frac{1}{2}\cos\tilde{\tau} \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.31}$$

where all quantities in the right-hand-side are non-dimensional, except

$$D_0 \;=\; \frac{1}{m^2}C_0\frac{\Omega^{-1}}{\lambda} \;=\; \frac{1}{m^2}\frac{4\,n\,e^4}{\sqrt{\sigma}} \;=\; \frac{2\sqrt{2}\,n\,e^4}{m^{3/2}\,\sqrt{k_B T}} \tag{8.32}$$

The functions $\tilde{F}_{\{\perp,\parallel\}} = \tilde{F}_{\{\perp,\parallel\}}(\phi;\tilde{v}_\parallel)$ were defined in (8.29); $\phi = \phi(x\,\tau) = \phi(x\,\tilde{\tau})$ in their argument is now given by:

$$\phi = \frac{\lambda}{2}\,x\,\tilde{\tau}$$

Notice that D_0 is scaled as $\sim T^{-1/2}$ as a function of temperature T. Later on, we may set, say

$$D_0 \;=\; D'_0\,T^{-1/2}$$

in order to study the dependence of coefficients on temperature.

[21] Remember that $\Omega^{-1} = \frac{\lambda}{k_D\sqrt{\sigma}}$.

Frictions vectors $\mathcal{F}_i$

Let us evaluate the exact (non-dimensional) form of the dynamical friction vectors:

$$\begin{aligned}\mathcal{F}_\perp &= (1+\mu)\,\frac{\partial D_\perp}{\partial v_\perp} \equiv (1+\mu)\,\sigma^{-1/2}\frac{\partial D_\perp}{\partial \tilde{v}_\perp} \\ \mathcal{F}_\parallel &= (1+\mu)\,\frac{\partial D_\parallel}{\partial v_\parallel} \equiv (1+\mu)\,\sigma^{-1/2}\frac{\partial D_\parallel}{\partial \tilde{v}_\parallel}\end{aligned} \tag{8.33}$$

($\mu = m/m_1$).

Notice, in the previous paragraph, that the role of the two velocity components, $v_\perp$ and $v_\parallel$, is clearly distinguished, as their appearance in the formulae is limited to the Bessel function $J_0(\tilde{v}_\perp; \ldots)$ and the analytical functions $F_{\{\perp,\parallel\}}(\tilde{v}_\parallel; \ldots)$, respectively.

First, using the property:

$$\frac{d\,J_0(x)}{dx} = -J_1(x)$$

we have:

$$\frac{d}{dx}J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2}) \;=\; -2\lambda\,\sqrt{x^2-1}\,\sin\frac{\tilde{\tau}}{2}\,J_1(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})$$

and thus, from (8.31) (actually the $\perp -$ part therein):

$$\begin{aligned}\mathcal{F}_\perp &= (1+\mu)\,\frac{1}{m^2}\,\frac{\lambda}{k_D\,\sqrt{\sigma}}\,\int_0^{\Omega t\to\infty} d\tilde{\tau}\,\frac{1}{\sqrt{\sigma}}\,\frac{\partial}{\partial\tilde{v}_\perp}C_{\{\perp,\parallel\}}(\tilde{\tau};\,\tilde{v}_\perp,\tilde{v}_\parallel)\,\frac{1}{2}\cos\tilde{\tau} \\ &= (1+\mu)\,\frac{1}{m^2}\,\frac{\lambda}{k_D\,\sigma}\,\int_0^{\Omega t\to\infty} d\tilde{\tau}\,\int_0^{\infty} dx\,e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\tilde{\tau}}{2}}\left(1-\frac{1}{x^2}\right) \\ &\qquad e^{-\tilde{v}_\parallel^2}\left[-2\lambda\,\sqrt{x^2-1}\,\sin\frac{\tilde{\tau}}{2}\,J_1(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})\right]\tilde{F}_\perp\,\frac{1}{2}\,\cos\tilde{\tau} \\ &\equiv -\,(1+\mu)\,\frac{D_0}{\sqrt{\sigma}}\,\lambda^2\,\int_0^{\Omega t\to\infty} d\tilde{\tau}\,\int_0^{\infty} dx\,e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\tilde{\tau}}{2}}\left(x^2-1\right)^{3/2}\frac{1}{x^2} \\ &\qquad e^{-\tilde{v}_\parallel^2}\,J_1(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})\,\tilde{F}_\perp\,\cos\tilde{\tau}\,\sin\frac{\tilde{\tau}}{2}\end{aligned} \tag{8.34}$$

On the other hand, from (8.31) (i.e. the $\parallel -$ part this time):

$$\begin{aligned}\mathcal{F}_\parallel &= (1+\mu)\,\frac{1}{m^2}\,\frac{\lambda}{k_D\,\sqrt{\sigma}}\,\int_0^{\Omega t\to\infty} d\tilde{\tau}\,\frac{1}{\sqrt{\sigma}}\,\frac{\partial}{\partial\tilde{v}_\parallel}C_{\{\perp,\parallel\}}(\tilde{\tau};\,\tilde{v}_\perp,\tilde{v}_\parallel) \\ &= (1+\mu)\,\frac{1}{m^2}C_0\,\frac{\lambda}{k_D\,\sigma}\,\int_0^{\Omega t\to\infty} d\tilde{\tau}\,\int_0^{\infty} dx\,e^{\lambda^2\,\sin^2\frac{\tilde{\tau}}{2}}\left(1-\frac{1}{x^2}\right) \\ &\qquad e^{-\tilde{v}_\parallel^2}\,J_0(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})\,\frac{\partial}{\partial\tilde{v}_\parallel}\tilde{F}_\parallel\end{aligned}$$

The derivative of $\tilde{F}_{\parallel}$ reads (cf. (8.29)):

$$\frac{\partial \tilde{F}_{\parallel}(\phi, \tilde{v}_{\parallel})}{\partial \tilde{v}_{\parallel}} =$$

$$\begin{aligned} &= \frac{\partial}{\partial \tilde{v}_{\parallel}} \Big\{ -\sqrt{\pi}\,\phi \ + \frac{\pi}{4} \sum_{s=+1,-1} \Big[e^{(\phi + s\tilde{v}_{\parallel})^2} \left(1 + 2\,\phi^2 + s2\,\phi\,\tilde{v}_{\parallel}\right) Erfc\big(\phi + s\,\tilde{v}_{\parallel}\big)\Big]\Big\} \\ &= -\frac{\sqrt{\pi}}{2} \Big\{ -4\,\phi\,\tilde{v}_{\parallel} \\ &\quad + \sqrt{\pi} \sum_{s=+1,-1} \Big[e^{(\phi + s\tilde{v}_{\parallel})^2}\, s\left(2\,\phi^3 - s\tilde{v}_{\parallel} + 4s\,\phi^2\,\tilde{v}_{\parallel} + 2\,\phi\,\tilde{v}_{\parallel}^2\right) Erfc\big(\phi + s\,\tilde{v}_{\parallel}\big)\Big]\Big\} \\ &\equiv -\tilde{F}'_{\parallel}(\phi, \tilde{v}_{\parallel}) \end{aligned} \tag{8.35}$$

so, finally:

$$\mathcal{F}_{\parallel} = -(1+\mu)\frac{D_0}{\sqrt{\sigma}}\,\lambda \int_0^{\Omega t \to \infty} d\tilde{\tau} \int_0^{\infty} dx\, e^{\lambda^2 \sin^2 \frac{\tilde{\tau}}{2}} \left(1 - \frac{1}{x^2}\right) e^{-\tilde{v}_{\parallel}^2}\, J_0(2\lambda\sqrt{x^2-1}\,\tilde{v}_{\perp}\, \sin\frac{\tilde{\tau}}{2})\, \tilde{F}'_{\parallel} \tag{8.36}$$

Once more, let us remind that the functions $\tilde{F} = \tilde{F}(\phi; \tilde{v}_{\parallel})$ were defined in (8.29); $\phi = \phi(x\,\tau) = \phi(x\,\tilde{\tau})$ in their argument is now given by:

$$\phi = \frac{\lambda}{2}\,x\,\tilde{\tau}\,.$$

8.4.6 Conclusions (same-species case)

The previous few sections refer to the case where the *t.p.* and the field ($R-$) particles considered are of the same species: $\alpha = \alpha'$. This might refer to an electron plasma, with a small population of 'astray' particles moving out of equilibrium and evolving against an immobile (*'frozen'*) background of ions, or simply to the term $Q^{\alpha,\alpha}$ in the evaluation of any quantity of the form: $Q^{\alpha} = \sum_{\alpha'} Q^{\alpha,\alpha'}$.

The result one should retain from them is mainly the set of non-dimensional expressions in the preceding paragraph (i.e. (8.31) and related expressions). This expressions suggests clearly the following conclusions.

1. The relaxation time in this case is scaled as suggested by (8.32) i.e.[22]:

$$\tau_R \sim \frac{n_\alpha\, e_\alpha^4}{m^{1/2}\, T^{3/2}} \tag{8.37}$$

[22] As stated elsewhere in this document, from simple dimensional arguments, a quantity with dimensions of time (the *'relaxation time'*) τ_R is related to D_0 by: $D_0 = v_{th}^{-1}\,\tau_R^{-1}$.

(up to a numerical factor); we thus recover the known expression presented in the bibliography for electrostatic plasma; see e.g. [5], [22][23].

2. As expected, the numerical factor, defined as a double integral in (8.31), is a function of velocity components $v_\perp$, $v_\parallel$, the bath characteristics (temperature T_α, density n_α) *and* the magnitude of the magnetic field $\mathbf{B}$. The influence of the latter intervenes simply through the dimensionless parameters $\lambda \sim \rho_L/r_D$. Studying the behaviour of coefficients versus this parameter (see in the next chapter), we shall see the existence of different regimes, say roughly for ρ_L *below*, *around* and *above* r_D[24].

3. The influence of the magnetic field on the $D^{\alpha\alpha}$ (e.g. *electron-electron* D^{ee}) term depends on the value of the λ parameter. However, from simple qualitative arguments, we see that the λ parameter for ions will be well above its value for electrons, i.e. in a realistic situation: $\lambda_{ee} \approx 1$ but $\lambda_{ii} \gg 1$, implying that the *ion-ion* terms may *not* depend on the field so strongly. This is rather expected from physical intuition, since ions are heavy and their trajectory is hardly curved within a Debye interaction sphere.

8.5 Collisions between different-species: $\alpha \neq \alpha'$

Let us now carry out the calculation presented in the previous section in the general case, when the test-particle is of different species than the reservoir particles: $\alpha \neq \alpha'$. Once more, our starting point will be expressions (8.7); the vector p_i therein was defined in (8.6).

8.5.1 The Fourier-integral(s) $\int d^3\mathbf{k}... = I^{\{\perp,\parallel\}}_{\mathbf{k}^{(3)}}$

The quantity in brackets in (8.7) is given by:

$$-\sigma_j^{\alpha'} {p_j^{\alpha'}}^2/4 + ip_j^\alpha v_j = \left(-\sigma_j^{\alpha'} {p_j^{\alpha'}}^2/4 + ip_j^{\alpha'} v_j\right) + i\left(p_j^\alpha - p_j^{\alpha'}\right) v_j = ... =$$

$$\equiv -\sigma_j^{\alpha'} {q_j^{\alpha'}}^2 - v_j^2/\sigma_j^{\alpha'} + i\tilde{q}_j^{\alpha\alpha'} v_j$$

so that

$$\prod_i \left[e^{-\sigma_i^{\alpha'} {p_i^{\alpha'}}^2/4} e^{ip_i^\alpha v_i} \right] = e^{\sum_j \left[-\sigma_j^{\alpha'} {q_j^{\alpha'}}^2 - v_j^2/\sigma_j^{\alpha'} + i\tilde{q}_j^{\alpha\alpha'} v_j\right]}$$

$$= e^{-\sigma_\perp^{\alpha'} {q_\perp^{\alpha'}}^2 - \sigma_\parallel^{\alpha'} {q_\parallel^{\alpha'}}^2} e^{-v_\perp^2/\sigma_\perp^{\alpha'} - v_\parallel^2/\sigma_\parallel^{\alpha'}} e^{+i\tilde{q}_\perp^{\alpha\alpha'} v_\perp + i\tilde{q}_\parallel^{\alpha\alpha'} v_\parallel}$$

[23] The main difference, of course, is the substitution of the (numerical) Coulomb logarithm therein by the (numerical) quantity defined as a double integral in (8.31).

[24] As more or less expected, and indeed confirmed in the next chapter, in the latter case (low-field-level), one recovers the unmagnetized plasma values.

(assuming that: $\sigma_1 = \sigma_2 = \sigma_\perp$, $\sigma_3 = \sigma_\parallel$) [25] where $q_j^{\alpha'}$ is defined as in (8.8):

$$q_m^{\alpha'}(\mathbf{k};\tau;v) \;=\; p_m^{\alpha'} - i\,\frac{2v_m}{\sigma_m^{\alpha'}} \;\equiv\; \sum_{n=1}^{3} k_n N_{nm}^{\alpha'}(\tau) - i\,\frac{2v_m}{\sigma_m^{\alpha'}} \qquad (8.38)$$

and $\tilde{q}_j^{\alpha\alpha'}$ is defined by:

$$\tilde{q}_j^{\alpha\alpha'} \;=\; p_j^{\alpha} - p_j^{\alpha'}$$

Therefore, the quantity within parenthesis in relation (8.9) (see (8.10)) now becomes:

$$I_{\mathbf{k}^{(3)}}^{\{\perp,\parallel\}} \;\equiv\; \int d\mathbf{k}\; e^{-\sigma_{\alpha'}\, q_{\alpha'}^2/4}\, e^{i\tilde{q}_j^{\alpha\alpha'} v_j}\; \tilde{V}_k^2 \left\{ \begin{array}{c} k_x^2 + k_y^2 \\ k_z^2 \end{array} \right\} \qquad (8.39)$$

(so (8.10) is recovered if $\alpha = \alpha'$). This relation is valid for *any* particular form of $V(r)$. As we have already mentioned, the Fourier transform of the interaction potential does *not* depend on the angle β ($\equiv (\hat{x}, \mathbf{k}_\perp)$) [26], so the corresponding angle integration inside the triple integral $I_{\mathbf{k}^{(3)}}^{\{\perp,\parallel\}}$ $(= \int_0^{+\infty} dk_\perp \int_{-\infty}^{+\infty} dk_\parallel \int_0^{2\pi} d\beta \,...)$ can be performed straightaway, giving a final double integration in $k_\perp$, $k_\parallel$. Following the procedure adopted in §8.4.1 step-by-step, we obtain an equation similar to (8.14) [27], yet with an extra factor $e^{i\tilde{q}_j^{\alpha\alpha'} v_j}$ multiplying the quantity in parenthesis therein:

$$\begin{aligned}
I_{\mathbf{k}^{(3)}}^{(\perp,\parallel)} \;&=\; \cdots \\
&=\; \int_0^\infty dk_\perp \left\{ \begin{array}{c} k_\perp^3 \\ k_\perp \end{array} \right\} \left(\int_0^{2\pi} d\beta\; e^{-\sigma_\perp^{\alpha'}\, {q_\perp^{\alpha'}}^2/4}\, e^{i\tilde{q}_j^{\alpha\alpha'} v_j} \right) \\
&\qquad \left[\int_{-\infty}^{\infty} dk_\parallel\; e^{-\sigma_\parallel^{\alpha'}\, {q_\parallel^{\alpha'}}^2/4}\, \tilde{V}^2_{k\equiv(k_\perp^2+k_\parallel^2)^{1/2}} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \right] \\
&\equiv\; \int_0^\infty dk_\perp \left\{ \begin{array}{c} k_\perp^3 \\ k_\perp \end{array} \right\} I_\beta\; I_{k_\parallel}^{\{\perp,\parallel\}} \qquad (8.40)
\end{aligned}$$

where the angle integral [28] (in parenthesis above) reads:

$$I_\beta = \int_0^{2\pi} d\beta\, e^{-\sigma_\perp^{\alpha'}\, {q_\perp^{\alpha'}}^2/4}\, e^{i\tilde{q}_j^{\alpha\alpha'} v_j}$$

(compare I_α in §8.4.2) and the $\parallel$ $-$integral (in brackets) $I_{k_\parallel}^{\{\perp,\parallel\}}$ is *exactly* as given in (8.14).

[25] Of course:

$$a_\perp b_\perp \equiv a_x b_x + a_y b_y \qquad a_\parallel b_\parallel \equiv a_z b_z \qquad \forall \mathbf{a}, \mathbf{b} \;\in\; \Re^3$$

[26] Remember that, in principle, $V = V(|\mathbf{r}|) = V(r)$ implies that $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$ ($= V(k_\perp^2 + k_\parallel^2)$ in a cylindrical-symmetric problem).

[27] A superscript α' is understood in σ and $q_{\perp,\parallel}$ in brackets therein.

[28] The angle-variable α will be replaced by β here, in order not to be misunderstood for the species symbol.

8.5.2 The β- integration

Let us evaluate the integrand in the angle integration (in parenthesis above). As a matter of fact, (8.11) to (8.13) still hold[29] so:

$$\begin{aligned} q_\perp^{\alpha'\,2} &= 2\,\frac{k_\perp^2}{\Omega_{\alpha'}^2}\,(1-\cos\Omega_{\alpha'}\tau) - 4\,\frac{v_\perp^2}{\sigma_\perp^{\alpha'\,2}} \\ &\quad -i\,\frac{4}{\sigma_\perp^{\alpha'}}\,\frac{k_\perp v_\perp}{\Omega_{\alpha'}}\,s\,\Big[\sin(\theta-\beta)-\sin(\theta-\beta-s\Omega_{\alpha'}\tau)\Big] \\ &= 4\,\frac{k_\perp^2}{\Omega_{\alpha'}^2}\,\sin^2\frac{\Omega_{\alpha'}\tau}{2} - 4\,\frac{v_\perp^2}{\sigma_\perp^{\alpha'\,2}} \\ &\quad -i\,\frac{4}{\sigma_\perp^{\alpha'}}\,\frac{k_\perp v_\perp}{\Omega_{\alpha'}}\,2\,s\,\sin(s\frac{\Omega_{\alpha'}\tau}{2})\cos(\theta-\beta-s\frac{\Omega_{\alpha'}\tau}{2}) \end{aligned} \tag{8.41}$$

Furthermore, we find that:

$$\sum_j \tilde{q}_j^{\alpha\alpha'} v_j = \ldots = \left(p_x^\alpha v_x + p_y^\alpha v_y\right)\,(1-\mathcal{P}_{\alpha\alpha'}) \;\equiv\; \mathbf{q_\perp}^{\alpha\alpha'}\cdot\mathbf{v}_\perp$$

($\mathcal{P}$ is a permutation operator, shifting α to α')[30]; finally:

$$\ldots = \frac{k_\perp v_\perp}{\Omega^\alpha}\,2\,\sin(\frac{\Omega^\alpha\tau}{2})\cos(\theta-\beta-s\frac{\Omega^\alpha\tau}{2})\,(1-\mathcal{P}_{\alpha\alpha'})$$

so that the exponent in the integrand in I_β becomes:

$$-\sigma_\perp^{\alpha'}\,q_\perp^{\alpha'\,2}/4 + i\tilde{q}_j^{\alpha\alpha'}v_j =$$

$$-\sigma_\perp^{\alpha'}\,\frac{k_\perp^2}{\Omega_{\alpha'}^2}\,\sin^2\frac{\Omega_{\alpha'}\tau}{2} + \frac{v_\perp^2}{\sigma_\perp^{\alpha'}} + i\,\frac{k_\perp v_\perp}{\Omega_{\alpha'}}\,2\,s\,\sin(s\frac{\Omega_{\alpha'}\tau}{2})\cos(\theta-\beta-s\frac{\Omega_{\alpha'}\tau}{2}) \tag{8.42}$$

Following the procedure described in §8.4.2, the angle $\beta-$integral finally gives its place to:

$$I_\beta = (2\,\pi)\,e^{-\sigma_\perp^{\alpha'}\,\frac{k_\perp^2}{\Omega_{\alpha'}^2}\,\sin^2\frac{\Omega_{\alpha'}\tau}{2}}\;e^{\frac{v_\perp^2}{\sigma_\perp^{\alpha'}}}\;J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha}\,\sin\frac{\Omega_\alpha\tau}{2}) \tag{8.43}$$

($J_O(x)$ is the zeroth-order Bessel function of the first kind). Comparing to (8.16) we see a separation in role between test- and field- particle species; however, the influence of the former disappears in both infinite- and vanishing- field limits (see the discussion in §8.4.4).

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[29] α' is understood everywhere therein; also replace the angle α by β in (8.13).

[30] The $\|$ $-$ part disappears in this expression, as it does *not* differ from species α to species α'.

8.5.3 Form of the coefficients for an arbitrary potential

Relation(s) (8.17) now give their place to:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \frac{n_{\alpha'}}{m_\alpha^2} (2\pi)^4 e^{-v_\parallel^2/\sigma_\parallel^{\alpha'}} \int_0^t d\tau \int_0^\infty dk_\perp$$

$$\left[\int_{-\infty}^{\infty} dk_\parallel \, k_\parallel^{\{0,2\}} e^{-\sigma_\parallel^{\alpha'} \left(k_\parallel \tau - i \frac{2v_\parallel}{\sigma_\parallel^{\alpha'}} \right)^2 /4} \tilde{V}_k^2 \right]$$

$$k_\perp^{\{3,1\}} \, e^{-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'} \tau}{2}} J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2}) \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_\alpha \tau \\ (-s) \frac{1}{2} \sin \Omega_\alpha \tau \\ 1 + \frac{1}{2} \cos \Omega_\alpha \tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.44}$$

Obviously, m (n) in $\{m, n\}$ correspond to the upper (lower) i.e. $\perp$ $(\|)$ parts respectively; a summation $\sum_{\alpha'=e,i,...}$ is understood.

8.5.4 Debye potential - final form of the coefficients

Following the procedure presented in the previous section, we can evaluate the $k_\parallel-$ integral, by explicitly taking V to be a Debye potential (see definitions in §8.4.4).

Our coefficients are now functions of $\{v_\perp, v_\parallel, t;\, \sigma_\perp^{\alpha'}, \sigma_\parallel^{\alpha'}, \Omega_\alpha, \Omega_{\alpha'}\}$ given by:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} 4 \, \frac{n_{\alpha'}}{m_\alpha^2} \, e_\alpha^2 e_{\alpha'}^2 \int_0^t d\tau$$

$$\int_0^\infty dk_\perp \, e^{-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'} \tau}{2}} J_O(2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2})$$

$$e^{-\hat{v}_\parallel^2} \left(1 - \frac{k_D^2}{k_D^2 + k_\perp^2} \right)^{\{3/2, 1/2\}} \left\{ \begin{array}{c} \left\{ \; \tilde{F}_\perp^{\alpha'} \; \right\} \\ \tilde{F}_\parallel^{\alpha'} \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega_\alpha \tau \\ \frac{-s_\alpha}{2} \sin \Omega_\alpha \tau \\ 1 + \frac{1}{2} \cos \Omega_\alpha \tau \end{array} \right\} \\ 1 \end{array} \right\} \tag{8.45}$$

where the functions $\tilde{F} = \tilde{F}_{\{\perp,\|\}}^{\alpha'}(k_\perp, v_\parallel, \tau;\, \sigma_\parallel^{\alpha'})$ are given by (8.26) i.e.:

$$\tilde{F}_{\{\perp,\|\}}^{\alpha'} = \pm\sqrt{\pi}\,\phi + \frac{\pi}{4} \sum_{s=+1,-1} \left[e^{(\phi + s\,\hat{v}_\parallel)^2} \left(1 \mp 2\,\phi^2 \mp s2\,\phi\,\hat{v}_\parallel \right) Erfc\big(\phi + s\,\hat{v}_\parallel\big) \right] \tag{8.46}$$

The argument variables are:

$$\phi^{\alpha'} = \frac{1}{2}\sqrt{\sigma_{\parallel}^{\alpha'}}\, k_\perp \tau \,, \qquad \hat{v}_{\parallel}^{\alpha'} = \frac{v_\parallel}{\sqrt{\sigma_{\parallel}^{\alpha'}}} \tag{8.47}$$

the upper (lower) signs corresponding to the $\perp$ ($\|$)- parts respectively; $\tilde{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$. $Erfc(x)$ is the complementary error function.

8.6 General case: scaling & discussion

Recall the discussion made in the end of §8.3.1 on our physical system: the *t.p.* (of species α) interacts with background particles (of species α', generally $\neq \alpha$). Also remember that the coefficients concerning the t.p. d.f. f^α are $D_\alpha = \sum_{\alpha'} D_{\alpha\alpha'}$. Since terms $D_{\alpha\alpha'}$ are essentially expressed as a product of a (dimensional) fixed quantity D_0 times a numerical factor (= a double integral, actually a function of parameter λ; see above), one would be interested in examining the relative magnitude between terms, say, $D_{\alpha\beta}$ and $D_{\alpha\gamma}$, trying to draw simple conclusions when e.g. $m_\beta \ll m_\gamma$, or vice versa. In the following we shall suggest a general scaling, and will then give a simple example of its application.

Recall expressions (8.45) and (8.46). The latter bears no dimensions. Let us see how the former can be conveniently re-scaled.

First, the Fourier integration variable: $k_\perp$ can be shifted to the non-dimensional variable:

$$x = \frac{\tilde{k}_\perp}{k_D} = \left(1 + \frac{k_\perp^2}{k_D^2}\right)^{1/2} \tag{8.48}$$

so that: $k_\perp^2 = k_D^2\,(x^2 - 1)^{1/2}$ and $k_\perp\, dk_\perp = k_D^2\, x\, dx$. Remember that the coefficients we are after, will appear in a kinetic equation for the *t.p.* described by f^α. Therefore, the scaling of the velocity components in the argument is rather prescribed:

$$\tilde{v}_\perp = v_\perp / \sqrt{\sigma_\alpha}\,, \qquad \tilde{v}_\parallel = v_\parallel / \sqrt{\sigma_\alpha}$$

Remember that

$$\sigma_\alpha = 2\, k_B\, T_\alpha / m_\alpha = 2\, v_{th,\alpha}^2$$

is related to the thermal velocity i.e. to the temperature T[31]. Now, a dimensionless 'time' may be defined as[32]:

$$\tau' = \Omega_\gamma\, \tau$$

while a dimensionless characteristic parameter λ is given by:

$$\lambda = \sqrt{\sigma'_\alpha}\, \frac{k_D}{\Omega_\delta}$$

[31] T_α denotes the final (equilibrium) temperature to which the t.p. will relax.
[32] Ω_γ is the cyclotron (gyroscopic) frequency: $\Omega_\gamma = e_\gamma\, B / m_\gamma\, c$.

(species γ, δ need not be specified yet). We may now re-scale (8.45) appropriately; the long exponent becomes:

$$-\sigma_\perp^{\alpha'} \frac{k_\perp^2}{\Omega_{\alpha'}^2} \sin^2 \frac{\Omega_{\alpha'}\tau}{2} = -\frac{\sigma_\perp^{\alpha'}}{\sigma_\parallel^{\alpha'}} \frac{\Omega_\delta^2}{\Omega_{\alpha'}^2} \lambda^2 (1-x^2) \frac{\Omega_\delta^2}{\Omega_{\alpha'}^2} \sin^2\left(\frac{\Omega_{\alpha'}}{\Omega_\gamma}\frac{\tau'}{2}\right)$$

the argument in the Bessel function becomes

$$2\frac{k_\perp v_\perp}{\Omega_\alpha} \sin \frac{\Omega_\alpha \tau}{2} = \frac{\sqrt{\sigma_\perp^\alpha}}{\sqrt{\sigma_\parallel^{\alpha'}}} \frac{\Omega_\delta}{\Omega_\alpha} 2\,\lambda\,(x^2-1)^{1/2}\,\tilde{v}_\perp \sin\left(\frac{\Omega_\alpha}{\Omega_\gamma}\frac{\tau'}{2}\right)$$

and the argument in the cosine/sine in the end is modified as: $\Omega_\alpha \tau = \frac{\Omega_\alpha}{\Omega_\gamma} \tau'$. Expression (8.45) now takes the form:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} = \sum_{\alpha'} D_0^{\alpha,\alpha'}\, \mathcal{G}_*^{\alpha,\alpha'}(\tilde{v}_\perp,\, \tilde{v}_\parallel;\, \lambda) \qquad (8.49)$$

(cf. (8.31)). The quantity D_0 reads:

$$D_0 = \left(4\,\frac{n_{\alpha'}\, e_\alpha^2\, e_{\alpha'}^2}{m_\alpha^2}\right) \frac{\Omega_\delta}{\Omega_\gamma} \frac{\sqrt{\sigma_\parallel^\alpha}}{\sqrt{\sigma_\parallel^{\alpha'}}} \qquad (8.50)$$

The dimensionless factor $\mathcal{G}_*$ ($* = \perp,\, \angle,\, XX,\, \parallel$) reads:

$$\begin{aligned} \mathcal{G}_*^{\alpha,\alpha'} = \lambda \int_0^t d\tau \int_0^\infty dk_\perp\, & e^{\frac{\sigma_\perp^{\alpha'}}{\sigma_\parallel^{\alpha'}} \frac{\Omega_\delta^2}{\Omega_{\alpha'}^2} \lambda^2 (x^2-1) \sin^2\left(\frac{\Omega_{\alpha'}}{\Omega_\gamma}\frac{\tau'}{2}\right)} \\ & J_0\left(\frac{\sqrt{\sigma_\perp^\alpha}}{\sqrt{\sigma_\parallel^{\alpha'}}} \frac{\Omega_\delta}{\Omega_\alpha} 2\,\lambda\,(x^2-1)^{1/2}\,\tilde{v}_\perp \sin\left(\frac{\Omega_\alpha}{\Omega_\gamma}\frac{\tau'}{2}\right)\right) \\ & e^{-\hat{v}_\parallel^2} \left(1-\frac{1}{x^2}\right)^{\{3/2,1/2\}} \left\{ \begin{array}{c} \left\{ F_\perp^{\alpha'} \right\} \\ \tilde{F}_\parallel^{\alpha'} \end{array} \right\} \end{aligned} \qquad (8.51)$$

where the variables ϕ, $\hat{v}_\parallel$ in the argument in $F_{\perp,\parallel} = F_{\perp,\parallel}(\phi,\, \hat{v}_\parallel)$ read:

$$\phi = \frac{\sqrt{\sigma_\alpha'}}{2} k_D\, x\, \tau = \frac{1}{2}\frac{\Omega_\delta}{\Omega_\gamma} \lambda\, x\, \tau' \quad , \qquad \hat{v}_\parallel = \frac{v_\parallel}{\sqrt{\sigma_\parallel^{\alpha'}}} = \frac{v_\parallel}{\sqrt{\sigma_\parallel^\alpha}} \frac{\sqrt{\sigma_\parallel^\alpha}}{\sqrt{\sigma_\parallel^{\alpha'}}} .$$

According to each case studied, one may now seek simplified versions of the above relation(s), via arguments about orders of magnitude of one quantity or another.

8.6.1 A simple example

Consider an isothermal plasma consisting of electrons e^- and ions ${}^Z_A X$, of charge $-e$ and $+Z\,e$ and mass m_e and $m_i \approx A\,m_p \equiv A'\,m_e$, respectively[33]. Global electro-neutrality imposes: $n_e = Z\,n_i$. We assume that $T_e/T_i \approx 1$ implying[34]: $v^e_{th}/v^i_{th} \sim A^{1/2} \gg 1$.

Now assume that a few particles (either electrons or ions, not interacting with one another, but only with the plasma) find themselves off equilibrium. In order to study their rates of relaxation back to equilibrium, characterized by $D^\alpha = \sum_{\alpha'} D^{\alpha,\alpha'}$, we are interested in examining the order of magnitude of, say, $D^{\alpha,\beta}$ as compared to $D^{\alpha,\alpha}$, where $\alpha \neq \beta$ is either e or i. Some simple arguments for the $(e-i)$ and $(i-e)$ contributions will be presented in the following paragraphs.

Ion - electron term $(i-e)$**.** Let us take $\alpha = i$ and $\alpha' = e$, so that $m_\alpha/m_{\alpha'} = A \gg 1$, $e_\alpha/e_{\alpha'} = Z$ and $n_\alpha/n_{\alpha'} = n_i/n_e = Z^{-1}$: the *t.p.* is now an energetic heavy ion interacting with surrounding electrons.

Let $\gamma = \delta = \alpha' = e$. Relation (8.50) now becomes:

$$D_0^{ie} = \left(4\,\frac{n_i\,e_i^4}{m_i^2}\right)\frac{n_e}{n_i}\,\frac{e_e^2}{e_i^2}\,\frac{\Omega_e}{\Omega_e}\,\frac{\sqrt{\sigma^i_\parallel}}{\sqrt{\sigma^e_\parallel}} = D_0^{ii}\,\frac{n_e}{n_i}\,\frac{e_e^2}{e_i^2}\,1\,\frac{T^{i\,1/2}_\parallel}{T^{e\,1/2}_\parallel}\,\frac{m_e^{1/2}}{m_i^{1/2}} \sim D_0^{ii}\,Z^{-1}\,A^{-1/2} \tag{8.52}$$

The numerical factor $\mathcal{G}_*$ bears a *finite* value (below 1, see plots in the next chapter). In addition, notice that the argument of the Bessel function inside (8.51) is now of the order of $A^{1/2}\,Z^{-1/2} \gg 1$, implying that J_0 takes values corresponding to a *high* value of $\tilde{v}_\perp$ [35]. As we expect[36] and indeed confirm in the next chapter, $D(v)$ tends to zero for large values of either $v_\perp$ or $v_\parallel$. Combining these two remarks, therefore, we see that:

$$D_0^{ie} \ll D_0^{ii}$$

so that:

$$D_0^i = D_0^{ii} + D_0^{ie} \approx D_0^{ii}$$

We see that the *ion - electron* contribution is negligible, implying that *ion - ion* collisions are more efficient for relaxation.

Electron - ion term $(e-i)$**.** Let us now take $\alpha = e$ and $\alpha' = i$, so that $m_\alpha/m_{\alpha'} = m_e/m_i = A^{-1} \ll 1$, $e_e/e_i = Z^{-1}$; recall that $n_e/n_i = Z$: the *t.p.* is now an electron interacting with heavy ion targets.

[33] Do not forget that $m_p/m_e \approx 1840$, so $m_i/m_e \gg 1$ even for $A = 1$.
[34] Of course, this *is* a somewhat restricting hypothesis, used as an example.
[35] To see this, absorb the scale inside the velocity variable.
[36] from the known previous result in the unmagnetized case.

Let $\gamma = \alpha = e$ and $\delta = \alpha' = i$. Relation (8.50) now scales as:

$$D_0^{ei} \left(4 \, \frac{n_e \, e_e^4}{m_e^2} \right) \frac{n_i}{n_e} \, \frac{e_i^2}{e_e^2} \, \frac{\Omega_i}{\Omega_e} \, \frac{\sqrt{\sigma_{\|}^e}}{\sqrt{\sigma_{\|}^i}} =\sim \; D_0^{ee} \, Z^2 \, A^{-1/2} \tag{8.53}$$

Also, notice that the velocity argument inside $F_{\perp,\|}$ in (8.51) is now of the order of $A^{1/2} \gg 1$, implying that $\mathcal{G}_*$ now attains values corresponding to a *high* value of $\tilde{v}_{\|}$. As discussed above, we expect the numerical value of $\mathcal{G}$ to be very low. Therefore, we see that:

$$D_0^{ei} \ll D_0^{ee}$$

so that:

$$D_0^e = D_0^{ee} + D_0^{ei} \approx D_0^{ee}$$

Once more, we see that the different species term (*electron - ion* contribution) is negligible, implying that *electron - electron* collisions are more efficient for relaxation.

This simple qualitative analysis therefore suggests that the *same-species* terms are more significant than the *different-species* ones, in both $e - i$ and $i - e$ cases[37].

8.7 Conclusion

We have derived a set of new exact formulae for the diffusion coefficients in magnetized plasma. These formulae suggest an explicit dependence on: (a) time, (b) particle velocity coordinates, perpendicular and parallel to the magnetic field, (c) the reservoir temperature and - the point we wanted to focus upon: (d) the magnitude of the magnetic field. The field-dependence is included in a dimensionless parameter λ, roughly equal to the Larmor radius to Debye length ratio. Therefore, the influence of the magnetic field needs to be traced in the order of magnitude of λ.

[37] A heuristic way to see why this is somewhat expected, is to consider the classical picture of a elastic (hard-sphere) head-on collision between a moving test - particle m_1 and a target m_2 (assumed at rest). It is an elementary textbook exercise to see that the percentage of energy loss, say $\Delta E_1 / E_0$ suffered by particle 1 can be found equal to $4\mu/(1+\mu)^2$, where $\mu = m_2/m_1$. Now see that this function tends to zero at both extreme cases $\mu \gg 1$ ($m_2 \gg m_1$) and $\mu \ll 1$ ($m_2 \ll m_1$), while it reaches a maximum for equal masses ($m_2 = m_1$), suggesting that collisions between equal mass particles are more efficient in slowing down the test-particle.

Part C
Analysis of Results

- *That, ..., is the whole truth, pure and simple.*
- *The truth is rarely pure and never simple.*

`Oscar Wilde`
in *The Importance of being Earnest*

Chapter 9

Diffusion coefficients: a parametric study

Summary

The non-dimensional form for the force correlations and the diffusion coefficients are studied numerically. We investigate their dependence on: (a) time, (b) particle velocity coordinates, perpendicular and parallel to the magnetic field and (c) the magnitude of the field. We focus, in particular, on the latter.

Everyone one knows what a curve is,
until he has studied enough mathematics
to become confused through
the countless number of possible exceptions.

Felix Klein

We have previously derived a set of exact *non-dimensional* expressions for the coefficients in the kinetic equation. In the following, we shall briefly summarize those formulae (for the equal-species case: $\alpha = \alpha'$) and then proceed to a numerical study of the coefficients.

9.1 Non-dimensional formulae - summary

The (interaction) force auto-correlations are functions of the normalized time $\tilde{\tau} = \Omega\tau$ given by (8.28) :

$$C_{\{\perp,\parallel\}}\,(\tilde{\tau}, \tilde{v}_\perp, \tilde{v}_\parallel;\, \lambda) \;=\; C_0 \int_1^\infty dx\; e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\tilde{\tau}}{2}} \left(1 - \frac{1}{x^2}\right)^{\{1,0\}} J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2})\, e^{-\tilde{v}_\parallel^2}\,\tilde{F}_{\{\perp,\parallel\}} \qquad (9.1)$$

and the diffusion coefficients are given by (8.31) :

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\parallel \end{array} \right\} =$$

$$\frac{1}{m^2}\,\frac{\lambda}{k_D\sqrt{\sigma}} \int_0^{\Omega t\to\infty} d\tilde{\tau}\; C_{\{\perp,\parallel\}}(\tilde{\tau};\,\tilde{v}_\perp, \tilde{v}_\parallel) \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\tilde{\tau} \\ \frac{-s}{2}\sin\tilde{\tau} \\ 1+\frac{1}{2}\cos\tilde{\tau} \end{array} \right\} \\ 1 \end{array} \right\}$$

$$\equiv\; D_0\,\lambda \int_0^{\Omega t\to\infty} d\tilde{\tau} \int_1^\infty dx\, e^{\lambda^2\,(1-x^2)\,\sin^2\frac{\tilde{\tau}}{2}} \left(1 - \frac{1}{x^2}\right)^{\{1,0\}}$$

$$e^{-\tilde{v}_\parallel^2}\, J_O(2\lambda\,\sqrt{x^2-1}\,\tilde{v}_\perp\,\sin\frac{\tilde{\tau}}{2}) \left\{ \begin{array}{c} \{\ \tilde{F}_\perp\ \} \\ \tilde{F}_\parallel \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2}\cos\tilde{\tau} \\ \frac{-s}{2}\sin\tilde{\tau} \\ 1+\frac{1}{2}\cos\tilde{\tau} \end{array} \right\} \\ 1 \end{array} \right\} \qquad (9.2)$$

Finally, the drift vectors $\mathcal{F}_{\{\perp,\parallel\}}$ are given by (8.33).

The functions $\tilde{F}_{\{\perp,\parallel\}} = \tilde{F}_{\{\perp,\parallel\}}(\phi(x,\tilde{\tau}), \tilde{v}_\parallel)$ in the above expressions are given by:

$$\tilde{F}_{\{\perp,\parallel\}}(\phi, \tilde{v}_\parallel) = \pm\sqrt{\pi}\,\phi + \frac{\pi}{4} \sum_{s=+1,-1} \left[e^{(\phi+s\tilde{v}_\parallel)^2} \left(1 \mp 2\,\phi^2 \mp s2\,\phi\,\tilde{v}_\parallel\right) Erfc\big(\phi + s\,\tilde{v}_\parallel\big) \right] \qquad (9.3)$$

Note the definitions:

$$\phi = \frac{\lambda}{2}\,\tilde{\tau}\,x\,, \qquad \tilde{v}_\perp = v_\perp/\sqrt{\sigma}\,, \qquad \tilde{v}_\parallel = v_\parallel/\sqrt{\sigma}\,, \qquad \sigma = 2\,k_B\,T/m = 2\,v_{th}^2 \qquad (9.4)$$

The field is 'hidden' in the dimensionless parameter λ [1]:

$$\lambda = \sqrt{\sigma}\,\frac{k_D}{\Omega} = \cdots = \sqrt{2}\,\frac{\omega_p}{\Omega} = \sqrt{2}\,\frac{\rho_L}{r_D} \tag{9.5}$$

The physical interpretation of λ is obvious: it expresses the relative magnitude between Larmor radius and Debye length; it may take a wide range of values, depending on the plasma characteristics and the field (remember the discussion in Ch. 2; see fig. 2.4).

The *only* quantities bearing dimensions in the above expressions are:

$$C_0 = 4\,n\,e^4\,k_D\,, \qquad D_0 = \frac{1}{m^2}C_0\frac{\Omega^{-1}}{\lambda} = \frac{1}{m^2}\frac{4\,n\,e^4}{\sqrt{\sigma}} = \frac{2\sqrt{2}\,n\,e^4}{m^{3/2}\sqrt{k_B T}} \tag{9.6}$$

(obviously k_D is the Debye wave-number: $k_D = \left(\frac{4\pi e^2 n}{k_B T}\right)^{1/2}$). The factor D_0 can be identified as v_{th}^2/τ_R, where τ_R is a characteristic *relaxation time*:

$$\tau_R = \frac{\sqrt{2}\,m^{1/2}\,(k_B T)^{3/2}}{n\,(e^2/\epsilon_0)^2} \tag{9.7}$$

(returning to SI units). See that this expression is exactly equal to the inverse of the Coulomb collision frequency (defined in Chapter 2; see (2.3) there)[2]. We therefore recover, up to a numerical factor, to the standard expression for the plasma relaxation time found in plasma textbooks[3]; see e.g. in [5], [22].

The correlations $C(\tilde{\tau}, \tilde{v}_\perp, \tilde{v}_\parallel; \lambda)$ are expressed as a definite integral in x from 1 to ∞. For a given set of parameter values (temperature T, particle density n, mass m, charge e and field magnitude B), one needs to determine the value of λ; the above formulae for $C(\tilde{\tau})$ can then be evaluated as functions of $\tilde{\tau}$ (i.e. $\Omega\tau$).

In the same manner, the diffusion coefficients are functions of time t, particle velocity and λ; they are defined as double integrals (over x and $\tilde{\tau}$) and can be evaluated appropriately.

---

[1] Remember that λ denotes the ratio between the plasma (Langmuir) frequency ω_p:

$$\omega_p = \omega_{p,\alpha} = \left(\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha}\right)^{1/2}$$

and the cyclotron (gyroscopic) frequency Ω:

$$\Omega = \Omega_\alpha = e_\alpha\,B/m_\alpha\,c$$

so λ essentially expresses the relative strength between collisional and magnetic field scales, as $\lambda \approx \frac{T_{gyro}}{T_{coll}}$.

[2] As a matter of fact, this expression can also be predicted from simple dimensional arguments (see in Appendix A; check that A.6, setting $\xi \sim \lambda$, gives exactly $D_0\,\lambda$; cf. (9.2)).

[3] for the *unmagnetized* case, that is; as for the influence of the magnetic field here, it is contained in the appearance of λ everywhere.

9.2 Numerical study - parameter set

For the following we have chosen a temperature of $T = 10\,KeV$ and a particle density of $n = 10^{14}\,cm^{-3} = 10^{20}\,m^{-3}$; these values are typical of fusion plasmas [9], [43], [51] (cf. fig. 2.1).
This choice implies:
- a *plasma frequency* $\omega_{p,e} = 5.64 \cdot 10^{11}\,s^{-1}$

and
- a *cyclotron (gyro-)frequency* of: $\Omega_e = 1.76\,10^{11} \times B \quad s^{-1}$, B being expressed in Tesla.

In terms of length scales, we have:
- a *mean inter-particle distance*: $\langle r \rangle = n^{-1/3} = 2.15 \cdot 10^{-7}\,m$

and
- a *Debye length*: $\lambda_D = v_{th}/\omega_p = 7.43 \cdot 10^{-5}\,m$.

As a consequence, the resulting numerical value of the *plasma parameter* $\mu_p = \left(\frac{4}{3}\pi\lambda_D^3 n\right)^{-2/3}$ [5] is of the order of $3.2 \cdot 10^{-6} \ll 1$, so this is indeed a *weakly coupled* plasma.
Finally, the electron *Larmor radius* is $2.38\,10^{-4} \times B^{-1} \quad m$ (B in Tesla). For a value of, say, $B = 1\,T$, this implies a Larmor radius to Debye length ratio of only 3.2, which suggests a rather non-negligible particle trajectory curvature within the size of a Debye sphere (hence the importance of taking into account the magnetic field in calculating trajectories between collisions); see figure 2.4[4].

The numerical parameter λ defined above is now equal to:

$$\lambda = 4.531 \times B^{-1}$$

(B expressed in Tesla).

9.3 Evolution in time

9.3.1 Correlations $C(\tau)$ vs τ

We have studied the perpendicular correlation function $C_\perp(\tau)$ as a function of τ (measured in gyration periods i.e. $\Omega\tau$), for different values of the magnitude of the magnetic field B ($\sim \Omega$); see the first figure below.

Correlations decrease very fast in time: they are seen to reach, say, 2% of their initial value within a *quarter* of a cyclotron period. The magnetic field seems to enhance correlation, since the higher its magnitude B ($\sim \Omega$), the higher the value of $C_\perp(\tau)$; see figure 9.1. Physically speaking, this fact reflects particle confinement by the magnetic field, since particles 'stick' to their helicoidal trajectory around the magnetic field lines and thus 'feel' each other for longer periods of time. Notice the short peaks appearing every gyration period, actually smoothed out very fast as time goes by. Eventually, particle interactions seem to be completely decorrelated after a few gyration periods.

[4] Qualitatively speaking, we are in case (b) therein.

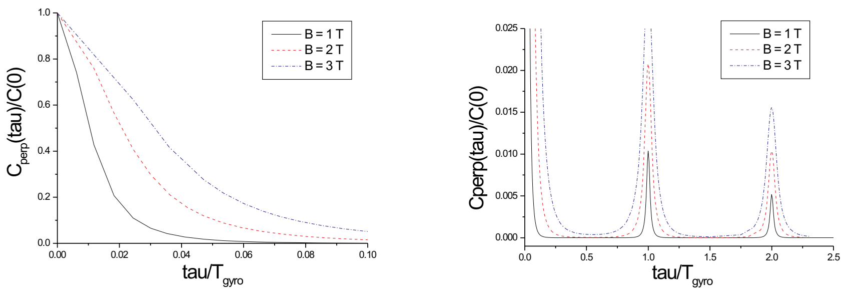

Figure 9.1: The perpendicular force correlation function $C_\perp(\tau; v_\perp, v_\parallel, B)$ (normalized) as a function of time τ (scaled over a cyclotron period T_c). In ascending order, the magnitude of the magnetic field is set to $B = 1, 2, 3 \quad T$ respectively. Both velocity components are taken equal to $v_{th} = (T/m)^{1/2}$. $C_\perp$ can be seen to decrease very fast in time, still bearing a 'tail' of gradually smoothed out peaks every gyration period (actually a signature of the magnetic field; see figure 9.1b).

The same remarks are valid for the parallel auto-correlation function $C_\parallel(\tau)$, which is seen to decay quickly in time (*faster than* $C_\perp(\tau)$, see fig. 9.3). It yields a negative tail which also presents a fine structure of spikes every gyration period (see fig. 9.2) similar to the one in $C_\perp(\tau)$. These spikes disappear almost immediately (they attain values as low as 10^{-6} of the initial value within 2 periods); particles are practically de-correlated in the direction parallel to the field[5].

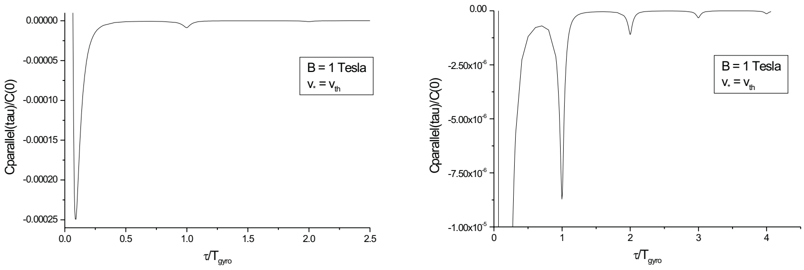

Figure 9.2: $C_\parallel(\tau)$ vs. time τ. Notice the negative spikes (of rather negligible value), which are smeared out after a few gyration periods.

[5]unless close to $\tau = 0$: the situation here is close to the widely used assumption of a $\delta-$correlated - *white noise* - process for interactions.

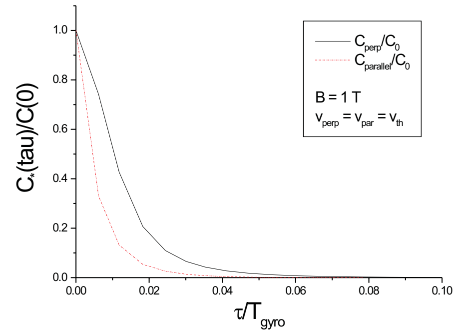

Figure 9.3: $C_\perp(\tau)$ (upper curve) and $C_\parallel(\tau)$ (lower curve) vs. time τ (normalized over a cyclotron period); both decrease very fast in time, practically disappearing after a small fraction of a gyration period.

9.3.2 Coefficients $D_{ij}(t)$ vs t

The evolution in time of the diffusion coefficients is depicted in figure 9.4. It starts from zero and soon evolves towards a final asymptotic value $D_\perp(\infty)$ which remains practically constant after a few gyration periods. Notice the short *'kinks'* every cyclotron period, in fact a consequence of the thin *'spikes'* in the correlations (cf. fig 9.1b).

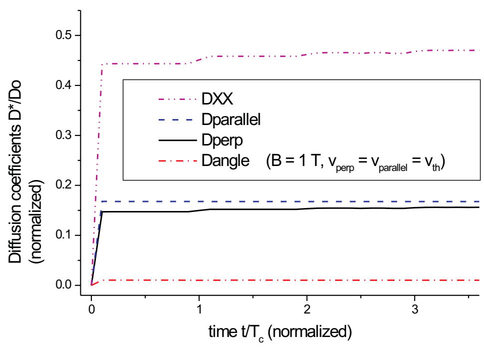

Figure 9.4: Diffusion coefficients $D_{ij}(t)$ vs. time t for $B = 1T$. The small 'kinks' at every gyration (absent in $D_\parallel$) reflect the form of $C_\perp$ (cf. fig. 9.1b). Remember that $D_{XX} = D_\perp + Q$, so Q seems to be practically $\approx 2D_\perp$.

See that $D_\perp(t)$ and $D_\parallel(t)$ are quite close in magnitude. As a matter of fact, every cyclotron period the former increases a little towards its asymptotic value, so that it eventually joins the latter after a few periods.

9.4 Coefficients D_{ij}, F_i vs. velocity

9.4.1 D_{ij}, F_i vs. $v_\perp$

In figures 9.5 to 9.6 we have represented the diffusion coefficients D_* ($* = \perp, \angle, \|, XX$) as functions of the perpendicular component of the velocity $v_\perp$. All coefficients come out to depend on the magnitude of the magnetic field B; nevertheless, this dependence is rather limited in magnitude. Notice the zero-field limit (circle-line).

The (modulus of the) friction vectors $F_\dagger$ ($\dagger = \perp, \|$) [6] is depicted in figures 9.7. The existence of the field seems to slightly *increase* friction.

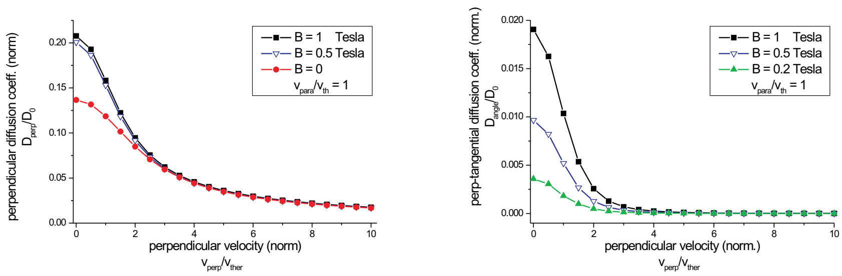

Figure 9.5: (a) $D_\perp$ vs. $v_\perp$. Notice the dependence on the magnitude of the magnetic field B (actually rather weak); compare to figure 9.11; remember that $\lambda = 4.531/B$ (B in Tesla). (b) The perp-tangential diffucion coefficient vs. $v_\perp$. The value depends on B, tending to zero as the field vanishes.

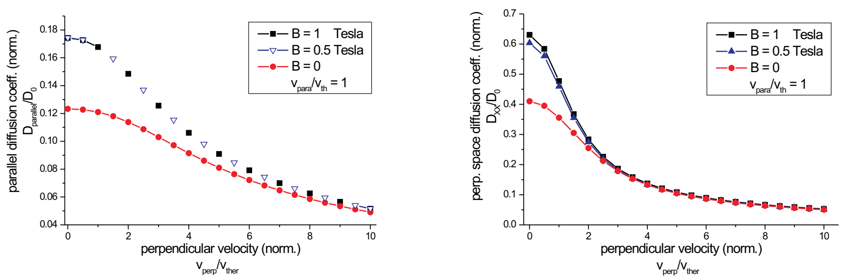

Figure 9.6: (a) $D_\|$ and (b) D_{XX} vs. $v_\perp$ for different values of B. Remember that the value of $D_{XX} = D_\perp + Q$ is multiplied by Ω^{-2} in the kinetic equation, so the zero-field-limit is provided only as an indication (otherwise, the limit is infinite; see discussion in Part B.

[6] Remember that the latter are related to a derivative of the former (essentially the slope of the curve in the appropriate plot).

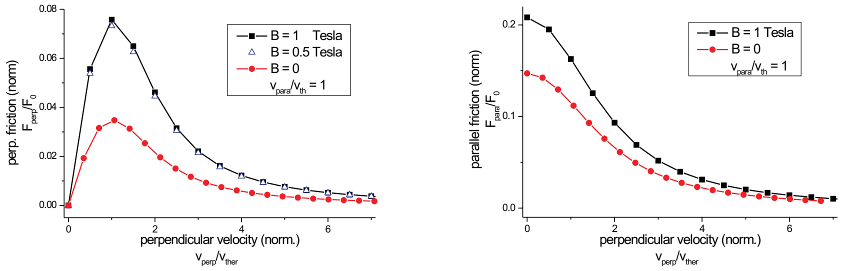

Figure 9.7: $F_\perp$, $F_\parallel$ vs. $v_\perp$. See that values obtained for different values of $B \neq 0$ practically coincide.

9.4.2 D_{ij}, F_i vs. $v_\parallel$

In analogy with the previous paragraph, figures 9.8 to 9.10 represent, respectively, the diffusion coefficients D_* $(* = \perp, \angle, \parallel, XX)$ and the friction vectors $F_\dagger$ $(\dagger = \perp, \parallel)$ as functions of the parallel component of the velocity $v_\parallel$.

Notice the variation of the friction coefficients $\mathcal{F}_\parallel$ with respect to $v_\parallel$; it is similar to that of $\mathcal{F}_\perp$ with respect to $v_\perp$ (compare figures 9.7a and 9.10b): both are qualitatively reminiscent of the unmagnetized case (circle-line)[7]. However, the way $\mathcal{F}_\perp$ varies with respect to $v_\parallel$ (and, independently, $\mathcal{F}_\parallel$ with respect to $v_\perp$) is quite different; compare figures 9.7b and 9.10a.

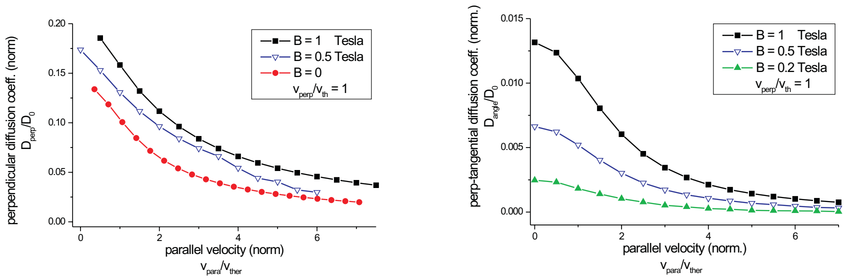

Figure 9.8: (a) The perpendicular and (b) the perp-tangential diffusion coefficient diffusion coefficients vs. the parallel velocity component $v_\parallel$. Notice the dependence on the magnitude of the magnetic field B.

[7] Also see in the Appendix; cf. figure J.3 therein, in particular.

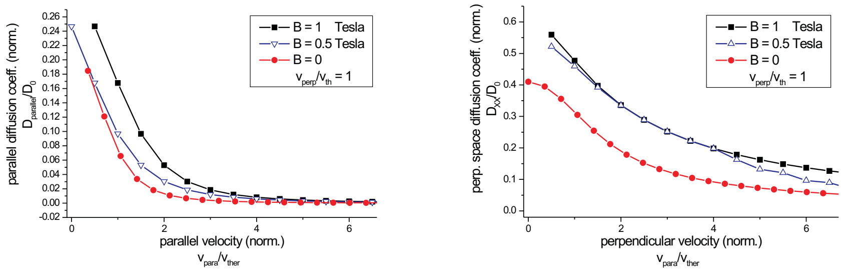

Figure 9.9: $D_{||}$ and D_{XX} vs. $v_{||}$ for different values of B. Remember that the correct value of D_{XX} is multiplied by Ω^{-2} in the kinetic equation, so the zero-field-limit is provided only as an indication (otherwise, the limit is infinite; see discussion in Part B.

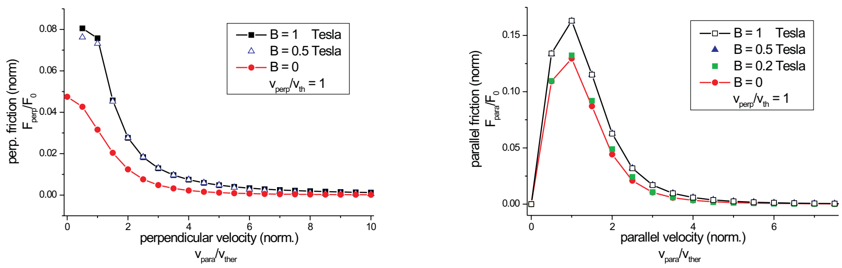

Figure 9.10: $F_{\perp}$, $F_{||}$ vs. $v_{||}$. See that values obtained for different values of B practically coincide.

9.5 Dependence on the magnetic field

A very important remark can be made on figures 9.11 a, b, where we have depicted one of the coefficients, $D_{\perp}$, against the value of the dimensionless parameter λ. In the first plot, we have taken different values of the upper limit of the integration, ranging from a few to a hundred gyration periods, to point out the rapid convergence of the integrand in time t. We clearly distinguish three different behaviours, corresponding to the three different regimes depicted in figure 2.4.

(a) For values of λ above one, i.e. in the region where the cyclotron frequency Ω_c is lower than the plasma frequency ω_p (or, equivalently, the Larmor radius ρ_L is greater than the Debye length r_D), $D_{\perp}$ converges fast (in time) towards the asymptotic value, which is actually very close to the one obtained for $\lambda \to \infty$ (i.e. $\Omega \to 0$): see the dash-dot line. We see that, as expected, the numerical

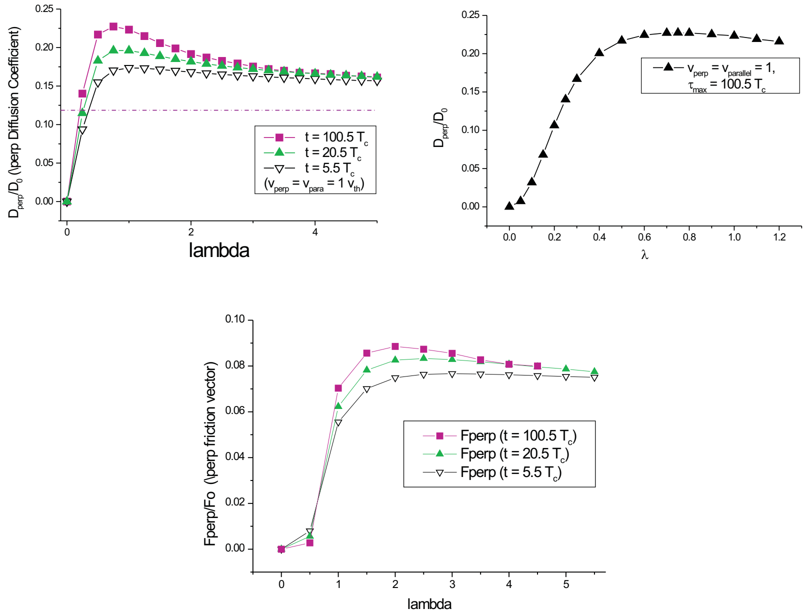

Figure 9.11: The perpendicular diffusion coefficient $D_\perp$ (top: in the right plot, we have focused on the region around $\lambda \approx 1$) and the friction vector (norm) $\mathcal{F}_\perp$ (bottom), plotted against the dimensionless parameter λ ($\sim 1/B$), at different instants of t. $D_\perp$ slightly increases in time, yet only around $\lambda \approx 1$ (i.e. $\rho_L \approx r_D$), above which it practically remains constant. The field-dependence is smoothed out, as $D_\perp$ approaches the asymptotic value for $\Omega \to 0$ (dash-dot line). .

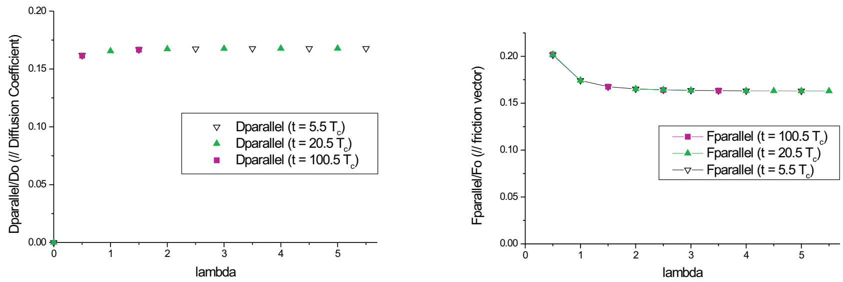

Figure 9.12: $D_\parallel$ comes out to be independent of the field *and* of time and so does $\mathcal{F}_\parallel$.

value of the diffusion coefficient[8] will be practically constant, for high values of

[8]In specific, this applies to the asymptotic value of $D_\perp$, i.e. for high values of time t.

λ (low-field-limit), and quite close to the value obtained in the unmagnetized case (as, more or less expected; see discussion about magnetized plasma regimes in Chapter 2; the situation described here corresponds to regime (a) in figure 2.4).

Notice that $D_\perp$ seems to be *linear* in lambda close to zero, in full agreement with experimental evidence indicating that the diffusion coefficient scales as $1/B$ for strong magnetic fields, as suggested by the Bohm model diffusion coefficient [2] [9].

(b) In the region below 1, where $\Omega_c > \omega_p$ (or $\rho_L < r_D$: strong magnetic fields), the value of the diffusion coefficient seems to depend quite strongly on the value of λ (i.e. the magnetic field)[10].

(c) In the region around $\lambda \approx 1$, where $\Omega_c \approx \omega_p$ (or $\rho_L \approx r_D$), the value of the diffusion coefficient changes (increases) in time, attaining the highest values of all. This fact is in qualitative agreement to our discussion in Chapter 2 (see regime (b) in figure 2.4), where we argued that gyration-related phenomena are expected to be more important when the Larmor radius is comparable to the binary interaction typical scale, e.g. the Debye length.

The friction vector $\mathcal{F}_\perp \sim \partial D_\perp/\partial v_\perp$ behaves in a similar way (fig. 9.11c). However, their $\|-$ counterparts (fig. 9.12a, b) are practically time- (and field-) independent. This is quite expected (and yet, rather not visible in the analytic form of these coefficients), since the magnetic field is not supposed to have a strong influence on phenomena in the direction *parallel* to it.

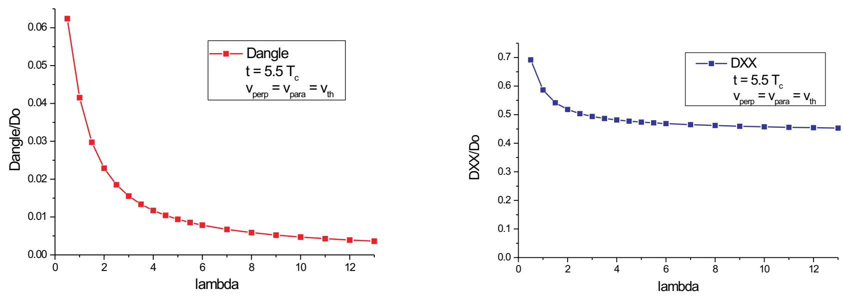

Figure 9.13: The perp-tangential and space- (D_{XX}) diffusion coefficients plotted against λ for $t = 5.5T_c$. Remember that these coefficients are multiplied by Ω^{-1}, Ω^{-2}, respectively, in the non-dimensional form of the FPE (see in A).

[9] See p. 120 in ref. [2].

[10] Remember that a factor λ is incorporated in the dimensionless expressions for $D_\perp/D_0$ - see in the beginning of this chapter - hence the vanishing limit at $\lambda \to 0$.

9.6 Conclusion

The expressions derived for the diffusion coefficients the previous chapter were studied in terms of their physical parameters, namely: (a) time, (b) particle velocity coordinates, perpendicular and parallel to the magnetic field and (c) the magnitude of the magnetic field. In the first two aspects, the qualitative features of the result obtained are rather expected from intuition: correlations are seen to decrease in time and coefficients tend to an asymptotic value attained at infinite time. Furthermore, the qualitative behaviour of all coefficients against velocity is reminiscent of the free-of-field (Landau) case.

We witness a dependence of coefficients on the magnitude of the magnetic field, in the parameter region where particle trajectory is strongly curved within a Debye sphere, in agreement with our qualitative expectations (discussed in Ch. 2).

We see that the magnetic field may *influence* relaxation towards a maxwellian state. This seems to agree with physical intuition: the more 'confined' the particles, the more they influence each other, so the more efficient collisions are towards relaxation. This fact seems to be in agreement with arguments appearing in [16], yet seems to contradict the standard description used in the past, where the influence of the magnetic field on the collision term is either under-estimated [86] or neglected when discussing the physical - transport - properties of plasma [5].

Chapter 10

Approximate analytic solution of the plasma-FPE - a 6-dimensional Ornstein-Uhlenbeck process

Summary

The plasma-Fokker-Planck equation derived previously is analytically solved by considering specific assumptions for the coefficients. These assumptions, which consist in taking diffusion coefficients to be constant (i.e. velocity-independent) and drift terms to be linear (in $\mathbf{v}$), formally define an Ornstein-Uhlenbeck process, for which an exact theory exists. An exact solution for the *pdf* $f(\mathbf{x}, \mathbf{v}; t)$ is obtained and its behaviour in time is discussed. Exact expressions are obtained for displacement and velocity moments.

She went on, "Would you tell me please
which way I ought to go from here?"
"That depends a good deal on where you want to get to" said the Cat.
"I don't much care where-" said Alice.
"Then it doesn't matter which way you go", said the Cat.
"-as long as I get somewhere", Alice added as an explanation.
"Oh, you are sure to do that", said the Cat,
"if you only walk long enough. "

`Lewis Carroll` in *Alice in Wonderland*

10.1 Introduction

Let us consider a special case, in which the FPE (7.1) can be solved exactly. It consists in assuming that:

- the diffusion coefficients D_* are *constant* (i.e. independent from $\mathbf{v}$): $D = const. \in \Re$

and that

- the friction vectors $\mathcal{F}_*$ are *linear* in the velocity coordinate: $\mathcal{F}_*(v) = -\gamma_* v_*$ ('$*$' may denote either x, y, z or $\perp, \|$ throughout this chapter; cf. previous definitions; $\gamma_* > 0$). In this case, the (generally *nonlinear* in its coefficients) FPE reduces to a *linear multivariate F.P. equation*[1]. These assumptions define an *Ornstein-Uhlenbeck process* [15], [42], actually generalized to a multi-dimensional problem.

It should be emphasized that these assumptions are physically plausible; upon simple inspection from figures in the previous chapter[2], we see that they are indeed practically satisfied in the region where the particle velocity is very small, compared to the thermal velocity[3]. This is close to the initial Ornstein-Uhlenbeck picture of a heavy (and slow) particle moving in a medium of light particles.

With these considerations, the FPE [4] takes the form:

$$\frac{\partial f}{\partial t} = - \sum_{i,j=1}^{d} A_{ij} \frac{\partial}{\partial y_i}(y_i\, f) + \sum_{i,j=1}^{d} D_{ij} \frac{\partial^2 f}{\partial y_i \partial y_j} \tag{10.1}$$

where $f = f(\mathbf{y}; t)$; $\mathbf{y} \in \Re^d$ is the position vector $(\mathbf{x}, \mathbf{v})$ in phase space Γ; d is the dimensionality of Γ (e.g. $1, 2, ...$). $\mathbf{A}$ and $\mathbf{D}$ are $d \times d$ (square) matrices. In particular, the *diffusion matrix* $\mathbf{D}$ is symmetric and positive definite (as mentioned before); as a matter of fact, $\mathbf{D}$ is related to the *symmetric part* of the 2nd order matrix defined previously in this text; the contribution of all elements of the anti-symmetric (skew-symmetric) part disappears, given the form of (10.1). *All* coefficients will be assumed to be *real*.

[1] We adopt the terminology used in the book by Van-Kampen [49], where the methodology followed here is exposed, in a formal context; notation will be trivially modified though.

[2] See figs. 9.5a, 9.7a for $D_\perp$, $\mathcal{F}_\perp$ vs. $v_\perp$ respectively; similarly figs. 9.9a, 9.10b for $D_\|$, $\mathcal{D}_\|$ vs. $v_\|$.

[3] As a matter of fact, this hypothesis may be verified analytically by going back to relations (9.2) and expanding the integrand(s) therein in the (dimensionless) velocity components $\tilde{v}_\perp$, $\tilde{v}_\|$. Keeping up to first order terms in $\tilde{v}_*$ ($* = \perp, \|$), one indeed finds that:

$$D_* = D(0) + \mathcal{O}(\tilde{v}_*^2)$$

$$\mathcal{F}_* \sim \frac{\partial D_*}{\partial v_*} = \gamma_* \, v_* + \mathcal{O}(\tilde{v}_*^3)$$

where $D(0)$, γ_* are *independent* of v_*. Note that this is plainly a consequence of the integrand(s) in (9.2) being *even* in v_*.

[4] Under the above assumptions, this remark concerns all equations derived in this thesis, both in general and for plasma, in specific.

10.2 Solution for $f(t)$: general method

We shall follow the formal guidelines given in [49] (see §VIII.6 therein) (and, separately, in [42]). This is a standard method which will only be outlined briefly, for the sake of conciseness, yet limiting details to a minimum, in order not to burden the presentation.

Let us define the characteristic function $G(\mathbf{y};\, t)$:

$$G(\mathbf{k};t) = \frac{1}{(2\pi)^d} \int d\mathbf{y}\ f(\mathbf{y};t)\ e^{-i\mathbf{k}\mathbf{y}(t)} \tag{10.2}$$

This is actually a Fourier transformation of f, which may be reversed as:

$$f(\mathbf{y};t) = \int d\mathbf{k}\ G(\mathbf{k};t)\ e^{i\mathbf{k}\mathbf{y}} \tag{10.3}$$

Note that f is a *real* function $f(\mathbf{y}) \in \Re$, so G is a real function $G(\mathbf{y};t) \in \Re$ itself.

The evolution equation for G reads:

$$\frac{\partial G}{\partial t} = A_{ij}\, k_i\, \frac{\partial G}{\partial k_j} - D_{ij}\, k_i\, k_j\, G \tag{10.4}$$

The characteristic curves for this PDE are determined by [12], [36]:

$$\frac{\partial k_j}{\partial t} = -\sum_{i=1}^{d} k_i\, A_{ij} \tag{10.5}$$

The formal solution reads:

$$\mathbf{k} = e^{-t\,\mathbf{A}^T}\, \mathbf{c} \equiv \mathbf{M}(t)\, \mathbf{c} \tag{10.6}$$

where $\mathbf{c} \in \Re^d$ is a *constant* (real) vector characterizing the state of $\mathbf{k}$ at $t = 0$ [5]; also

$$\mathbf{c} = e^{t\,\mathbf{A}^T}\, \mathbf{k} = \mathbf{M}^{-1}(t)\, \mathbf{k}.$$

Along its characteristics, G obeys:

$$\begin{aligned} \frac{\partial}{\partial t} \ln G &= -\mathbf{k}^T\, \mathbf{D}\, \mathbf{k} \;=\; -\mathbf{c}^T\, e^{-t\,\mathbf{A}}\mathbf{D}\, e^{-t\,\mathbf{A}^T}\, \mathbf{c} \;=\; -\mathbf{c}^T \left[\mathbf{M}^T(t)\, \mathbf{D}\, \mathbf{M}(t)\right] \mathbf{c} \\ &\equiv -\mathbf{c}^T\, \mathcal{N}(t)\, \mathbf{c} \;\equiv\; -\hat{\Lambda}(t;\, \mathbf{c}) \end{aligned} \tag{10.7}$$

where we used (10.6); let us keep in mind the definition of the matrix:

$$\mathcal{N}(t) \;=\; \mathbf{M}^T(t)\, \mathbf{D}\, \mathbf{M}(t) \tag{10.8}$$

[5] Note that the matrix $\mathbf{M}$, via $\mathbf{A}$, may incorporate information related to the external field (if one is present): $\mathbf{M} = \mathbf{M}(t;\, field)$.

The solution for G now reads:

$$\begin{aligned}
G(\mathbf{k},\, t) &= G(\mathbf{k},\, 0)\, \exp\big[-\int_0^t \mathbf{c}^T\, e^{-t'\,\mathbf{A}}\, \mathbf{D}\, e^{-t'\,\mathbf{A}^T}\, \mathbf{c}\, dt'\big] \\
&= G(\mathbf{k},\, 0)\, \exp\big[-\int_0^t \mathbf{k}^T\, e^{(t-t')\,\mathbf{A}}\, \mathbf{D}\, e^{(t-t')\,\mathbf{A}^T}\, \mathbf{k}\, dt'\big] \\
&= G(\mathbf{k},\, 0)\, \exp\big\{-k_i\, \big[\int_0^t N_{ij}(t'-t)\, dt'\big]\, k_j\big\} \\
&\equiv G(\mathbf{k},\, 0)\, \exp\big[-k_i\, \Lambda_{ij}(t)\, k_j\big] \qquad (10.9)
\end{aligned}$$

where we defined:

$$\Lambda_{ij}(t) \;=\; \int_0^t N_{ij}(t'-t)\, dt' \qquad (10.10)$$

Obviously, $\Lambda_{ij}(0) = 0$. The factor $G(\mathbf{k},\, 0)$ is defined as:

$$G(\mathbf{k},\, 0) = \exp[-i\mathbf{k}^T \exp(t\mathbf{A})\, \mathbf{y}'] = \exp[-i\mathbf{k}^T \mathbf{M}^T(-t)\, \mathbf{y}'] \qquad (10.11)$$

for a given *'initial value'* $\mathbf{y}'$ (corresponding to the characteristic curve followed). Combining (10.9), (10.11) we obtain:

$$G(\mathbf{k},\, t) = e^{-k_i\, M_{ji}(-t)\, y'_j}\, e^{-k_i\, \Lambda_{ij}(t)\, k_j} \qquad (10.12)$$

where M_{ij}, Λ_{ij} were defined in (10.6), (10.10) respectively. Finally, having obtained $G(\mathbf{k},\, t)$, we may use (10.3) to revert to $f(\mathbf{y},\, t)$.

10.2.1 En resumé

This is essentially a Green function method. Summarizing, for a given problem one has to follow the following steps:

Step 1. Identify $\mathbf{A}$ and $\mathbf{D}$ (cf. (10.1)).

Step 2. Define $\mathbf{M}(t) = \exp(-t\mathbf{A}^T)$ by solving (10.5):

$$\dot{\mathbf{k}} = -\mathbf{A}^T\mathbf{k} \qquad \Rightarrow \qquad \mathbf{k}(t) = \mathbf{M}\,\mathbf{c}$$

(cf. (10.6)).

Step 3. Define $\mathcal{N}(t)$ as in (10.8):

$$N_{ij}(t) \;=\; M_{li}(t)\, D_{lk}\, M_{kj}(t)$$

and also define $\Lambda(t)$ as in (10.10):

$$\Lambda_{ij}(t) \;=\; \int_0^t N_{ij}(t'-t)\, dt'$$

Step 4. G is now explicitely given by expression (10.12). Once substituted in (10.3) for f it gives, finally:

$$\begin{aligned} f(\mathbf{y};\,t) &= \frac{1}{(2\pi)^d}\int d\mathbf{k}\,e^{i\mathbf{k}\mathbf{y}}\int d\mathbf{y}'\,f_0(\mathbf{y}')\,G(\mathbf{k};\,\mathbf{y}';\,t) \\ &= \frac{1}{(2\pi)^d}\int d\mathbf{k}\,e^{ik_i y_i}\int d\mathbf{y}'\,f_0(\mathbf{y}')\,e^{-k_i\,M_{ji}(-t)\,y'_j}\,e^{-k_i\,\Lambda_{ij}(t)\,k_j} \\ &= \ldots \\ &= \frac{1}{(2\pi)^d}\int d\mathbf{y}'\,f_0(\mathbf{y}')\int d\mathbf{k}\,e^{i\mathbf{k}^T[\mathbf{y}-\mathbf{M}^T(-t)\mathbf{y}']}\,e^{-\mathbf{k}^T\,\mathbf{\Lambda}(t)\mathbf{k}} \end{aligned} \tag{10.13}$$

For the sake of clarity and reference, we will explicitly carry out the complete calculation in the (simplest) case of a one-dimensional problem, in the following section. The method will then be applied, in the section following that one, to our case of interest: magnetized plasma.

10.2.2 Explicit formal solution in 1 dimension

Let us first display the formalism by considering a well-known $1d$ problem ($\Gamma = \{x\}$):

$$\frac{\partial f}{\partial t} = -a\frac{\partial}{\partial x}(x\,f) + d\frac{\partial^2 f}{\partial x^2}$$

Note that, in order for

$$f_{eq}(x) = f(0)\,e^{-\beta_0 x^2} \tag{10.14}$$

($f(0) = \sqrt{\beta_0/\pi}$ for normalization) to be a stationary state (i.e. satisfy: $\partial_t\,f_{eq} = 0$), we need to impose the condition:

$$a = -2\beta_0 d \equiv -\gamma < 0 \tag{10.15}$$

(for $a > 0$ there is no stationary solution).

The procedure elaborated above now gives:

$$\begin{aligned} f(x;\,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\,f_0(x')\int_{-\infty}^{\infty} dk\,e^{ik[x-M(-t)]x'}\,e^{-\Lambda(t)k^2} \\ &= \int dx'\,f_0(x')\,\frac{1}{\sqrt{4\pi\Lambda(t)}}\,e^{-[x-M(-t)x']^2/4\Lambda(t)} \end{aligned} \tag{10.16}$$

(notice that the *positivity* of $f(t)$ is immediately ensured, *provided that* $f(0) > 0$) where[6]:

- $M(t) = e^{-at} = e^{\gamma t}$

and

- $\Lambda(t) = \int_0^t dt'\,d\,M^2(t'-t) = \ldots = \frac{d}{2\gamma}(1 - e^{-2\gamma t})$.

Notice that $\Lambda(t) \to d/2\gamma = 1/4\beta_0 = const.$ for $t \to \infty$.

[6]These expressions coincide with the formulae obtained in [42] pp. 100 - 101.

Now, for a δ-function initial state: $f_0(x') = \delta(x' - x_0)$, one obtains a time-dependent Maxwell distribution:

$$f(x;\, t) = \frac{1}{\sqrt{4\pi \Lambda(t)}}\, e^{-[x - e^{-\gamma t} x_0]^2 / 4\Lambda(t)} \tag{10.17}$$

Notice that both the distribution's width and average (yet *not* normalization!) change in time, as f tends to the asymptotic equilibrium solution (10.14) for $t \to \infty$.

Let us now consider a Maxwellian distribution at $t = 0$: $f_0(x') = \sqrt{\frac{\beta}{\pi}}\, e^{-\beta x^2}$; once more, we obtain a time-dependent Maxwellian of the form:

$$f(x;\, t) = \sqrt{\frac{\beta}{\pi \tilde{\Lambda}(t)}}\, e^{-\beta x^2 / 4\tilde{\Lambda}(t)} \tag{10.18}$$

where:

$$\tilde{\Lambda}(t) = 4\beta\Lambda(t) + e^{-2\gamma t} = \ldots = \frac{\beta}{\beta_0}(1 - e^{-2\gamma t}) + e^{-2\gamma t}\,.$$

Once more, for $t \gg \gamma^{-1}$, the distribution relaxes to the equilibrium state (10.14). See however, that if the initial state was *already* at equilibrium temperature i.e. if $\beta = \beta_0$, then the system will never depart from f_0 (see that $\beta/\tilde{\Lambda} = \beta_0$ in this case)[7].

10.3 Moments of variables $y(t)$: general method of calculation

An alternative 'shortcut' to actually solving the *FPE* is the following[8]. Since it is well known that the solution of the *FPE* (10.1) for an initial condition of the form:

$$f(y, t = 0) = \prod_{i=1}^{d} \delta(y_i - y_{i0}) = \delta^d(\mathbf{y} - \mathbf{y_0})$$

is Gaussian, all moments of y_i are fully determined by the first and second moments. Standard methods allow the calculation of mean values $\langle y_i \rangle$ and and covariance matrix elements $\langle\langle y_k\, y_l \rangle\rangle = \langle y_k\, y_l \rangle - \langle y_k \rangle \langle y_l \rangle \equiv \Xi_{kl}$ *directly* from the coefficients A_{ij}, D_{ij} in (10.1). In brief, the former are given by:

$$\langle \mathbf{y} \rangle_t = e^{t\,\mathbf{A}}\, \mathbf{y_0}$$

while the latter are equal to:

$$\Xi(t) = 2 \int_0^t e^{(t-t')\,\mathbf{A}}\, \mathbf{D}\, e^{(t-t')\,\mathbf{A}^T}$$

[7] A similar result is obtained for a non-zero-mean Maxwellian initial state f_0.

[8] This method is less systematic yet quicker than (yet equally rigorous as) the one presented in the previous paragraph. Once again, the formalism is standard and will not be extensively presented here; details on the method can be found in the bibliography e.g. [49] (see §VIII.6 therein).

Given these formulae, the corresponding Gaussian distribution can be directly constructed[9].

Now see that, according to definitions in the preceding section: $e^{t\,\mathbf{A}^T} \equiv \mathbf{M}(-t)$ so:

$$\langle \mathbf{y} \rangle_t \equiv \mathbf{M}^T(-t)\,\mathbf{y_0} \qquad \textit{and} \qquad \Xi_{ij}(t) \equiv 2\,\Lambda_{ij}(t) \tag{10.19}$$

exactly. Therefore, following steps 1 to 3 described in the previous section, relations (10.19) immediately provide the average and covariances that characterize the collisional mechanism as a Gaussian process, given a sharp peaked ($\delta-$) initial velocity and position distribution.

10.4 Plasma FPE - calculation of mean values (via M_{ij})

Let us now apply the method described in the preceding sections to the plasma-FPE derived in Chapter 7.

We will assume that all diffusion coefficients D_* are constant (i.e. velocity independent) and that the friction vectors $\mathcal{F}_*$ are linear in the velocities v_*:

$$\frac{\partial D_{ij}}{\partial v_k} = 0\,, \qquad \frac{\partial \mathcal{F}_i}{\partial v_i} = -\gamma_\perp\,(\delta_{i1} + \delta_{i2}) - \gamma_\parallel\,\delta_{i3}$$

($i, j, k \in \{1, 2, 3\} = \{x, y, z\}$; $\gamma_* > 0$). Of course, the real constants D_*, γ_* may depend on other parameters e.g. the magnetic field, in specific.

Before we start, let us ensure the existence of an equilibrium solution. We anticipate the existence of an equilibrium state in the form:

$$f_{eq}(\mathbf{v}) = f_{eq}(\mathbf{0})\, e^{-\sum \beta_0^{(i)} v_i^2} = f_{eq}(\mathbf{0})\, e^{-\beta_0^\perp v_\perp^2}\, e^{-\beta_0^\parallel v_\parallel^2}$$

where $\beta_0^* = m/2T_*^0$ ($* = \perp, \parallel$) (we have considered two different temperatures, for generality); as we know, normalization ($\int d\mathbf{v} f = 1$) implies: $f_{eq}(\mathbf{0}) = \prod_{i=1}^3 \sqrt{\beta_0^{(i)}/\pi} = \beta_0^\perp \sqrt{\beta_0^\parallel}/\pi^{3/2}$. Substituting in the FPE, we obtain the equilibrium condition:

$$\gamma_* = 2\beta_0^* D_* \qquad (* = \perp, \parallel) \tag{10.20}$$

which defines the equilibrium plasma temperature(s) $T_*^0 = mD_*/\gamma*$. Now, let us proceed, as in the previous section, to the solution for $f(t)$.

Step 1: Coefficients. As one may easily verify, given the above assumptions, the FPE equation (7.4) can indeed be cast into the general form (10.1) in the 6-dimensional phase space $\{\mathbf{x}, \mathbf{v}\}$. The 6×6 diffusion matrix $\mathbf{D}$ then reads:

---

[9] i.e.

$$f(y;\,t) = (2\pi)^{-1/2}\,(\det \Xi)^{-1/2}\,\exp[-\frac{1}{2}\,(\mathbf{y} - \langle \mathbf{y} \rangle)^T\,\Xi^{-1}\,(\mathbf{y} - \langle \mathbf{y} \rangle)]$$

[49]. Yet strictly correct, this calculation is too lengthy in a multi-dimensional case like ours, and will rather be avoided here.

$$
\mathbf{D} = \begin{pmatrix}
\Omega^{-2}(D_\perp + Q) & 0 & 0 & 0 & -s\,\Omega^{-1}D_\perp & 0 \\
0 & \Omega^{-2}(D_\perp + Q) & 0 & s\,\Omega^{-1}D_\perp & 0 & 0 \\
0 & 0 & D_\parallel^{(XX)} & 0 & 0 & D_\parallel^{(VX)}/2 \\
0 & s\,\Omega^{-1}D_\perp & 0 & D_\perp & 0 & 0 \\
-s\,\Omega^{-1}D_\perp & 0 & 0 & 0 & D_\perp & 0 \\
0 & 0 & D_\parallel^{(VX)}/2 & 0 & 0 & D_\parallel
\end{pmatrix}
= \begin{pmatrix} \mathbf{D_{XX}} & \mathbf{D_{VX}}^T \\ \mathbf{D_{VX}} & \mathbf{D_{VV}} \end{pmatrix}
\tag{10.21}
$$

where the definition of the (three) 3×3 matrices $\mathbf{D_{VV}}$, $\mathbf{D_{VX}}$ and $\mathbf{D_{XX}}$ is obvious. The third line and the third column (related to the $z-$ direction) - provided here for completeness - are omitted, according to our previous considerations; see comment 4 in §7.2.1. The $\mathbf{A}$ matrix is:

$$
\mathbf{A} = \begin{pmatrix}
0 & 0 & 0 & 1 & s\Omega^{-1}\gamma_\perp & 0 \\
0 & 0 & 0 & -s\Omega^{-1}\gamma_\perp & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\gamma_\perp & s\Omega & 0 \\
0 & 0 & 0 & -s\Omega & -\gamma_\perp & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma_\parallel
\end{pmatrix}
= \begin{pmatrix} \mathbf{0} & \mathbf{A_{VX}} \\ \mathbf{0} & \mathbf{A_{VV}} \end{pmatrix}
\tag{10.22}
$$

Notice that the flow term including the external (Lorentz) force[10] has been incorporated in the $45-$, $54-$ elements of $\mathbf{A}$.

Strategy. Let us now define our target when carrying out the lengthy calculation that will follow. We aim in comparing (the results obtained in) three distinct cases, in terms of the form of FP equation derived above.

1st) **Case I -** the $3d-$ FPE obtained in the spatially homogeneous case: $\mathbf{D}$, $\mathbf{A}$ simply reduce to the 3×3 matrices $\mathbf{D_{VV}}$, $\mathbf{A_{VV}}$, respectively.

[10]i.e.

$$
\frac{e}{mc}\,(\mathbf{v} \times \mathbf{B})\,\frac{\partial f}{\partial \mathbf{v}} = s\,\Omega\,\epsilon_{ijk}\,v_j\,B_k\,\delta_{k3}\,\frac{\partial}{\partial v_i}\,f = s\,\Omega\left(v_x\,\frac{\partial}{\partial v_y} - v_y\,\frac{\partial}{\partial v_x}\right) f
$$

see e.g. the *lhs* in (7.1).

2nd) **Case II -** the full $6d-$ FPE obtained in the general case: $\mathbf{D}$, $\mathbf{A}$ are just as defined above.

3rd) **Case III -** the 'reduced' $6d-$ FPE obtained by neglecting space inhomogeneity terms in the collision term (but keeping the flow term - $\mathbf{v}\nabla f$ - in the *lhs*, as often done '*in practice*'): the corresponding matrices are obtained by formally cancelling terms containing Ω^{-1}, Ω^{-2} (only!) wherever it appears in the formulae for $\mathbf{D}$, $\mathbf{A}$ defined above. The matrices then read:

$$\mathbf{D}^{(III)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & D_\perp & 0 \\ 0 & 0 & 0 & 0 & 0 & D_\parallel \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D_{VV}} \end{pmatrix} \tag{10.23}$$

and

$$\mathbf{A}^{(III)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\gamma_\perp & s\Omega & 0 \\ 0 & 0 & 0 & -s\Omega & -\gamma_\perp & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma_\parallel \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{A_{VV}} \end{pmatrix} \tag{10.24}$$

Step 2: Definition of M. We have to solve (10.5):

$$\dot{\mathbf{k}} = -\mathbf{A}^T\mathbf{k} \qquad \Rightarrow \qquad \mathbf{k}(t) = \mathbf{M}(t)\,\mathbf{c}$$

(cf. (10.6)) in order to define the matrix $\mathbf{M}$ for our problem.
Let $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6) \in \Re^6$; (10.5) corresponds to a system of 6 *ODE*s:

$$\begin{aligned} \dot{k}_1 &= \dot{k}_2 = \dot{k}_3 = 0 \\ \dot{k}_4 &= -k_1 + s\Omega^{-1}\gamma_\perp k_2 + \gamma_\perp k_4 + s\Omega k_5 \\ \dot{k}_5 &= -k_2 - s\Omega^{-1}\gamma_\perp k_1 - s\Omega k_4 + \gamma_\perp k_5 \\ \dot{k}_6 &= +\gamma_\parallel k_6 - k_3 \end{aligned} \tag{10.25}$$

This is the system obtained in the general case (II). By omitting the second term in the *rhs* of the second and third lines ($\pm s\Omega^{-1}\gamma_\perp k_{2,1} \to 0$) we would obtain case (III); otherwise, plainly by setting $k_1 = k_2 = k_3 = 0$, we obtain the homogeneous case (I).

Let us assume a known initial condition: $k_i(t=0) = c_i, \quad (i = 1, 2, ..., 6)$. The first three equations readily give:

$$k_i(t) = c_i = const. \quad \forall t, \qquad (i = 1, 2, 3)$$

However, the remaining (4th to 6th) equations have to be solved separately, in detail, in cases I to III defined above.

In Case I, the system of the 4th to 6th equations reduces to:

$$\begin{aligned}\dot{k}_4 &= \gamma_\perp k_4 + s\Omega k_5\\ \dot{k}_5 &= -s\Omega k_4 + \gamma_\perp k_5\\ \dot{k}_6 &= +\gamma_\| k_6\end{aligned} \tag{10.26}$$

which has the solution:

$$\begin{aligned}k_4(t) &= e^{\gamma_\perp t}\big(k_4 \cos\Omega t + k_5 s \sin\Omega t\big)\\ k_5(t) &= e^{\gamma_\perp t}\big(-k_4 s \sin\Omega t + k_5 \cos\Omega t\big)\\ k_6(t) &= c_6 e^{\gamma_\| t}\end{aligned} \tag{10.27}$$

defining the matrix (see on top of this paragraph):

$$\mathbf{M_{VV}}(t) = \begin{pmatrix} e^{\gamma_\perp t}\cos\Omega t & e^{\gamma_\perp t} s\sin\Omega t & 0\\ -e^{\gamma_\perp t} s\sin\Omega t & e^{\gamma_\perp t}\cos\Omega t & 0\\ 0 & 0 & e^{\gamma_\| t}\end{pmatrix} \tag{10.28}$$

In (the general) Case II, the (complete) system (10.25) has the solution:

$$\begin{pmatrix} k_1(t)\\ k_2(t)\\ k_3(t)\\ k_4(t)\\ k_5(t)\\ k_6(t)\end{pmatrix} = \left(\begin{matrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ -\Omega^{-1}e^{\gamma_\perp t}\sin\Omega t & -s\Omega^{-1}(1-e^{\gamma_\perp t}\cos\Omega t) & 0\\ s\Omega^{-1}(1-e^{\gamma_\perp t}\cos\Omega t) & -\Omega^{-1}e^{\gamma_\perp t}s\sin\Omega t & 0\\ 0 & 0 & \frac{1}{\gamma_\|}(1-e^{\gamma_\| t})\end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ e^{\gamma_\perp t}\cos\Omega t & e^{\gamma_\perp t}s\sin\Omega t & 0\\ -e^{\gamma_\perp t}s\sin\Omega t & e^{\gamma_\perp t}\cos\Omega t & 0\\ 0 & 0 & e^{\gamma_\| t}\end{matrix}\right)\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\\ c_5\\ c_6\end{pmatrix} \tag{10.29}$$

defining therefore the 6×6 evolution matrix:

$$\mathbf{M}^{(II)}(t) = \begin{pmatrix} \mathbf{I} & \mathbf{0}\\ \mathbf{M_{VX}^{(II)}} & \mathbf{M_{VV}}\end{pmatrix} \tag{10.30}$$

where the definitions of all 3×3 matrices in it are obvious.

In Case III, we have the reduced system:

$$\begin{aligned}\dot{k}_1 &= \dot{k}_2 = \dot{k}_3 = 0\\ \dot{k}_4 &= -k_1 + \gamma_\perp k_4 + s\Omega k_5\\ \dot{k}_5 &= -k_2 - s\Omega k_4 + \gamma_\perp k_5\\ \dot{k}_6 &= +\gamma_\| k_6 - k_3\end{aligned} \tag{10.31}$$

which has a solution of a similar form:

$$\mathbf{M}^{(III)}(t) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{M}_{\mathbf{VX}}^{(\mathbf{III})} & \mathbf{M}_{\mathbf{VV}} \end{pmatrix} \tag{10.32}$$

that is, only modifying ${\mathbf{M}_{\mathbf{VX}}}^{(III)}$:

$$\begin{aligned} &{\mathbf{M}_{\mathbf{VX}}}^{(III)}(t) = \\ &\frac{1}{\Omega^2+\gamma_\perp^2}\begin{pmatrix} \gamma_\perp(1-e^{\gamma_\perp t}\cos\Omega t)-\Omega e^{\gamma_\perp t}\sin\Omega t & -s\Omega(1-e^{\gamma_\perp t}\cos\Omega t)-s\gamma_\perp e^{\gamma_\perp t}\sin\Omega t & 0 \\ s\Omega(1-e^{\gamma_\perp t}\cos\Omega t)+s\gamma_\perp e^{\gamma_\perp t}\sin\Omega t & \gamma_\perp(1-e^{\gamma_\perp t}\cos\Omega t)-\Omega e^{\gamma_\perp t}\sin\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+\frac{1-e^{\gamma_\parallel t}}{\gamma_\parallel}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{10.33}$$

Notice that, in both cases II and III,

- $\mathbf{M} \to \mathbf{I}$ for $t \to 0$,
- $Det\mathbf{M} = Det\mathbf{M}_{\mathbf{VV}} = e^{2\gamma_\perp t}\, e^{\gamma_\parallel t}$;

also,

- $Det{\mathbf{M}_{\mathbf{VX}}}^{(II)} = Det{\mathbf{M}_{\mathbf{VX}}}^{(III)} = \frac{1-e^{\gamma_\perp t}\cos 2\Omega t}{\Omega^2}\,\frac{1-e^{\gamma_\parallel t}}{\gamma_\parallel}$.

Step 3: given the complexity of the calculation, we will devote a separate section to each case, in the following.

10.5 Plasma FPE - Λ_{ij} and covariance matrix

10.5.1 Case I - homogeneous plasma

We shall now focus on the homogeneous case, where the phase space is 3-dimensional: $\Gamma = \{v_x, v_y, v_z\}$, so $\mathbf{A}$, $\mathbf{D}$ reduce to the 3×3 matrices $\mathbf{A}_{(\mathbf{VV})}$, $\mathbf{D}_{(\mathbf{VV})}$ defined in (10.21), (10.22) above. Let us calculate the matrices $\mathcal{N}_{ij}(t)$ and $\Lambda_{ij}(t)$, as required in Step 3, in the methodology described above.

Step 3: Matrices. The matrix $\mathcal{N}(t)$ was defined in (10.8); substituting the matrix $\mathbf{M}_{\mathbf{VV}}(t)$ from (10.28), the calculation leads to the diagonal form:

$$\mathcal{N}(t) = \begin{pmatrix} e^{2\gamma_\perp t}D_\perp & 0 & 0 \\ 0 & e^{2\gamma_\perp t}D_\perp & 0 \\ 0 & 0 & e^{2\gamma_\parallel t}D_\parallel \end{pmatrix} \tag{10.34}$$

Notice that the influence of Ω disappears!

Substituting this result in (10.10) for the matrix $\Lambda(t)$; we obtain:

$$\Lambda(t) = \int_0^t dt' \mathcal{N}(t'-t) = \frac{D_\perp}{2\gamma_\perp}(1 - e^{-2\gamma_\perp t}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{D_\parallel}{2\gamma_\parallel}(1 - e^{-2\gamma_\parallel t}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.35}$$

Remember that $D_*/2\gamma_* = 1/4\beta_*^0 = T_*^0/2m_*$ $(* = \perp, \parallel)$ (T_*^0 is the equilibrium temperature) due to condition (10.20). We shall henceforth define:

$$\lambda_*(t) \equiv \frac{1}{2\gamma_*}(1 - e^{-2\gamma_* t}) \qquad * = \perp, \parallel \tag{10.36}$$

We will also later need the product: $\mathbf{M}^T(-t)\mathbf{v}'$ (see Step 4 above); we have:

$$\begin{aligned} \mathbf{M}^T(-t)\mathbf{v}' &= \begin{pmatrix} e^{-\gamma_\perp t}\cos\Omega t & s e^{-\gamma_\perp t}\sin\Omega t & 0 \\ -s e^{-\gamma_\perp t}\sin\Omega t & e^{-\gamma_\perp t}\cos\Omega t & 0 \\ 0 & 0 & e^{-\gamma_\parallel t} \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} \\ &= \begin{pmatrix} v'_x e^{-\gamma_\perp t}\cos\Omega t + v'_y s e^{-\gamma_\perp t}\sin\Omega t \\ -s v'_x e^{-\gamma_\perp t}\sin\Omega t + v'_y e^{-\gamma_\perp t}\cos\Omega t \\ v'_z e^{-\gamma_\parallel t} \end{pmatrix} \end{aligned} \tag{10.37}$$

Setting $(v'_x, v'_y, v'_z) = (v'_\perp \cos\alpha', v'_\perp \sin\alpha', v'_\parallel)$, this vector takes the elegant form:

$$\mathbf{M}^T(-t)\mathbf{v}' = \begin{pmatrix} v'_\perp\, e^{-\gamma_\perp t}\cos(\alpha' - s\Omega t) \\ v'_\perp\, e^{-\gamma_\perp t}\sin(\alpha' - s\Omega t) \\ v'_\parallel\, e^{-\gamma_\parallel t} \end{pmatrix} \tag{10.38}$$

10.5.2 Case II - inhomogeneous plasma/complete description : $\mathcal{K} = \mathcal{K}^\Phi\{f(\mathbf{x}, \mathbf{v}; t)\}$

Let us now consider the general (inhomogeneous) Case II. The phase space is now 6-dimensional: $\Gamma = \{x, y, z; v_x, v_y, v_z\}$, so $\mathbf{A}$, $\mathbf{D}$ are the 6×6 matrices defined in (10.21), (10.22) above. Let us now calculate the matrices $\mathcal{N}_{ij}(t)$ and $\Lambda_{ij}(t)$, as required in Step 3, in the methodology described above (compare to the calculation in the previous section).

Step 3: Matrices. The matrix $\mathcal{N}(t)$ was defined in (10.8); substituting the 6×6 matrix $\mathbf{M}(t)$ from (10.30), the lengthy calculation leads to the form:

$$\mathcal{N}^{(II)}(t) = \begin{pmatrix} \mathbf{N}_{\mathbf{XX}}^{(\mathbf{II})} & \mathbf{N}_{\mathbf{VX}}^{(\mathbf{II})\,T} \\ \mathbf{N}_{\mathbf{VX}}^{(\mathbf{II})} & \mathbf{N}_{\mathbf{VV}} \end{pmatrix} \tag{10.39}$$

where

$$\begin{aligned}
\mathbf{N_{XX}^{(II)}} &= \Omega^{-2}(Q + D_\perp e^{2\gamma_\perp t})\,\mathbf{1}_\perp \,+\, D_\parallel \left(\frac{1-e^{\gamma_\parallel t}}{\gamma_\parallel}\right)^2 \mathbf{1}_\parallel \\
\mathbf{N_{VX}^{(II)}} &= s\Omega^{-1} e^{2\gamma_\perp t}\, D_\perp\, \mathbf{1}_\angle \,+ e^{\gamma_\parallel t}\, \frac{1-e^{\gamma_\parallel t}}{\gamma_\parallel}\, D_\parallel\, \mathbf{1}_\parallel \\
\mathbf{N_{VV}} &= e^{2\gamma_\perp t}\, D_\perp\, \mathbf{1}_\perp \,+ e^{2\gamma_\parallel t}\, D_\parallel\, \mathbf{1}_\parallel
\end{aligned} \tag{10.40}$$

Remember the definitions:

$$\begin{aligned}
\mathbf{1}_\perp &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 \\
\mathbf{1}_\angle &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{e}_1\hat{e}_2 - \hat{e}_2\hat{e}_1 \\
\mathbf{1}_\parallel &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{e}_3\hat{e}_3
\end{aligned}$$

Substituting this result in (10.10) for the matrix $\mathbf{\Lambda}(t)$; we obtain:

$$\mathbf{\Lambda}^{(II)}(t) = \begin{pmatrix} \mathbf{\Lambda_{XX}^{(II)}} & \mathbf{\Lambda_{VX}^{(II)}}^T \\ \mathbf{\Lambda_{VX}^{(II)}} & \mathbf{\Lambda_{VV}} \end{pmatrix} \tag{10.41}$$

where

$$\begin{aligned}
\mathbf{\Lambda_{XX}^{(II)}} &= \Omega^{-2}(\lambda_\perp D_\perp + Qt)\,\mathbf{1}_\perp \,+\, \frac{D_\parallel}{2\gamma_\parallel^3}\left(-3 - e^{-2\gamma_\parallel t} + 4\,e^{-\gamma_\parallel t} + 2\gamma_\parallel t\right)\mathbf{1}_\parallel \\
\mathbf{\Lambda_{VX}^{(II)}} &= s\Omega^{-1}\lambda_\perp\, D_\perp\, \mathbf{1}_\angle \,+\, \frac{D_\parallel}{2}\left(\frac{1-e^{-\gamma_\parallel t}}{\gamma_\parallel}\right)^2 \mathbf{1}_\parallel \\
\mathbf{\Lambda_{VV}} &= \lambda_\perp\, D_\perp\, \mathbf{1}_\perp \,+ \lambda_\parallel\, D_\parallel\, \mathbf{1}_\parallel
\end{aligned} \tag{10.42}$$

Remember that $\lambda_* = \lambda_*(t)$ were defined in (10.36). Consequently,

$$\begin{aligned}
\mathbf{k}\,\mathbf{\Lambda}(t)\,\mathbf{k} &= \mathbf{k_x}\,\mathbf{\Lambda_{xx}^{(II)}}(t)\,\mathbf{k_x} + 2\,\mathbf{k_v}\,\mathbf{\Lambda_{vx}^{(II)}}(t)\,\mathbf{k_x} + \mathbf{k_v}\,\mathbf{\Lambda_{vv}^{(II)}}(t)\,\mathbf{k_v} \\
&= \lambda_\perp\, D_\perp\,(k_4^2 + k_5^2) \,+ \lambda_\parallel\, D_\parallel\, k_6^2 \\
&\quad + 2\,s\,\Omega^{-1}\lambda_\perp\, D_\perp\,(k_2k_4 - k_1k_5) \,+\, \left(\frac{1-e^{-\gamma_\parallel t}}{\gamma_\parallel}\right)^2 D_\parallel\, k_3k_6 \\
&\quad + \Omega^{-2}(\lambda_\perp D_\perp + Qt)\,(k_1^2 + k_2^2) +\, \gamma_\parallel^{-2}\lambda_\parallel^{(X)}\, D_\parallel\, k_3^2
\end{aligned} \tag{10.43}$$

where we defined:

$$\mathbf{k_x} = (k_1, k_2, k_3), \qquad \mathbf{k_v} = (k_4, k_5, k_6)$$

and

$$\lambda_{\parallel}^{(X)}(t) = \frac{1}{2\gamma_{\parallel}}\left(-3 - e^{-2\gamma_{\parallel} t} + 4\,e^{-\gamma_{\parallel}\,t} + 2\gamma_{\parallel} t\right) \tag{10.44}$$

for brevity. Check that, by setting $k_1 = k_2 = k_3 = 0$, one recovers immediately the homologous relation from Case I (previous section); the similar test should be made in the following, as well, for confirmation.

10.6 Plasma FPE - Final results for mean values $\langle y_i \rangle$

Let us re-capitulate the results of §10.4 and then proceed by computing mean values of x_i and v_i. For convenience, we shall consider the Larmor frequency *inside the collision term* as Ω', i.e. setting $\Omega' = 0$ in (10.21) (Case II) gives directly (10.24) (Case III). Therefore, keep in mind the definition:

$$\zeta = 1 - \Omega\,{\Omega'}^{-1} \tag{10.45}$$

which will help us switch from Case II ($\zeta = 0$) to Case III ($\zeta = 1$) in the formulae that will follow.

Recall the results for $\mathbf{M}(t)$ in §10.4:

$$\mathbf{M}^{(II)}(t) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{M_{VX}^{(II)}} & \mathbf{M_{VV}} \end{pmatrix}$$

(see (10.30); remember that $\mathbf{M_{VV}}$ is the same for all cases I - III (see (10.28), while $\mathbf{M_{VX}}(t)$ is given by:

$$\mathbf{M_{VX}}(t) = \begin{pmatrix} M_{41}(t) & M_{42}(t) & 0 \\ -M_{42}(t) & M_{41}(t) & 0 \\ 0 & 0 & M_{63}(t) \end{pmatrix} \tag{10.46}$$

where[11]

$$\begin{aligned} M_{41}(t) &= \frac{1}{\Omega^2 + \gamma_\perp^2}\,\zeta\,[\gamma_\perp(1 - e^{\gamma_\perp t}\cos\Omega t) + e^{\gamma_\perp t}\sin\Omega t\,\gamma_\perp^2/\Omega] \\ &\qquad -s\,\Omega^{-1}\,e^{\gamma_\perp t}\sin\Omega t \\ M_{42}(t) &= \frac{1}{\Omega^2 + \gamma_\perp^2} \times \zeta\,[-\gamma_\perp e^{\gamma_\perp t}\,s\sin\Omega t + s\,(1 - e^{\gamma_\perp t}\cos\Omega t)\,\gamma_\perp^2/\Omega] \\ &\qquad -s\,\Omega^{-1}\,(1 - e^{\gamma_\perp t}\cos\Omega t) \\ M_{63}(t) &= \frac{1 - e^{\gamma_\parallel t}}{\gamma_\parallel} \end{aligned}$$

[11] Obviously, we have "merged" (10.30) and (10.33) (recovered for $\zeta = 0$ and $\zeta = 1$ respectively).

Now, recalling (10.19) for $\langle y_i \rangle_t$, we have the result:

$$\begin{aligned}
\langle v_x \rangle_t &= e^{-\gamma_\perp t} \cos \Omega t \, v_x^0 \; + \; e^{-\gamma_\perp t} s \sin \Omega t \, v_y^0 \\
\langle v_y \rangle_t &= -e^{-\gamma_\perp t} s \sin \Omega t \, v_x^0 \; + \; e^{-\gamma_\perp t} \cos \Omega t \, v_y^0 \\
\langle v_z \rangle_t &= e^{-\gamma_\parallel t} \, v_z^0 \\
\langle x \rangle_t &= x_0 + M_\perp(t) \, v_x^0 - M_\angle(t) \, y_y^0 \\
\langle y \rangle_t &= y_0 + M_\angle(t) \, v_x^0 + M_\perp(t) \, v_y^0 \\
\langle z \rangle_t &= z_0 + M_\parallel(t) \, v_z^0
\end{aligned} \tag{10.47}$$

where:

$$\begin{aligned}
M_\perp(t) \equiv M_{41}(-t) &= \frac{1}{\Omega^2 + \gamma_\perp^2} \, \zeta[\gamma_\perp(1 - e^{-\gamma_\perp t} \cos \Omega t) - e^{\gamma_\perp t} \sin \Omega t \, \gamma_\perp^2 / \Omega] \\
&\qquad + e^{-\gamma_\perp t} s \sin \Omega t \\
M_\angle(t) \equiv M_{42}(-t) &= \frac{1}{\Omega^2 + \gamma_\perp^2} \times \\
&\quad \zeta[\gamma_\perp e^{-\gamma_\perp t} s \sin \Omega t + s \, (1 - e^{-\gamma_\perp t} \cos \Omega t) \, \gamma_\perp^2 / \Omega] \\
&\qquad - s \, \Omega^{-1} \, (1 - e^{-\gamma_\perp t} \cos \Omega t) \\
M_\parallel(t) \equiv M_{63}(-t) &= \frac{1 - e^{-\gamma_\parallel t}}{\gamma_\parallel}
\end{aligned}$$

In conclusion, for $t \ll \gamma_*^{-1}$ we obtain:

$$\begin{aligned}
\langle \mathbf{v} \rangle_t &= \mathbf{0} \\
\langle x \rangle_t &= x_0 + s \, \Omega^{-1} \, v_y^0 \\
\langle y \rangle_t &= y_0 - s \, \Omega^{-1} \, v_x^0 \\
\langle z \rangle_t &= z_0 + \frac{v_z^0}{\gamma_\parallel}
\end{aligned} \tag{10.48}$$

in Case II ($\zeta = 0$ i.e. from the *complete FP* collision term), while in Case III ($\zeta = 1$ i.e. from the *reduced HOM-FP* collision term), we obtain:

$$\begin{aligned}
\langle \mathbf{v} \rangle_t &= \mathbf{0} \\
\langle x \rangle_t &= x_0 + \frac{\gamma_\perp}{\Omega^2 + \gamma_\perp^2} \, v_x^0 + s \, \frac{\Omega}{\Omega^2 + \gamma_\perp^2} \, v_y^0 \\
\langle y \rangle_t &= y_0 - s \, \frac{\Omega}{\Omega^2 + \gamma_\perp^2} \, v_x^0 + \frac{\gamma_\perp}{\Omega^2 + \gamma_\perp^2} \, v_y^0 \\
\langle z \rangle_t &= z_0 + \frac{v_z^0}{\gamma_\parallel}
\end{aligned} \tag{10.49}$$

As we see, both cases give the same result for $\Omega \gg \gamma_\perp$, i.e. $\langle \mathbf{x} \rangle_t = \mathbf{0}$, while they differ substantially for $\Omega \ll \gamma_\perp$.

10.7 Plasma FPE - Final results for covariances Ξ_{ij}

Let us study of the covariances $\Xi_{ij} = 2\Lambda_{ij}$, defined above. The symmetric matrix Λ_{ij} has been evaluated above; using that result, we obtain, for velocity variables:

$$\begin{aligned} \Xi_{44} = \Xi_{55} &\equiv \langle\langle v_x\, v_x \rangle\rangle = \frac{D_\perp}{\gamma_\perp}\,(1 - e^{-2\gamma_\perp\, t}) \equiv 2\,\lambda_\perp(t)\, D_\perp = \frac{T_\perp(t)}{m} \\ \Xi_{66} &\equiv \langle\langle v_z\, v_z \rangle\rangle = \frac{D_\parallel}{\gamma_\parallel}\,(1 - e^{-2\gamma_\parallel\, t}) \equiv 2\,\lambda_\parallel(t)\, D_\parallel = \frac{T_\parallel(t)}{m} \\ \Xi_{45} = \Xi_{46} &= \Xi_{56} = 0 \end{aligned} \qquad (10.50)$$

(all definitions are obvious; we have made use of relation (10.20)). At high times, these quantities 'relax' to the equilibrium (thermal) velocity *MSD*, as expected.

For the position variables we obtain:

$$\begin{aligned} \Xi_{11} = \Xi_{22} &\equiv \langle\langle x\, x \rangle\rangle = \langle\langle y\, y \rangle\rangle = \\ &= 2\,\Omega^{-2}\,[\lambda_\perp(t)\, D_\perp + 2\,Q\, t] \approx 2\,\Omega^{-2}\left(\frac{T_{eq}}{m} + 2\,Q\,t\right) \\ &\rightarrow_{t\to\infty} \quad \frac{4\,Q}{\Omega^2}\,t \end{aligned} \qquad (10.51)$$

in Case II (the asymptotic behaviour for $t \ll \gamma_\perp^{-1}$ and $t \to \infty$ was considered), while Case III, for comparison, gives:

$$\Xi_{11} = \Xi_{22} = \frac{2D_\perp}{\Omega^2 + \gamma_\perp^2}\,t + \frac{D_\perp\,(\Omega^2 - 3\gamma_\perp^2)}{\gamma_\perp(\Omega^2 + \gamma_\perp^2)^2} + \mathcal{O}(e^{-\gamma_\perp\, t}) \rightarrow_{t\to\infty} \quad \frac{2\,D_\perp}{\Omega^2 + \gamma_\perp^2}\,t \qquad (10.52)$$

In both cases, we obtain a diffusive asymptotic behaviour[12]. The other elements are:

$$\begin{aligned} \Xi_{33} &\equiv \langle\langle z\, z \rangle\rangle = \frac{D_\parallel}{\gamma_\parallel^3}\,(2\,\gamma_\parallel t - 3 + 4\,e^{-\gamma_\parallel\, t} - e^{-2\gamma_\parallel\, t}) \\ \Xi_{12} &= \Xi_{13} = \Xi_{23} = 0 \end{aligned} \qquad (10.53)$$

Finally, the cross-velocity-position covariances read:

$$\begin{aligned} \Xi_{41} = \Xi_{52} &\equiv \langle\langle v_x\, x \rangle\rangle = \langle\langle v_y\, y \rangle\rangle = 0 \\ \Xi_{42} = \Xi_{51} &\equiv \langle\langle v_x\, y \rangle\rangle = \langle\langle v_y\, x \rangle\rangle = 2\,s\,\Omega^{-1}\,\lambda_\perp(t)\, D_\perp \end{aligned}$$

[12] The difference, in the asymptotic limit, between Cases II and III seems to be rather only quantitative (remember that $Q \approx 2\,D_\perp$, see fig. 9.4). Not surprisingly, this behaviour will be recovered later, by solving for the *df* $f(t)$.

For an infinite Ω, the *MSD* tends to zero, as physically expected. For a vanishing Ω, Case II gives infinity, as the kinetic operator is not valid anymore (see in ch. 7).

$$
\begin{aligned}
&= s\,\Omega^{-1}\,\frac{T_{eq}}{m}(1-e^{-2\gamma_\perp t}) \\
\Xi_{43}=\Xi_{53} &\equiv \langle\langle v_x\,z\rangle\rangle = \langle\langle v_y\,z\rangle\rangle = 0 \\
\Xi_{63} &\equiv \langle\langle v_z\,z\rangle\rangle = D_{\|}\left(\frac{1-e^{-\gamma_{\|}t}}{\gamma_{\|}}\right)^2
\end{aligned}
\tag{10.54}
$$

in Case II. For comparison, in Case III:

$$
\begin{aligned}
\Xi_{41}=\Xi_{52} &\equiv \langle\langle v_x\,x\rangle\rangle = \langle\langle v_y\,y\rangle\rangle = \frac{D_\perp}{\Omega^2+\gamma_\perp^2} \\
\Xi_{42}=\Xi_{51} &\equiv \langle\langle v_x\,y\rangle\rangle = \langle\langle v_y\,x\rangle\rangle \\
&= D_\perp\,\frac{\Omega}{(\Omega^2+\gamma_\perp^2)\gamma_\perp}(1-e^{-2\gamma_\perp t}-2\frac{\gamma_\perp}{\Omega}\sin\Omega t) \\
\Xi_{43}=\Xi_{53} &\equiv \langle\langle v_x\,z\rangle\rangle = \langle\langle v_y\,z\rangle\rangle = 0 \\
\Xi_{63} &\equiv \langle\langle v_z\,z\rangle\rangle = D_{\|}\left(\frac{1-e^{-\gamma_{\|}t}}{\gamma_{\|}}\right)^2
\end{aligned}
\tag{10.55}
$$

(no difference in the direction $\|$ to the field).

10.8 Calculation of the final expression for $f(t)$: Case I - homogeneous plasma

According to the method exposed above, the solution to the FPE is given by (10.13):

$$
f(\mathbf{v};\,t) = \frac{1}{(2\pi)^3}\int d\mathbf{v}'\,f_0(\mathbf{v}')\int d\mathbf{k}\,e^{i\mathbf{k}[\mathbf{v}-\mathbf{M}^T(-t)\mathbf{v}']}\,e^{-\mathbf{k}^T\,\mathbf{\Lambda}(t)\mathbf{k}} \tag{10.56}
$$

It seems appropriate to work in cylindrical coordinates. Let us define:

$$
\mathbf{v}=v_c\angle\alpha, \qquad \mathbf{v}'=v_c'\angle\alpha', \qquad \mathbf{k}=k_c\angle\theta
$$

where $\mathbf{w}=w_c\angle\beta \quad \in \Re^3$ denotes the vector

$$
\mathbf{w}=(w_x,w_y,w_z)=(w_\perp\cos\beta, w_\perp\sin\beta, w_{\|})\,.
$$

In this way, we obtain:

$$
\begin{aligned}
\mathbf{k}\cdot\mathbf{v} &= k_\perp v_\perp\cos(\alpha-\theta)+k_{\|}v_{\|} \\
\mathbf{k}^T\!\cdot\mathbf{M}^T(-t)\!\cdot\mathbf{v}' &= k_\perp v_\perp' e^{-\gamma_\perp t}\cos(\alpha'-\theta-s\Omega t)+k_{\|}v_{\|}e^{-\gamma_{\|}t} \\
\mathbf{k}^T\!\cdot\mathbf{\Lambda}(t)\!\cdot\mathbf{k} &= \frac{1-e^{-2\gamma_\perp t}}{2\gamma_\perp}D_\perp k_\perp^2+\frac{1-e^{-2\gamma_{\|}t}}{2\gamma_{\|}}D_{\|}k_{\|}^2 \\
&\equiv \lambda_\perp(t)D_\perp k_\perp^2+\lambda_{\|}(t)D_{\|}k_{\|}^2
\end{aligned}
$$

The integrals may also be expressed in polar coordinates appropriately as:

$$\int d\mathbf{w} = \int_0^\infty dw_\perp\, w_\perp \int_{-\infty}^\infty dw_\parallel \int_0^{2\pi} d\beta \equiv \int d^c\mathbf{w} \int_0^{2\pi} d\beta.$$

Substituting in the above expression for f we have:

$$f(\mathbf{v};\, t) = \frac{1}{(2\pi)^3} \int d\mathbf{v}'\, f_0(\mathbf{v}') \int d\mathbf{k}\, e^{ik_\perp v_\perp \cos(\alpha-\theta)}\, e^{ik_\parallel v_\parallel}$$
$$e^{-ik_\perp v'_\perp e^{-\gamma_\perp t} \cos(\alpha'-\theta-s\Omega t)}\, e^{ik_\parallel v_\parallel e^{-\gamma_\parallel t}}\, e^{-\lambda_\perp(t) D_\perp k_\perp^2}\, e^{-\lambda_\parallel(t) D_\parallel k_\parallel^2} \tag{10.57}$$

First, let us isolate the two exponents containing θ; using the Bessel function identity:

$$e^{i\,x\,\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\,\phi} \qquad\qquad \forall x, \phi \in \Re \tag{10.58}$$

and the fact that

$$\int_0^{2\pi} d\alpha'\, e^{in\alpha'} = 2\pi\delta_{n,0}^{Kr} \qquad\qquad n = 0, \pm1, \pm2, ... \tag{10.59}$$

the integration in θ can be carried out immediately. We thus obtain:

$$\int_0^{2\pi} d\theta\, e^{ik_\perp v_\perp \cos(\alpha-\theta)}\, e^{-ik_\perp v'_\perp e^{-\gamma_\perp t} \cos(\alpha'-\theta-s\Omega t)} = ...$$

$$= 2\pi \sum_{n=-\infty}^{\infty} J_n(k_\perp v_\perp)\, J_n(k_\perp v'_\perp e^{-\gamma_\perp t})\, e^{in(\alpha-\alpha')}\, e^{ins\Omega t}$$

Let us name this quantity, for instance, Q. Now, assuming a gyrotropic initial state f_0, i.e. $\partial f_0/\partial\alpha' = 0$, the angle integration in α' (inside $\int d\mathbf{v}'...$) can also be carried out in a similar way; the result is:

$$\int_0^{2\pi} d\alpha'\, Q = (2\pi)^2\, J_0(k_\perp v_\perp)\, J_0(k_\perp v'_\perp e^{-\gamma_\perp t})$$

Notice that the d.f. f gyrophase α completely disappears, now. Therefore, given an initial state which is gyrotropic (independent of α'), the evolving state f will remain such at any instant t; of course, this is a consequence of the very structure of the (kinetic) evolution equation itself.

So far, we have obtained:

$$f(\mathbf{v};\, t) = \frac{1}{2\pi} \int d^c\mathbf{v}'\, f_0(\mathbf{v}') \int d^c\mathbf{k}\; J_0(k_\perp v_\perp)\, J_0(k_\perp v'_\perp e^{-\gamma_\perp t})$$
$$e^{ik_\parallel(v_\parallel - v'_\parallel e^{-\gamma_\parallel t})}\, e^{-\lambda_\perp(t) D_\perp k_\perp^2}\, e^{-\lambda_\parallel(t) D_\parallel k_\parallel^2} \tag{10.60}$$

The $\perp$ and $\|$ parts in $\mathbf{k}$ may clearly be separated now[13]:

$$\begin{aligned} f(\mathbf{v};\,t) &= \frac{1}{2\pi}\int_0^\infty dv'_\perp\, v'_\perp \int_{-\infty}^\infty dv_\|\, f_0(v'_\perp, v'_\|) \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^\infty dk_\| \\ &\qquad J_0(k_\perp v_\perp)\, J_0(k_\perp v'_\perp e^{-\gamma_\perp t})\, e^{ik_\|(v_\| - v'_\| e^{-\gamma_\| t})} \\ &\qquad e^{-\lambda_\perp(t) D_\perp k_\perp^2}\, e^{-\lambda_\|(t) D_\| k_\|^2} \\ &= \left[\int_0^\infty dv'_\perp\, v'_\perp \int_{-\infty}^\infty dv_\|\, f_0(v'_\perp, v'_\|)\right. \\ &\qquad \left.\int_0^\infty dk_\perp\, k_\perp\, J_0(k_\perp v_\perp)\, J_0(k_\perp v'_\perp e^{-\gamma_\perp t})\, e^{-\lambda_\perp(t) D_\perp k_\perp^2}\right] \\ &\qquad \times \left[\frac{1}{2\pi}\int_{-\infty}^\infty dk_\|\; e^{ik_\|(v_\| - v'_\| e^{-\gamma_\| t})}\, e^{-\lambda_\|(t) D_\| k_\|^2}\right] \end{aligned} \tag{10.61}$$

and the integrals in $k_\perp$, $k_\|$ may be carried out by using, respectively:

$$\int_0^\infty dx\, x\, e^{-\rho^2 x^2}\, J_0(ax)\, J_0(bx) = \frac{1}{2\rho^2}\, e^{-(a^2+b^2)/4\rho^2}\, I_0\!\left(\frac{ab}{2\rho^2}\right)$$

(see 6.633.2 in [19]) where $I_0(x)$ is the *modified* Bessel function: $I_0(x) = J_0(ix)$ and

$$\int_{-\infty}^\infty e^{-iax}\, e^{-bx^2}\, dx \;=\; e^{-\frac{a^2}{4b}} \int_{-\infty}^\infty e^{-b(x+i\frac{a}{2b})^2}\, dx \;=\; e^{-a^2/4b}\frac{\sqrt{\pi}}{\sqrt{b}}$$

The result reads:

$$\begin{aligned} f(\mathbf{v};\,t) &= \int_0^\infty dv'_\perp\, v'_\perp \int_{-\infty}^\infty dv_\|\, f_0(v'_\perp, v'_\|) \\ &\quad \frac{1}{2\lambda_\perp(t) D_\perp}\, e^{-v_\perp^2/4\lambda_\perp(t)D_\perp}\, e^{-v'^2_\perp e^{-2\gamma_\perp t}/4\lambda_\perp(t)D_\perp}\, I_0\!\left(\frac{v_\perp v'_\perp e^{-\gamma_\perp t}}{2\lambda_\perp(t) D_\perp}\right) \\ &\quad \frac{1}{\sqrt{4\pi\lambda_\|(t) D_\|}}\, e^{-(v_\| - v'_\| e^{-\gamma_\| t})^2/4\lambda_\|(t) D_\|} \end{aligned} \tag{10.62}$$

Notice that:

$$\begin{aligned} 4\lambda_\perp D_\perp &= 4\,\frac{1-e^{-2\gamma_\perp t}}{2\gamma_\perp}\, D_\perp = \ldots = \frac{1-e^{-2\gamma_\perp t}}{\beta_\perp^0} \equiv \frac{2T_\perp^0}{m}(1-e^{-2\gamma_\perp t}) \\ &\equiv \frac{2T_\perp^0(t)}{m} \equiv \frac{1}{\beta_\perp(t)} \end{aligned} \tag{10.63}$$

where we have used condition (10.20) to define the equilibrium temperature $T_\perp^0$. The analogous expression holds for the parallel part: $4\lambda_\| D_\perp = \ldots \equiv \frac{1}{\beta_\|(t)}$; see that $\beta_*(t) \to \beta_*^0 = \frac{m}{2T_\perp^0}$ as $t \to \infty \quad (* = \perp, \|)$.

[13] The whole expression may be appropriately factorized, actually, if the corresponding parts in f_0 are separable too i.e. $f_0 = f_0^{(\perp)}\, f_0^{(\|)}$; nevertheless, the integrals in $\mathbf{k}$ and $\mathbf{v}'$ are *not* decoupled.

Final expression for an arbitrary f_0. Assuming that

$$f_0 = f_0^{(\perp)}(v_\perp)\, f_0^{(\|)}(v_\|)$$

and re-arranging a little, the above expression becomes:

$$\begin{aligned} f(\mathbf{v};\, t) &= 2\,\beta_\perp(t)\, e^{-\beta_\perp(t) v_\perp^2} \int_0^\infty dv'_\perp\, v'_\perp\, f_0^{(\perp)}(v'_\perp)\, e^{-\beta_\perp(t) e^{-2\gamma_\perp t} v'^2_\perp} \\ &\qquad\qquad I_0\left(2\,\beta_\perp(t)\, v_\perp v'_\perp e^{-\gamma_\perp t}\right) \\ &\times \quad \frac{{\beta_\|}^{1/2}(t)}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv'_\|\, f_0^{(\|)}(v'_\|)\, e^{-\beta_\|(t)(v_\| - v'_\| e^{-\gamma_\| t})^2} \\ &\equiv f_\perp(v_\perp;\, t)\, f_\|(v_\|;\, t) \end{aligned} \tag{10.64}$$

The d.f. f is thus separated to $\perp$ and $\|$ parts (i.e. corresponding respectively to the first and second lines in this expression).

Now we need to make an assumption for the initial state f_0.

10.8.1 Solution with a $\delta-$ function for f_0.

Let us assume that the initial state is given by:

$$f_0(\mathbf{v}') = \frac{1}{2\pi}\frac{1}{v'_\perp}\delta(v'_\perp - v_{0,\perp})\delta(v'_\| - v_{0,\|}) \tag{10.65}$$

(verify that $\int d\mathbf{v}'\, f_0(\mathbf{v}') = \int_0^\infty dv'_\perp\, v'_\perp \int_{-\infty}^\infty dv'_\| \int_0^{2\pi} d\alpha'\, f_0(v'_\perp, v'_\|) = 1$). The distribution function at any instant t is then given by:

$$\begin{aligned} f(\mathbf{v};\, t) &= \left[\frac{\beta_\perp(t)}{\pi}\, e^{-\beta_\perp(t) v_\perp^2}\, e^{-\beta_\perp(t) e^{-2\gamma_\perp t} v_{0,\perp}^2}\, I_0\left(2\,\beta_\perp(t)\, v_\perp v_{0,\perp} e^{-\gamma_\perp t}\right)\right] \\ &\quad \left[\left(\frac{\beta_\|(t)}{\pi^{1/2}}\right)^{1/2} e^{-\beta_\|(t)(v_\| - v'_{0,\|} e^{-\gamma_\| t})^2}\right] \\ &\equiv f_0^{(\perp)}(v_\perp;\, t)\, f_0^{(\|)}(v_\|;\, t) \end{aligned} \tag{10.66}$$

Notice that, for large times $t \gg \gamma_\perp^{-1}, \gamma_\|^{-1}$, we get $\beta_{\perp,\|}(t) \to \beta_{\perp,\|}^{(0)}$ and $I_0(...) \to I_0(0) = 1$, so the distribution loses its dependence on the initial condition v_0 and tends to the equilibrium condition anticipated previously:

$$\begin{aligned} f_\infty(\mathbf{v}) &= \frac{1}{\pi^{3/2}}\, \beta_\perp^0\, {\beta_\|^0}^{1/2}\, e^{-\beta_\perp^0 v_\perp^2}\, e^{-\beta_\|^0 v_\|^2} \\ &\equiv f_\infty^{(\perp)}(v_\perp)\, f_\infty^{(\|)}(v_\|) \end{aligned} \tag{10.67}$$

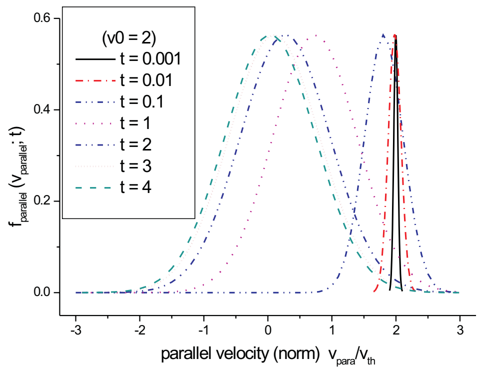

Figure 10.1: The evolution $f_\perp(v_\perp;\, t)$ of an initial $\delta(v_\| - v_\|^0)$ state (sharp solid line) ($\|$ $-$part, as defined in the text) versus velocity $v_\|$ ($v_\| \in \Re$) (normalized over the thermal velocity $\beta_\|^{0\,-1/2}$) for an initial velocity value of $2v_{th}$. The initial sharp profile spreads fast and attains the final zero-averaged thermalized state within a few time constants.

10.8.2 Solution with a Maxwellian function for f_0.

Let us now assume that the initial state is given by a Maxwellian function (yet at a temperature T different from the equilibrium temperature T_0):

$$f_0(\mathbf{v}) \quad = \quad \frac{1}{\pi^{3/2}}\, \beta_\perp\, {\beta_\|}^{1/2}\, e^{-\beta_\perp v_\perp^2}\, e^{-\beta_\| v_\|^2} \tag{10.68}$$

($\beta_* = m/2T_*$; remember that $\beta_*^0 = m/2T_*^0 = \gamma_*/2D_*$, cf. (10.20)). Inserting into (10.64) and carrying out the integrations, we find again a solution of the form: $f_0(\mathbf{v};\, t) = f_0^{(\perp)}(v_\perp;\, t)\, f_0^{(\|)}(v_\|;\, t)$; precisely:

$$f(\mathbf{v};t) \quad = \quad \frac{1}{\pi^{3/2}}\, \tilde\beta_\perp(t)\, \tilde\beta_\|^{1/2}(t)\, e^{-\tilde\beta_\perp(t) v_\perp^2}\, e^{-\tilde\beta_\|(t) v_\|^2} \tag{10.69}$$

where:

$$\tilde\beta_*(t) = \frac{\beta_*\, \beta_*^0}{\beta_*\,(1 - e^{-2\gamma_* t}) + \beta_*^0\, e^{-2\gamma_* t}}$$

($* = \perp, \|$) is the time-dependent *'inverse temperature'*; this relation can be cast in the reduced form:

$$\tilde\beta_*(t)/\beta_*^0 = \frac{\theta}{(1 - e^{-2\tau})\,\theta\, + e^{-2\tau}} \tag{10.70}$$

where

$$\theta = \beta_*/\beta_*^0\,, \qquad \tau = \gamma_* t$$

This relation has been depicted in figure 10.2. Regardless of the initial value $\beta_*(t=0) = \beta_*$, $\tilde\beta_*(t)$ tends to unity (so that T_* attains T_*^0) practically after a few times the time constant γ_*^{-1}.

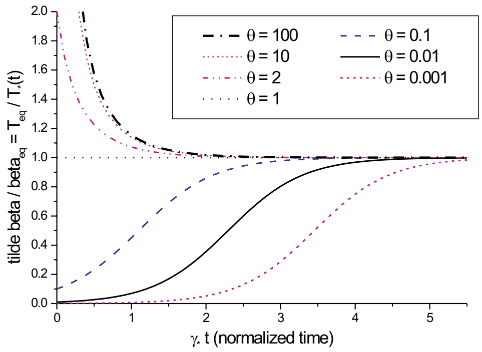

Figure 10.2: The inverse temperature $\tilde{\beta}_*(t)$ (normalized over $\tilde{\beta}_*(\infty) = \beta_*^0$) versus time t (normalized over γ_*^{-1}) for several values of $\theta = \beta_* / \beta_*^0 = T_{eq} / T_*(0)$. Notice that collisions seem to be more efficient in accelerating initially slow particles (notice the curves for $\theta > 1$) than slowing down fast ones ($\theta < 1$). A plasma which is initially at the equilibrium temperature ($\theta = 1$) will not be influenced. See that an upper limit is reached as θ approaches ∞, as suggested by its definition (10.70).

Notice that $\tilde{\beta}_*(t)$ tends to β_*^0 for large times, so $f(t)$ tends to the equilibrium condition $f_\infty = f_{eq}$ given above. Nevertheless, if the plasma was initially at the equilibrium temperature i.e. if $\beta_* = \beta_*^0$, then $\tilde{\beta}_*(t)$ remains equal to β_*^0 at all times t, as expected.

10.8.3 A remark concerning the field dependence of $f(t)$

Remember that $\gamma_* = 2\,\beta_*\,D_*$ (according to (10.20)). Also recall that the diffusion coefficients were seen to depend on the (magnitude of the) magnetic field B. Therefore, for a given equilibrium plasma temperature, all the (final) formulae obtained above for $f(t)$ will depend on the magnetic field through D_*.

In a qualitative manner, to see this dependence we may define a phenomenological relaxation time, say:

$$\tau_{R,*} \equiv \gamma_*^{-1} = \frac{1}{2\,\beta_*^0\,D*} = \frac{T_*^0}{m\,D*}$$

in agreement with [34]. Therefore, e.g. increasing D_* with increasing Ω (see ch. 10), we obtain a reduced relaxation time.

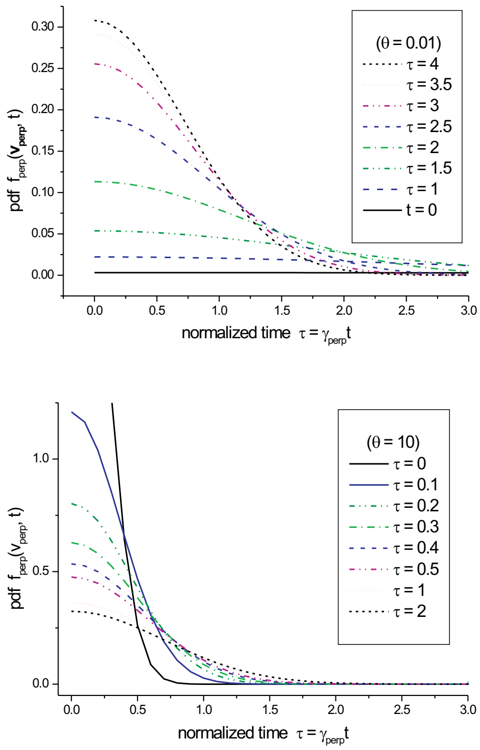

Figure 10.3: The evolution $f_\perp(v_\perp;\, t)$ of an initial Maxwellian state ($\perp$ −part, as given by (10.69) (also in (10.104)) versus velocity $v_\perp$ (normalized over the thermal velocity $\beta_\perp^{0\ -1/2}$) for two values of $\theta = \beta_*/\beta_*^0 = T_{eq}/T_*(0)$: (a) $\theta = 0.01$ (high initial plasma temperature, see black solid line at the bottom) and (b) $\theta = 10$ (low initial plasma temperature, see black solid line on top). In the second case, the initial distribution relaxes faster to the final equilibrium state (practically attained after 2 time constants) in agreement with the comment in fig. 10.2.

10.9 Calculation of the final expression for $f(t)$: Case II - general case

According to the method exposed previously, the solution to the complete (6d) FPE is given by (10.13):

$$f(\mathbf{x}, \mathbf{v};\, t) = \frac{1}{(2\pi)^6} \int d\mathbf{y}'\, f_0(\mathbf{y}') \int d\mathbf{k}\, e^{i\mathbf{k}[\mathbf{y} - \mathbf{M}^T(-t)\,\mathbf{y}']}\, e^{-\mathbf{k}^T \mathbf{\Lambda}(t)\mathbf{k}} \tag{10.71}$$

where all quantities now ‘live’ in a 6d-space.

The calculation is obviously lengthy, since $6 + 6 = 12$ integrals are involved! Nevertheless, it can be carried out till the end, in a way similar to case I (see in the previous section). For clarity, we have devoted the following subsection to the analytical calculation leading to a final expression for $f(t)$, which is then presented in the subsection after that. For a quick overview of this part, therefore, refer directly to the section *after* the following, that is to final expressions in §10.9.2 and forth.

10.9.1 Analytical calculation of $f(t)$

1. Re-expressing the above relation in terms of the sub-spaces $\{\mathbf{x}\}$, $\{\mathbf{v}\}$ (with the help of all the above definitions), we have:

$$
\begin{aligned}
f(\mathbf{x},\mathbf{v};\,t) = \frac{1}{(2\pi)^6} \int d\mathbf{x}' \int d\mathbf{v}'\, f_0(\mathbf{x}',\mathbf{v}') \int d\mathbf{k_x} \int d\mathbf{k_v}\, e^{i\mathbf{k_x}\mathbf{x}}\, e^{i\mathbf{k_v}\mathbf{v}} \\
e^{-i\mathbf{k_x}\mathbf{x}'}\, e^{-i\mathbf{k_x}\mathbf{M^T_{vx}}(-\mathbf{t})\,\mathbf{v}'}\, e^{-i\mathbf{k_v}\mathbf{M^T_{vv}}(-\mathbf{t})\,\mathbf{v}'} \\
e^{-\mathbf{k_x}^T\,\mathbf{\Lambda_{xx}}(t)\,\mathbf{k_x}}\, e^{-2\,\mathbf{k_v}^T\,\mathbf{\Lambda_{vx}}(t)\,\mathbf{k_x}}\, e^{-\mathbf{k_v}^T\,\mathbf{\Lambda_{vv}}(t)\,\mathbf{k_v}}
\end{aligned}
\tag{10.72}
$$

We will work in polar coordinates in both position and velocity space. We define:

$$
\mathbf{v} = v_c\angle\alpha, \qquad \mathbf{v}' = v_c'\angle\alpha', \qquad \mathbf{k_v} = (k_4,k_5,k_6) = k_\perp^c\angle\theta
$$

$$
\mathbf{x} = \rho_c\angle\beta, \qquad \mathbf{x}' = \rho_c'\angle\beta', \qquad \mathbf{k_x} = (k_1,k_2,k_3) = k_\rho^c\angle\phi
$$

where $\mathbf{w} = w_c\angle\beta \quad \in \Re^3$ denotes the vector

$$
\mathbf{w} = (w_x, w_y, w_z) = (w_\perp\cos\beta, w_\perp\sin\beta, w_\parallel) \quad .
$$

In this way, we obtain:

$$
\begin{aligned}
\mathbf{k_v}\cdot\mathbf{v} &= k_\perp v_\perp \cos(\theta-\alpha) + k_\parallel v_\parallel \\
\mathbf{k_x}\cdot\mathbf{x} &= k_\rho \rho\cos(\phi-\beta) + k_z\, z \\
\mathbf{k_x}\cdot\mathbf{x}' &= k_\rho \rho'\cos(\phi-\beta') + k_z\, z' \\
k_1\,k_4 + k_2\,k_5 &= k_\rho\, k_\perp\, \cos(\phi-\theta) \\
k_2\,k_4 - k_1\,k_5 &= k_\rho\, k_\perp\, \sin(\phi-\theta) \\
\mathbf{k_v}^T\!\cdot\!\mathbf{M_{vv}}^T(-t)\!\cdot\!\mathbf{v}' &= k_\perp v_\perp' e^{-\gamma_\perp t}\cos(\alpha'-\theta-s\Omega t) + k_\parallel v_\parallel e^{-\gamma_\parallel t} \\
\mathbf{k_x}^T\!\cdot\!\mathbf{M_{vx}}^T(-t)\!\cdot\!\mathbf{v}' &= \ldots \\
\mathbf{k_v}^T\!\cdot\!\mathbf{\Lambda_{vv}}(t)\!\cdot\!\mathbf{k_v} &= \frac{1-e^{-2\gamma_\perp t}}{2\gamma_\perp} D_\perp k_\perp^2 + \frac{1-e^{-2\gamma_\parallel t}}{2\gamma_\parallel} D_\parallel k_\parallel^2 \\
&\equiv \lambda_\perp(t) D_\perp k_\perp^2 + \lambda_\parallel(t) D_\parallel k_\parallel^2
\end{aligned}
\tag{10.73}
$$

and so forth. The integrals will also be expressed in polar coordinates appropriately as:

$$\int d\mathbf{w} = \int_0^\infty dw_\perp \, w_\perp \int_{-\infty}^\infty dw_\parallel \int_0^{2\pi} d\beta \equiv \int d^c\mathbf{w} \int_0^{2\pi} d\beta$$

($\mathbf{w} = \mathbf{x}', \mathbf{v}', \mathbf{k}'_\mathbf{x}$ or $\mathbf{k}'_\mathbf{v}$). Substituting in the above expression for f and separating $\perp$ and $\parallel$ parts we have:

$$\begin{aligned}
f(\mathbf{x},\mathbf{v};\,t) = \frac{1}{(2\pi)^4} \quad \times & \int_0^\infty d\rho'\,\rho' \int_0^{2\pi} d\beta'\;\tilde{f}_0^{(\perp)}(\rho') \\
& \int_0^\infty dv'_\perp\, v'_\perp \int_0^{2\pi} d\alpha'\; f_0^{(\perp)}(v'_\perp) \\
& \int_0^\infty dk_\rho\, k_\rho \int_0^{2\pi} d\phi\; e^{-is\frac{k_\rho v'_\perp}{\Omega}\left[\sin(\phi-\alpha') - e^{-\gamma_\perp t}\,\sin(\phi-\alpha'-s\Omega t)\right]} \\
& \quad e^{ik_\rho \rho \cos(\phi-\beta)}\, e^{-ik_\rho \rho' \cos(\phi-\beta')}\, e^{-\Omega^{-2}[\lambda_\perp(t)\,D_\perp + Q\,t]k_\rho^2} \\
& \int_0^\infty dk_\perp\, k_\perp \int_0^{2\pi} d\theta\, e^{ik_\perp v_\perp \cos(\theta-\alpha)}\, e^{-ik_\perp v'_\perp e^{-\gamma_\perp t}\cos(\alpha'-\theta-s\Omega t)} \\
& \quad e^{-\lambda_\perp(t) D_\perp k_\perp^2}\, e^{-2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t) D_\perp\,\sin(\phi-\theta)} \\
\times \frac{1}{(2\pi)^2} \quad \times & \int_0^\infty dv'_\parallel\, f_0(v'_\parallel) \int_{-\infty}^\infty dk_\parallel\, e^{ik_\parallel(v_\parallel - v'_\parallel e^{-\gamma_\parallel t})}\, e^{-\lambda_\parallel(t) D_\parallel k_\parallel^2} \\
& \int_0^\infty dz' \int_0^\infty dk_z\, e^{ik_z(z-z')}\, e^{-\gamma_\parallel^{-2}\lambda_\parallel^{(X)}\,D_\parallel k_z^2} \\
& \quad e^{-ik_z v'_\parallel \frac{1-e^{-\gamma_\parallel t}}{\gamma_\parallel} D_\parallel}\, e^{-ik_z k_\parallel\, D_\parallel \left(\frac{1-e^{-\gamma_\parallel t}}{\gamma_\parallel}\right)^2} \\
\equiv \quad & f_\perp(\mathbf{x}_\perp, \mathbf{v}_\perp;\,t)\, f_\parallel(v_\parallel;\,t)
\end{aligned} \tag{10.74}$$

where we assumed that the initial state f_0 is *gyrotropic* and separable as[14]:

$$f_0(\mathbf{x}', \mathbf{v}') = \tilde{f}_0^{(\perp)}(\rho')\, f_0^{(\perp)}(v'_\perp)\, f_0^{(\parallel)}(v'_\parallel)$$

This expression contains 12 (!) integrals, part of which are decoupled from the rest. We shall try to evaluate it carefully.

In the following, we shall often use the definitions:

$$\beta_*(t) = \frac{1}{4\lambda_*(t) D_*} = \frac{m}{2T_*(0)}(1 - e^{-2\gamma_* t}) \equiv \frac{m}{2T_*(t)}$$

$$\lambda_*(t) = \frac{1 - e^{-2\gamma_* t}}{2\gamma_* t} \qquad (* = \perp, \parallel) \tag{10.75}$$

and the equilibrium condition (10.20) i.e. $\gamma_* = 2\beta_0^* D_*$. Also recall the Bessel function identity (10.58) and property (10.59), which will be used repeatedly.

[14] Remember that we have previously assumed independence of f from z.

2. First, the parallel part is easy to evaluate: the $z'-$integration in it immediately yields:

$$\int_{-\infty}^{\infty} dz'\, e^{-ik_z z'} = 2\pi\delta(k_z)$$

and the subsequent k_z-integral thus trivially gives 1 (setting $k_z = 0$ everywhere). Now, the rest of the $\parallel -$ part has been evaluated in the preceding section[15]; it gives, just as in Case I:

$$\begin{aligned} f_\parallel(v_\parallel;\, t) &= \frac{1}{\sqrt{4\pi\lambda_\parallel(t)D_\parallel}} \int_{-\infty}^{\infty} dv'_\parallel\, f_0^{(\parallel)}(v'_\parallel)\; e^{-(v_\parallel - v'_\parallel e^{-\gamma_\parallel t})^2/4\lambda_\parallel(t)D_\parallel} \\ &= \frac{\sqrt{\beta_\parallel(t)}}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv'_\parallel\, f_0^{(\parallel)}(v'_\parallel)\, e^{-\beta_\parallel(t)(v_\parallel - v'_\parallel e^{-\gamma_\parallel t})^2} \end{aligned} \qquad (10.76)$$

and can be readily evaluated for a given initial condition e.g. a Maxwellian state or else; cf. case I; for a Maxwellian $f_0^{(\parallel)}$ in specific, see §10.8.2).

We see that there is no modification with respect to Case I, as far as the $\parallel -$ part is concerned. This was more or less expected, since the $\parallel -$ and $\perp -$ parts are not coupled, and no spatial ($z-$)dependence was considered in the former, as discussed above.

3. Now, let us isolate the (one) exponent in (10.74) containing β'; using the Bessel function identity (10.58) and property (10.59), we have:

$$\begin{aligned} \int_0^{2\pi} d\beta'\, e^{-ik_\rho \rho' \cos(\phi-\beta')} &= \sum_{n=-\infty}^{\infty} J_n(k_\rho\rho')\, e^{-in\,(\frac{\pi}{2}-\phi+\beta')} \\ &= \int_0^{2\pi} d\beta' \sum_{n=-\infty}^{\infty} J_n(k_\rho\rho')\, e^{-in\,(\frac{\pi}{2}-\phi)}\, (2\pi)\, \delta_{n,0}^{Kr} \\ &= (2\pi)\, J_0(k_\rho\rho') \end{aligned} \qquad (10.77)$$

We can try to evaluate the (three) remaining angle integrals, using the same technique. We shall set:

$$\begin{aligned} e^{ik_\perp v_\perp \cos(\theta-\alpha)} &= \sum_{n=-\infty}^{\infty} J_n(k_\perp v_\perp)\, e^{+in\,(\frac{\pi}{2}-\theta+\alpha)} \\ e^{-ik_\perp v'_\perp e^{-\gamma_\perp t} \cos(\alpha'-\theta-s\Omega t)} &= \sum_{n'=-\infty}^{\infty} J_{n'}(k_\perp v'_\perp e^{-\gamma_\perp t})\, e^{in'\,(-\frac{\pi}{2}-\theta+\alpha'-s\Omega t)} \\ e^{ik_\rho \rho \cos(\phi-\beta)} &= \sum_{m=-\infty}^{\infty} J_m(k_\rho\rho)\, e^{im\,(\frac{\pi}{2}-\phi+\beta)} \\ e^{-is\frac{k_\rho v'_\perp}{\Omega} \sin(\phi-\alpha')} &= \sum_{m'=-\infty}^{\infty} J_{m'}\left(s\frac{k_\rho v'_\perp}{\Omega}\right) e^{-im'\,(\phi-\alpha')} \end{aligned}$$

---

[15] See the second line in expression (10.64).

$$e^{is\frac{k_\rho v'_\perp}{\Omega}\,e^{-\gamma_\perp t}\,\sin(\phi-\alpha'-s\Omega t)} = \sum_{m''=-\infty}^{\infty} J_{m''}\left(s\frac{k_\rho v'_\perp}{\Omega}\,e^{-\gamma_\perp t}\right) e^{im''\,(\phi-\alpha'-s\Omega t)}$$

$$e^{-2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t)D_\perp\,\sin(\phi-\theta)} = \sum_{l=-\infty}^{\infty} J_l\left(2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t)D_\perp\right) e^{-il\,(\phi-\theta)}$$

Now, collecting terms containing angle variables and evaluating the corresponding integrals, we obtain:

$$\int_0^{2\pi} d\alpha'\, e^{i(m'-m''+n')\,\alpha'} \int_0^{2\pi} d\theta\, e^{i(-n-n'+l)\,\theta} \int_0^{2\pi} d\phi\, e^{i(-m-m'+m''-l)\,\phi}$$
$$e^{i(n-n'+m)\,\frac{\pi}{2}}\, e^{-i(n'+m)\,s\,\Omega\,t}$$
$$= (2\pi)^3\,\delta^{Kr}_{m'-m''+n',0}\,\delta^{Kr}_{n+n'-l,0}\,\delta^{Kr}_{m+m'-m''+l,0}\,e^{i(n-n'+m)\,\frac{\pi}{2}}\,e^{-i(n'+m)\,s\,\Omega\,t} \tag{10.78}$$

Therefore, setting $m \doteq -n$, $m'' \doteq n'+m'$ and $l \doteq n+n'$ in the summations and substituting in the first ($\perp$ −)part of (10.74), we are left with:

$$f_\perp(\mathbf{x}_\perp, \mathbf{v}_\perp;\, t) =$$

$$\int_0^\infty d\rho'\,\rho'\,\tilde f_0^{(\perp)}(\rho') \int_0^\infty dv'_\perp\, v'_\perp\, f_0^{(\perp)}(v'_\perp) \int_0^\infty dk_\rho\, k_\rho \int_0^\infty dk_\perp\, k_\perp\; J_0(k_\rho \rho')$$
$$\sum_{n,\,n',\,m'=-\infty}^{\infty} J_n(k_\perp v_\perp)\, J_{n'}(k_\perp v'_\perp e^{-\gamma_\perp t})\, J_{-n}(k_\rho \rho)$$
$$J_{m'}\left(s\frac{k_\rho v'_\perp}{\Omega}\right) J_{n'+m'}\left(s\frac{k_\rho v'_\perp}{\Omega}\,e^{-\gamma_\perp t}\right) J_{n+n'}\left(2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t)D_\perp\right)$$
$$e^{-in'\,\frac{\pi}{2}}\, e^{in\,(\alpha-\beta)}\, e^{-\Omega^{-2}[\lambda_\perp(t)\,D_\perp + Q\,t]k_\rho^2}\, e^{-\lambda_\perp(t)D_\perp k_\perp^2} \tag{10.79}$$

4. Notice that m' appears only in two Bessel functions and nowhere else. We may therefore evaluate the summation in m' by making use of *Neumann's addition theorem*:

$$\sum_{k=-\infty}^{\infty} J_{n\pm k}(u)\, J_k(v) = J_n(u \mp v) \tag{10.80}$$

(see 9.1.75, p. 363 in [1]); we thus obtain:

$$\sum_{m'=-\infty}^{\infty} J_{m'}\left(s\frac{k_\rho v'_\perp}{\Omega}\right) J_{n'+m'}\left(s\frac{k_\rho v'_\perp}{\Omega}\,e^{-\gamma_\perp t}\right) = J_{n'}\left(-s\frac{k_\rho v'_\perp}{\Omega}(1-e^{-\gamma_\perp t})\right) \tag{10.81}$$

so the summation in m' vanishes.

5. Substituting in (10.79), we obtain a slightly simpler expression containing the velocity integral:

$$\int_0^\infty dv'_\perp\, v'_\perp\, f_0^{(\perp)}(v'_\perp)\, J_{n'}(k_\perp v'_\perp e^{-\gamma_\perp t})\, J_{n'}\left(-s\frac{k_\rho v'_\perp}{\Omega}(1-e^{-\gamma_\perp t})\right) \equiv I_{v'_\perp} \quad (10.82)$$

which needs to be evaluated for a given initial state f_0. The calculation is simplified by choosing a Maxwell state:

$$f_0^{(\perp)}(v'_\perp) = \frac{\beta_\perp}{\pi}\, e^{-\beta_\perp\, v'^2_\perp} \quad (10.83)$$

where, of course, the initial (inverse) temperature $\beta_\perp$ need not be equal to the equilibrium temperature $\beta_\perp^0$ defined elsewhere[16]. The above integral can be evaluated by making use of:

$$\int_0^\infty e^{-\rho^2 x^2}\, J_p(ax)\, J_p(bx)\, x\, dx = \frac{1}{2\rho^2}\, e^{-\frac{a^2+b^2}{4\rho^2}}\, I_p\left(\frac{ab}{2\rho^2}\right) \quad (10.84)$$

(see 6.633.2 in [19]); we thus obtain:

$$I_{v'_\perp} = \frac{1}{2\pi}\, e^{-\frac{-e^{-2\gamma_\perp t} k_\perp^2}{4\beta_\perp}}\, e^{-\frac{(1-e^{-\gamma_\perp t})^2 k_\rho^2}{4\Omega^2\beta_\perp}}\, I_{n'}\left(-\frac{sk_\perp k_\rho}{2\Omega\beta_\perp} e^{-\gamma_\perp t}\,(1-e^{-\gamma_\perp t})\right) \quad (10.85)$$

Remember that $I_n(z)$ denotes the *modified* Bessel function: $I_n(z) = J_n(iz)$, which obeys: $I_{-n}(z) = I_n(z)$, $I_n(-z) = (-1)^n\, I_n(z)$.

5. Substituting in (10.79), we come up with the summation in n', which is now simplified:

$$\begin{aligned} S &\equiv \sum_{n'=-\infty}^{\infty} I_{n'}\left(-\frac{sk_\perp k_\rho}{2\Omega\beta_\perp} e^{-\gamma_\perp t}\,(1-e^{-\gamma_\perp t})\right) \\ &\qquad\qquad J_{n+n'}\left(2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t) D_\perp\right) e^{-i\,n'\,\pi/2} \\ &= \sum_{k=-\infty}^{\infty} I_{k-n}\left(-\frac{sk_\perp k_\rho}{2\Omega\beta_\perp} e^{-\gamma_\perp t}\,(1-e^{-\gamma_\perp t})\right) \\ &\qquad\qquad J_k\left(2s\frac{k_\rho k_\perp}{\Omega}\lambda_\perp(t) D_\perp\right) e^{-i\,(k-n)\,\pi/2} \\ &\equiv \left[\sum_{k=-\infty}^{\infty} I_{k-n}(u)\, J_k(v)\, e^{-i\,k\,\pi/2}\right] e^{i\,n\,\pi/2} \end{aligned} \quad (10.86)$$

[16] Keep in mind the distinction between:
- the variable $\beta_\perp(t)$ defined in (10.75),
- the (fixed) equilibrium value $\beta_\perp^0 = \lim_{t\to\infty} \beta_\perp(t)$

and
- the (arbitrary) initial value $\beta_\perp = \beta_\perp(0)$.

(all definitions are obvious) which can be evaluated via Gegenbauer's addition theorem (see 9.1.79 in [1]):

$$\sum_{k=-\infty}^{\infty} \mathcal{G}_{p+k}(u)\, J_k(v)\, e^{i\,k\,\alpha} = \mathcal{G}_p(w)\, e^{i\,p\,\chi}$$
$$w^2 = u^2 + v^2 - 2uv\cos\alpha$$
$$w\cos\chi = u - v\cos\alpha\,, \quad w\sin\chi = v\sin\alpha \tag{10.87}$$

($\mathcal{G}_n(z)$ may denote either $J_n(z)$ or $I_n(z)$ here). We thus obtain:

$$S = I_{-n}(w)\, e^{i\,n\,(\chi+\pi/2)} = I_n(w)\, e^{i\,n\,(\chi+\pi/2)} \tag{10.88}$$

where:

$$\begin{aligned} w &= \sqrt{u^2+v^2} = \frac{k_\rho\, k_\perp}{\Omega}\sqrt{(2\lambda_\perp\, D_\perp)^2 + \left[\frac{1}{2\beta_\perp} e^{-\gamma_\perp t}(1-e^{-\gamma_\perp t})\right]^2} \\ &\equiv \frac{k_\rho\, k_\perp}{\Omega}\,\xi(t) \\ \chi &= -\arctan\frac{v}{w} = \arctan\frac{4\beta_\perp \lambda_\perp\, D_\perp}{e^{-\gamma_\perp t}(1-e^{-\gamma_\perp t})} = \chi(t) \end{aligned}$$

Using definitions (10.75) these expressions become:

$$\begin{aligned} w(t) &= \frac{k_\rho\, k_\perp}{\Omega}\,\xi(t) \\ &= \frac{k_\rho\, k_\perp}{\Omega}\,\frac{1}{2\,\beta_\perp\,\beta_\perp^0}\left[\beta_\perp^2\,(1-e^{-2\gamma_\perp t})^2 + {\beta_\perp^0}^2\, e^{-2\gamma_\perp t}(1-e^{-\gamma_\perp t})^2\right]^{1/2} \\ \chi(t) &= \arctan\frac{\beta_\perp}{\beta_\perp^0}(1+e^{\gamma_\perp t}) \end{aligned} \tag{10.89}$$

See that

$$\lim_{t\to\infty}\xi(t) = 1/\beta_\perp^0\,, \qquad \lim_{t\to\infty}\chi(t) = \pi/2$$

So far, (10.79) has taken the form:

$$f_\perp^{(Max)}(\mathbf{v}_\perp;\, t) = \frac{1}{2\pi}\int_0^\infty d\rho'\,\rho'\,\tilde{f}_0^{(\perp)}(\rho')\int_0^\infty dk_\rho\, k_\rho\int_0^\infty dk_\perp\, k_\perp\; J_0(k_\rho\rho')$$
$$\sum_{n=-\infty}^{\infty} J_n(k_\perp v_\perp)\, J_{-n}(k_\rho\rho)\, I_n\!\left(\frac{k_\rho k_\perp}{\Omega}\xi(t)\right)\; e^{i\,n\,(\chi+\pi/2+\alpha-\beta)}\; e^{-\tilde{A}k_\rho^2}\, e^{-A(t)k_\perp^2} \tag{10.90}$$

where

$$\begin{aligned} A(t) &= \lambda_\perp(t)\, D_\perp + \frac{e^{-2\gamma_\perp t}}{4\beta_\perp} = \ldots = \frac{1}{4\,\beta_\perp\,\beta_\perp^0}\left[\beta_\perp\,(1-e^{-2\gamma_\perp t}) + \beta_\perp^0\, e^{-2\gamma_\perp t}\right] \\ \tilde{A}(t) &= \frac{1}{\Omega^2}\left[\lambda_\perp(t)\, D_\perp + Q\,t + \frac{(1-\, e^{-\gamma_\perp t})^2}{4\beta_\perp}\right] \end{aligned} \tag{10.91}$$

6. The integration in $k_\perp$ can now be carried out, with the help of:

$$\int_0^\infty e^{-\rho^2 x^2} I_p(ax)\, J_p(bx)\, x\, dx \;=\; \frac{1}{2\rho^2}\, e^{\frac{a^2-b^2}{4\rho^2}}\, J_p\left(\frac{ab}{2\rho^2}\right) \tag{10.92}$$

(see 6.633.4 in [19]); we thus obtain:

$$\begin{aligned} I_{k_\perp} &= \int_0^\infty dk_\perp\, k_\perp\, J_n(k_\perp v_\perp)\, I_n\left(\frac{k_\rho k_\perp}{\Omega}\,\xi\right) e^{-A\,k_\perp^2} \\ &= \frac{1}{2A}\, J_n\left(\frac{k_\rho v_\perp}{2\Omega A}\xi\right)\, e^{\xi^2 k_\rho^2/4A\Omega^2}\; e^{-v_\perp^2/4A} \end{aligned} \tag{10.93}$$

7. The remaining summation in n, say S, now reads:

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\rho v_\perp}{2\Omega A}\xi(t)\right) J_{-n}(k_\rho \rho)\, e^{i\,n\,(\chi+\frac{\pi}{2}+\alpha-\beta)} \\ &= J_n\left(\frac{k_\rho v_\perp}{2\Omega A}\xi(t)\right) J_n(k_\rho \rho)\, e^{i\,n\,(\chi+\frac{3\pi}{2}+\alpha-\beta)} \\ &= J_0\left(k_\rho \left[\rho^2 + \left(\frac{v_\perp\,\xi(t)}{2\Omega A}\right)^2 - 2\frac{v_\perp\,\xi(t)}{2\,\Omega\,A}\rho\,\sin(\chi+\alpha-\beta)\right]^{1/2}\right) \\ &\equiv J_0\Big(k_\rho\;\Xi(v_\perp,\rho;\Omega,\;t)\Big) \end{aligned} \tag{10.94}$$

where we have expressed $J_{-n}(k_\rho\rho)$ as:

$$J_{-n}(k_\rho\rho) = (-1)^n\, J_n(k_\rho\rho) = e^{in\pi}\, J_n(k_\rho\rho)$$

in the first step, while in the last we have used *Gegenbauer's theorem* (10.87) once again (setting $p = 0$ therein). The definition of Ξ is obvious.

8. So far, (10.90) has taken the form:

$$\begin{aligned} f_\perp^{(Max)}(\mathbf{v}_\perp;\,t) = \frac{1}{2\pi}\,\frac{1}{2A(t)}\, e^{-v_\perp^2/4A(t)} \int_0^\infty d\rho'\,\rho'\,\tilde f_0^{(\perp)}(\rho')\;\times \\ \int_0^\infty dk_\rho\;k_\rho\,J_0(k_\rho\rho')\,J_0\Big(k_\rho\;\Xi(v_\perp,\rho;\Omega,\;t)\Big)\, e^{-\tilde A k_\rho^2}\, e^{\xi(t)^2 k_\rho^2/4A(t)\Omega^2} \\ \equiv \frac{1}{2\pi}\,\frac{1}{2A(t)}\, e^{-v_\perp^2/4A(t)} \int_0^\infty d\rho'\,\rho'\,\tilde f_0^{(\perp)}(\rho') \\ \int_0^\infty dk_\rho\;k_\rho\,J_0(k_\rho\rho')\,J_0\big(k_\rho\;\Xi\big)\, e^{-\hat A k_\rho^2} \end{aligned} \tag{10.95}$$

Now, the integration in k_ρ can be carried out by using (10.84) once more; we thus obtain:

$$\int_0^\infty dk_\rho\;k_\rho\,J_0(k_\rho\rho')\,J_0\big(k_\rho\;\Xi\big)\, e^{-\hat A^2 k_\rho^2} \;=$$

$$\frac{1}{2\hat{A}(t)}\, e^{-\rho'^2/4\hat{A}(t)}\, e^{-\Xi^2(t)/4\hat{A}(t)}\, I_0\left(\frac{\Xi(t)\rho'}{2\hat{A}(t)}\right)$$

Substituting into (10.95) we obtain the final expression for $f(t)$, given an arbitrary initial spatial distribution $f_0^{(\perp)}(\rho')$:

$$f_\perp^{(Max)}(\mathbf{v}_\perp;\, t) = \frac{1}{2\pi}\,\frac{1}{2A(t)}\,\frac{1}{2\hat{A}(t)}\, e^{-v_\perp^2/4A(t)}\, e^{-\Xi^2(t)/4\hat{A}(t)}$$
$$\int_0^\infty d\rho'\,\rho'\,\tilde{f}_0^{(\perp)}(\rho')\; e^{-\rho'^2/4\hat{A}(t)}\, I_0\left(\frac{\Xi(t)\rho'}{2\hat{A}(t)}\right) \qquad (10.96)$$

where $A, \tilde{A}$, Ξ and $\hat{A}$ have been defined in §5, 7 and 8, above.

10.9.2 Final expression for a Maxwellian $f_0(\mathbf{v})$ and an arbitrary $f_0(\mathbf{x})$

Re-arranging the results of the previous subsection and summarizing, we obtain the concise expression:

$$f^{(Max)}(\mathbf{v};\, t) = f_\perp^{(Max)}(\mathbf{x}_\perp, \mathbf{v}_\perp;\, t)\;\, f_\parallel^{(Max)}(v_\parallel;\, t) \qquad (10.97)$$

where the superscript denotes that a Maxwellian initial velocity distribution has been assumed. The two functions $f_{\perp,\parallel}$ are given by:

$$f_\parallel^{(Max)}(v_\parallel;\, t) = \frac{1}{\pi^{1/2}}\,\tilde{\beta}_\parallel^{1/2}\, e^{-\tilde{\beta}_\parallel v_\parallel^2}$$

$$f_\perp^{(Max)}(\mathbf{x}_\perp, \mathbf{v}_\perp;\, t) = \frac{\tilde{\beta}_\perp(t)}{\pi}\, e^{-\tilde{\beta}_\perp(t)\, v_\perp^2}\;\times$$
$$\frac{\tilde{\beta}_\perp^{(X)}(t)}{\pi}\, e^{-\tilde{\beta}_\perp^{(X)}(t)\,\Xi^2(v_\perp,\rho;\,\Omega;\,t)}\;\times$$
$$2\pi\int_0^\infty d\rho'\,\rho'\,\tilde{f}_0^{(\perp)}(\rho')\; e^{-\tilde{\beta}_\perp^{(X)}(t)\rho'^2}\; I_0\Big(2\,\tilde{\beta}_\perp^{(X)}(t)\,\Xi(v_\perp,\rho;\,\Omega;\,t)\rho'\Big) \qquad (10.98)$$

where all quantities were defined in the text.

Definitions. Let us summarize the definitions of quantities in the above expression. They may be conveniently expressed in terms of:

$$\theta = \beta_*/\beta_*^0 \equiv \beta_*(0)/\beta_*^0 \equiv T_*^0/T_*(0)\,, \qquad \tau = \gamma_* t \qquad\qquad * = \perp,\, \parallel$$

We then have:

$$\tilde{\beta}_*(\tau) = \frac{1}{4A} = \frac{\theta}{(1-e^{-2\tau})\,\theta\, + e^{-2\tau}}\,\beta_*^0$$
$$\tilde{\beta}_\perp^{(X)}(\tau) = \frac{1}{4\hat{A}} = \Omega^2\,\beta_\perp^0\,\hat{a}^{-1}(\tau)$$

$$\begin{aligned}
\Xi(\tau) &= \Bigg[\rho^2 + \left(\frac{2\,\tilde{\beta}_\perp(\tau)\,\xi(\tau)}{\Omega}\,v_\perp\right)^2 \\
&\qquad - \frac{4\,\tilde{\beta}_\perp(\tau)\,\xi(\tau)}{\Omega}\,v_\perp\,\rho\,\sin(\chi+\alpha-\beta)\Bigg]^{1/2} \\
\xi(\tau) &= \frac{1}{\beta_\perp^0}\frac{1}{2\theta}\left\{\theta^2\,(1-e^{-2\tau})^2 + e^{-2\tau}\,(1-e^{-\tau})^2\right\}^{1/2} \\
\hat{a}(\tau) &= \Bigg\{(1-e^{-2\tau}) + \frac{(1-e^{-\tau})^2}{\theta} \\
&\qquad -\frac{1}{\theta}\,\frac{\theta^2(1-e^{-2\tau})^2 + e^{-2\tau}(1-e^{-\tau})^2}{\theta(1-e^{-2\tau}) + e^{-2\tau}} + 4\beta_\perp^0\,Q\,t\Bigg\} \\
\chi(\tau) &= \arctan\theta(1+e^{\tau})
\end{aligned} \tag{10.99}$$

Note the asymptotic values:

$$\lim_{\tau\to\infty}\chi(\tau) = \pi/2\,, \qquad \lim_{\tau\to\infty}\tilde{\beta}_*(\tau) = \beta_*^0$$

$$\lim_{\tau\to\infty}\tilde{\beta}_\perp^{(X)}(\tau) = \Omega^2\,\beta_*^0/\left[1/\theta + 4\beta_\perp^0\,Q\,t\right] = \Omega^2\,\beta_\perp/\left[1+4\beta_\perp\,Q\,t\right]$$

Physical interpretation - asymptotic behaviour. Notice that $\tilde{\beta}_*(t) \equiv 1/2V_{th}^2(t) \equiv m/2T_*(t)$ corresponds to a time-dependent inverse temperature, which tends monotonically to its asymptotic equilibrium value $\beta_*^0 \equiv m/2T_*^0$. As discussed in Case I, this time variation in time depends on the initial temperature - via $\theta = T_*^0/T_*(t)$ - (see figure); nevertheless, the system will not evolve if initially set at $\theta = 1$ (i.e. at equilibrium).

In the spatial part of $f(t)$, $\tilde{\beta}_\perp(t)^{(X)} \equiv 1/2L^2(t) \equiv 1/[2\,\hat{a}(t)\,(V_{th}(t)/\Omega)^2]$ expresses the effective width of the Maxwellian curve for the space variable ρ, which is also related to the time-varying temperature. $L^2(t)$ varies as $\sim t$ for high times, thus defining a classical diffusive process. We thus recover the behaviour encountered earlier, in the calculation of moments (yet .

Gyro-phase averaging. Notice that, even for a gyro-tropic initial state, a dependence on the gyro-phases α, β is induced by cross-V-X coupling (see Ξ). Nevertheless, it seems appropriate to consider an average over these angle variables, since we will be interested in the evolution in time of angle-*independent* observable quantities. For this, notice that the corresponding factor is of the form $e^{a\,\sin(\alpha-\beta+\chi)}$, where $a = 4\;\Omega^{-1}\,\tilde{\beta}_\perp^{(X)}(t)\,\tilde{\beta}_\perp(t)\,\xi(t)\,v_\perp\,\rho$. Therefore, we shall use (10.58) once more, in order to calculate:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{a\,\sin(\alpha-\beta+\chi)} &= \frac{1}{2\pi}\int_0^{2\pi} d\alpha \sum_{n=-\infty}^{\infty} J_n(ia)\,e^{-in\,(\alpha-\beta+\chi)} \\
&\equiv \frac{1}{2\pi}\int_0^{2\pi} d\alpha \sum_{n=-\infty}^{\infty} I_n(a)\,e^{-in\,(\alpha-\beta+\chi)}
\end{aligned}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} I_n(a)\, e^{in\,(\alpha-\beta+\chi)}\,(2\pi)\,\delta^{Kr}_{n,0}$$
$$= I_0(a) \qquad (10.100)$$

so the factor containing the angle variables in the final (expanded) form of (10.98) may be replaced by[17] $(2\pi)^2\, I_0(4\;\Omega^{-1}\,\tilde{\beta}^{(X)}_{\perp}(t)\,\tilde{\beta}_{\perp}(t)\,\xi(t)\,v_{\perp}\,\rho)$.

10.9.3 Final expression for a $\delta-$function for $f_0(\rho)$.

Let us assume that the initial state is given by[18]:

$$f_0^{(X)}(\rho', z) = \frac{1}{2\pi\rho'}\,\delta(\rho' - \rho_0)\,\delta(z) \qquad (10.101)$$

The distribution function at any instant t is then given by (10.98), where the last line, say $I_{\rho'}$, is to be replaced by:

$$I_{\rho'} = e^{-\tilde{\beta}^{(X)}_{\perp}(t)\rho_0{}^2}\; I_0\left(2\,\tilde{\beta}^{(X)}_{\perp}(t)\,\Xi(v_{\perp}, \rho;\, \Omega;\, t)\,\rho_0\right)$$

The initial position of the particle can be set at $\rho_0 = 0$, for instance, so this expression simply reduces to a factor 1 (one).

10.9.4 FINAL expression for $f(t)$ with Maxwellian $f_0(\rho)$ - analysis

Let us now assume that the system was initially at a Maxwellian spatial distribution:

$$f_0^{(\perp)}(\rho') = \frac{\Gamma}{\pi}\,e^{-\Gamma\,\rho'^2} \qquad (10.102)$$

The integration in the last line in (10.98), say $I_{\rho'}$ (cf. previous paragraph), can now be carried out according to the Bessel property (10.92) (set $b = 0$, $p = 0$ therein). The final result reads:

$$I_{\rho'} = \frac{\Gamma}{\Gamma + \tilde{\beta}^{(X)}_{\perp}(t)}\; e^{+\frac{[\tilde{\beta}^{(X)}_{\perp}(t)]^2}{\Gamma + \tilde{\beta}^{(X)}_{\perp}(t)}\,\Xi^2}$$

so the FINAL expression takes the elegant form (after a brief manipulation of the second part in (10.98)):

$$f^{(Max)}(\mathbf{x}, \mathbf{v};\, t) = f^{(Max)}_{\perp}(\mathbf{x}_{\perp}, \mathbf{v}_{\perp};\, t)\;\; f^{(Max)}_{\parallel}(v_{\parallel};\, t) \qquad (10.103)$$

[17] To see this, expand Ξ^2 therein, following (10.99)

[18] Once more, verify that $\int d\mathbf{x}'\, f_0(\mathbf{x}') = \int_0^{\infty} d\rho'\,\rho' \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\beta'\, f_0^{(X)}(\rho', z) = 1$.

where

$$\begin{aligned} f_{\parallel}^{(Max)}(v_{\parallel};\, t) &= \frac{1}{\pi^{1/2}}\, \tilde{\beta}_{\parallel}^{1/2}\, e^{-\tilde{\beta}_{\parallel} v_{\parallel}^2} \\ f_{\perp}^{(Max)}(\mathbf{x}_{\perp}, \mathbf{v}_{\perp};\, t) &= \frac{\tilde{\beta}_{\perp}(t)}{\pi}\, e^{-\tilde{\beta}_{\perp}(t)\, v_{\perp}^2} \\ &\quad \times \; \frac{\tilde{\gamma}_{\perp}(t)}{\pi}\, e^{-\tilde{\gamma}_{\perp}(t)\, \Xi^2(v_{\perp}, \rho;\, \Omega;\, t)} \end{aligned} \tag{10.104}$$

where

$$\tilde{\gamma}_{\perp}(t) = \frac{\Gamma\, \tilde{\beta}_{\perp}^{(X)}(t)}{\Gamma + \tilde{\beta}_{\perp}^{(X)}(t)} \qquad i.e. \qquad \frac{1}{\tilde{\gamma}_{\perp}(t)} = \frac{1}{\Gamma} + \frac{1}{\tilde{\beta}_{\perp}^{(X)}(t)} \tag{10.105}$$

and all the other quantities were defined in (10.99).

This form describes a Maxwellian distribution in both velocity and real (position) space. One may notice the modification in the effective width of the former, due to the latter, and also the coupling between the two, as a consequence of the magnetic field; to see this, use (10.99) in order to expand the above form (10.104b) for $f_{\perp}$ to:

$$\begin{aligned} f_{\perp}^{(Max)}(\mathbf{v}_{\perp}, \mathbf{x}_{\perp};\, t) = \frac{\tilde{\beta}_{\perp}(t)}{\pi}\, \frac{\tilde{\gamma}_{\perp}(t)}{\pi}\, e^{-\tilde{\beta}_{\perp}(t)\, v_{\perp}^2}\, e^{-\tilde{\gamma}_{\perp}(t)\, \rho^2} \\ e^{-\tilde{\gamma}_{\perp}(t) \frac{4\, \tilde{\beta}_{\perp}^2(t)\, \xi^2(t)}{\Omega^2}\, v_{\perp}^2}\, e^{\tilde{\gamma}_{\perp}(t) \frac{4\, \tilde{\beta}_{\perp}(t)\, \xi(t)}{\Omega}\, v_{\perp}\, \rho\, \sin(\alpha - \beta + \chi(t))} \end{aligned} \tag{10.106}$$

All time-dependent parameters here are actually function of $\tau = \gamma_{\perp} t$; see (10.99).

A few comments are in row.

1. Remember that the effective width of a Maxwellian in the form: $f(x) \sim e^{-ax^2} \equiv e^{-x^2/2x_0^2}$ is related to the constant $x_0 = (2a)^{-1/2}$ appearing in the exponent: the higher x_0 (the lower a) the wider the curve will be (since the MSD $\langle x^2 \rangle$ is here equal to a^2). Therefore, we see that the quantity $\tilde{\beta}_*(t)$ ($* = \perp, \parallel$) appearing in the velocity distribution, may be interpreted as, say, $1/2V_{th}^2(t)$, where $V_{th}(t)$ is a time-dependent thermal velocity which relaxes to its equilibrium value $v_{th}^0 = \lim_{t \to \infty} V_{th}(t) = (2\beta_*^0)^{-1/2}$ after a few times the time constant γ_*^{-1}. This behaviour is depicted in figure 10.3.

2. A similar comment can be done in the spatial part of the distribution (2nd exponential in the rhs of (10.106)): $\tilde{\gamma}_{\perp}(t)$ may be interpreted as, say, $1/2L^2(t)$, where $L(t)$ denotes the effective width of the Gaussian curve describing probability distribution in space (actually a measure of the lateral random walk of the particle). Notice that $L(t)$ will depend on the magnitude of the magnetic field (through Ω). Also note that a time dependence persists even for longer times (above the relaxation time γ_*^{-1}); in order to see this, remember the structure of

$\tilde{\beta}_\perp^{(X)}(\tau)$ (see (10.99)):

$$\tilde{\beta}_\perp^{(X)}(\tau) = \Omega^2 \, \beta_\perp^0 \, \hat{a}^{-1}(\tau) \equiv \frac{\Omega^2 \, \beta_\perp^0}{[a(\tau) + a' \, t]}$$

where $a(\tau)$ varies in time, relaxing towards θ^{-1} as time goes by, while $a' = 4\beta_\perp^0 \, Q$ is constant. Therefore, the characteristic length $L(t)$, defined through (10.105):

$$2L^2(t) \equiv \tilde{\gamma}_\perp^{-1}(t) \; = \frac{1}{\Gamma} + \frac{1}{\tilde{\beta}_\perp^{(X)}(t)} = \frac{1}{\Gamma} + \frac{[a(\tau) + a' \, t]}{\Omega^2 \, \beta_\perp^0} \tag{10.107}$$

will behave, for times $t \gg \gamma_\perp^{-1}$, as:

$$2L_{as}^2(t) \; = \frac{1}{\Gamma} + \frac{[a_{as}(\tau) + a' \, t]^{1/2}}{\Omega^2 \, \beta_\perp^0} = ... = \frac{1}{\Gamma} + \frac{\left[1 + 4\beta_\perp \; Q \, t\right]}{\Omega^2 \, \beta_\perp} \tag{10.108}$$

(*'as'* = *'asymptotic'*); remember that Γ (β) are related to the space (velocity) distribution at $t = 0$: a high value of Γ implies space localization of the particle, while a high value of β denotes low initial temperature T_0. In our case, we see that $L(t)$ grows with time (practically as $\sim t^1$ after some time), yet this growth is sort of suppressed by the field: the higher the value of Ω, the lower the value of L (in fact $L \sim \Omega^{-1}$ for high times).

3. The 3rd exponential in the *rhs* of (10.106) represents a modification of the time-dependent velocity distribution width (temperature) due to the magnetic field; combining with the first exponential, we see that the inverse temperature $\tilde{\beta}_\perp(t)$ now takes the modified value:

$$\begin{aligned}
\hat{\beta}_\perp(\tau) \;\; &= \;\; \tilde{\beta}_\perp(\tau) \left[1 + 4 \frac{\tilde{\beta}_\perp(\tau) \, \tilde{\gamma}_\perp(\tau) \, \xi^2(\tau)}{\Omega^2} \right] \\
&= \;\; ... \\
&= \;\; \tilde{\beta}_\perp(\tau) \left\{ 1 + \frac{4 \, \Gamma \, \beta_\perp^0}{\Gamma[a(\tau) + a' \, t]^{1/2} + \Omega^2 \, \beta_\perp^0} \, \tilde{\beta}_\perp(\tau) \, \xi^2(\tau) \right\}
\end{aligned} \tag{10.109}$$

Nevertheless, this modification disappears at infinite time (and also for an infinite Ω, which appears *only* in the denominator).

4. The 4th exponential in the *rhs* of (10.106) represents coupling between the space and velocity distributions. The corresponding factor is of the form, say: $e^{a(t) \, v_\perp \, \rho \, \sin(\alpha - \beta + \chi(t))}$ where:

$$a(t) = \frac{4 \, \tilde{\gamma}_\perp(t)}{\Omega} \, \tilde{\beta}_\perp(\tau) \, \xi(\tau) = ... = \frac{4 \, \Gamma \, \Omega \, \beta_\perp^0}{\Gamma[a(\tau) + a' \, t] + \Omega^2 \, \beta_\perp^0} \, \tilde{\beta}_\perp(\tau) \, \xi(\tau) \tag{10.110}$$

and χ was defined previously; remember that χ tends to $\pi/2$ at large times. Nevertheless, this factor also disappears at infinite time (and also for an infinite Ω), since a tends to zero.

In addition, as mentioned above, this factor seems to point out the appearance of an angular dependence of $f(t)$ (even if $f(0)$ is gyrotropic, i.e. independent of the angle variables i.e. α or β); this is due to velocity-position coupling through the rotating charged particle motion inside a magnetic field. However - as indicated previously - averaging over (one of) the angle variables, we obtain a factor of: $I_0(a(t)\, v_\perp\, \rho)$. Once more, notice that $\lim_{t\to\infty} a(t) = 0$, so $\lim_{t\to\infty} I_0(a(t)\, v_\perp\, \rho) = \lim_{x\to\infty} I_0(x) = 1$.

5. The final expression for $f(t)$ now allows the calculation of quantities like[19] e.g. $\langle x^2 + y^2 \rangle = \langle \rho^2 \rangle$ and $\langle v_x^2 + v_y^2 \rangle = \langle v_\perp^2 \rangle$. The calculation will not be displayed here since it simply recovers the behaviour mentioned earlier. For the mean square velocity one exactly recovers: $\langle v_\perp^2 \rangle = \beta_\perp^{-1}(t)$, in agreement with our previous result, while the mean square displacement $\langle \rho^2 \rangle$ is again found to behave as $\sim t^1$ for high values of time[20].

10.10 Calculation of particle density $n(\mathbf{x},\, t)$

Recalling expression (10.71) (or (10.72)) for $f(t)$, we may end this chapter by calculating an expression for the mean particle density in space:

$$n(\mathbf{x},\, t) = \int d\mathbf{v}\, f(\mathbf{x}, \mathbf{v};\, t)$$

For simplicity, we shall use (10.72) as a starting point, neglecting the direction $\|$ to the field[21], and will consider a particle initially located at a specific point in phase space:

$$f(\mathbf{x}', \mathbf{v}';\, t) = \delta(x' - x_0)\, \delta(y' - y_0)\, \delta(v_x' - v_x^0)\, \delta(v_y' - v_y^0)$$

The integration over $\mathbf{x}'$, $\mathbf{y}'$ in (10.72) then plainly results in shifting those variables to $\mathbf{x_0}$, $\mathbf{v_0}$ respectively[22]. Now, the integration over velocities (see the

[19]In general, a Gaussian expression for $f(x)$ enables the calculation of moments via the known expressions:

$$\int_0^\infty dx\, x^n\, e^{-bx^2} = \begin{cases} \frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{2^{k+1}} \frac{\sqrt{\pi}}{b^{k+1/2}} & n = 2k \\ \frac{k!}{2b^{k+1}} & n = 2k+1 \end{cases}.$$

[20]Nevertheless, the exact expression found for $\langle \rho^2 \rangle_t$ here differs *a little* from previously. Remember that a Maxwellian space distribution was considered in this section, while a $\delta-$function was taken in a previous one.

[21]The solution is already known in this direction - see the previous sections - and integrating over z would make no difference, since we have assumed that $\partial f/\partial z = 0$.

[22]Let us remark that, assuming this initial condition right from the beginning, i.e. in (10.72), and simply working in cartesian coordinates, would have provided a solution much faster to calculate, than the one in the previous section. The tedious calculation will not be provided here. Let us only mention that one thus obtains a product of (decoupled) integrals of a standard form. The final result confirms our conclusions on the asymptotic behaviour of both velocity and displacement.

above definition for n) will simply shift $\mathbf{k_v}$ to zero everywhere[23]. The result now reads:

$$n(\mathbf{x},\,t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1 x_0}\, e^{ik_2 y_0}$$
$$e^{-i(M_{VX})_\perp(-t)(k_1 v_x^0 + k_2 v_y^0)}\, e^{-i(M_{VX})_\angle(-t)(k_1 v_y^0 - k_2 v_x^0)}\, e^{-(\Lambda_{XX})_\perp(t)(k_1^2+k_2^2)} \tag{10.111}$$

The two integrations can now be carried out independently, using the general result:

$$\int_{-\infty}^{\infty} e^{-bx^2}\, e^{iax}\, dx = \frac{\sqrt{\pi}}{\sqrt{b}}\, e^{-a^2/4b} \qquad (b > 0)$$

The final result is the Maxwellian:

$$n(\mathbf{x}_\perp,\,t) = \frac{1}{2\pi}\frac{1}{L^2}\, e^{-[x-X(t)]^2/2L^2}\, e^{-[y-Y(t)]^2/2L^2} \tag{10.112}$$

where $L^2 = L^2(t) = 2\,(\Lambda_{XX})_\perp(t) \equiv \Xi_{11}(t)$ (see §10.51 above):

$$L^2(t) \;=\; 2\,\Omega^{-2}\,[\lambda_\perp(t)\,D_\perp + 2\,Q\,t] = \frac{T_\perp^0}{m\Omega^2}\left[1 - e^{-2\gamma_\perp t} + \frac{4\,mQ}{T_\perp^0}\,t\right] \approx \frac{4\,Q}{\Omega^2}\,t \tag{10.113}$$

and the mean values $X(t) = \langle x\rangle$, $Y(t) = \langle y\rangle$ are *exactly* as given in (10.47). We therefore exactly recover the classical diffusion mechanism anticipated earlier, in the plane perpendicular to the magnetic field.

We may now advance a little further, by averaging out the gyrophase ϕ. By expanding the exponent as:

$$\chi^2 \equiv [x - X(t)]^2 + [y - Y(t)]^2 = (x^2 + y^2) - 2\,X\,x - 2\,Y\,y + (X^2 + Y^2)$$
$$= \rho^2 - 2\,X\,\rho\,\cos\phi - 2\,Y\,\rho\sin\phi + R_0^2$$

where we set: $x = \rho\,\cos\phi\,,\quad y = \rho\,\sin\phi$ and

$$R_0 = [M_\perp(t)^2 + M_\angle(t)^2]^{1/2}\,(v_x^2 + v_y^2) = [\det\mathbf{M_{VX}}(t)]^{1/2}\,v_\perp^0 \equiv \alpha_0\,v_\perp^0$$

we obtain[24]:

$$f(\rho,\,t) = \int_0^{2\pi} d\phi\, n(\mathbf{x}_\perp, t) = \frac{1}{2\pi}\frac{1}{L^2(t)}\, e^{-\rho^2/2L^2(t)}\, e^{-R_0^2(t)/2L^2(t)}\, I_0\left(\frac{R_0(t)\,\rho}{L^2(t)}\right) \tag{10.114}$$

---

[23] i.e.

$$\int dv_i \int dk_i\, e^{ik_i v_i}\, F(k_i) = \int dk_i\, \delta(k_i)\, F(k_i) = F(0)$$

for an arbitrary function $F(k_i)$.

[24] We have used:

$$\int_0^{2\pi} e^{-a\cos\phi}\, e^{-b\sin\phi} d\phi = \int_0^{2\pi} e^{-P\cos(\phi-\theta)}$$

($P = a^2 + b^2$) so transforming the exponential by using the Bessel function identity (6.9), the integration is carried out yielding $(2\pi)\, I_0(P)$.

This is the final result for the radial particle distribution. Note that:

(i) R_0 is given by[25]:

$$R_0 = \frac{v_\perp^0}{\Omega}[1 + e^{-2\gamma_\perp t}(1 - 2\cos\Omega t)]^{1/2} \tag{10.115}$$

so $R_0 \approx \frac{v_\perp^0}{\Omega} \equiv \rho_L^0$, after a while. Also, for zero initial velocity, $R_0 = 0$ (and $e^0 = I_0(0) = 1$) so a pure Gaussian profile is recovered. On the other hand, for a higher $v_\perp^0$, the distribution spreads in space, as expected.

(ii) Remember that $L^2(t)$ asymptotically behaves as $\sim (t^1 + cst.)$. Therefore, at infinite time, $I_0(x) \approx I_0(0) = 1$.

(iii) The distribution may be expressed in terms of the re-scaled variables: $r = \rho/L(t)$ and $R = R_0/L(t)$, so that:

$$f(r(t), R(t)) = \frac{1}{2\pi}\frac{1}{L^2}\, e^{-r^2/2}\, e^{-R^2/2}\, I_0(R\,r) \tag{10.116}$$

The free variable $r(t)$ and the parameter $R(t)$ (determined by the initial velocity and the magnetic field) are treated on the same basis, in this expression. For $\gamma_\perp t \gg 1$ we obtain, approximately: $\rho/L(t) \approx 2\Omega\rho/(Qt)^{1/2}$, $R_0/L(t) \approx \frac{v_\perp^0}{(T_\perp^0/m)^{1/2}}$ and $L^2(t) \approx (4Qt/\Omega^2)^{1/2}$ so:

$$f(t) = \frac{1}{2\pi}\frac{\Omega^2}{4Qt}\, e^{-m v_\perp^{0\,2}/2T_\perp^0}\, e^{-\Omega^2\rho^2/8Qt} \tag{10.117}$$

so the distribution is a Maxwellian spreading in time, since it is characterized by a mean-square displacement $\Delta\rho^2 = \frac{4Qt}{\Omega^2}$.

10.11 Conclusion

Considering an approximate form of the plasma *FPE* derived in a previous chapter, we have been able to obtain a closed expression for the distribution function f and follow its evolution in time. Regardless of the initial condition, it was found to relax to a final state of thermal equilibrium quite fast, practically after a few time constants γ_*^{-1}.

The calculation of moments gives a (time-dependent) mean square velocity which relaxes to T_{eq}/m at equilibrium (as expected), and a mean-square displacement which grows *linearly* with time t for large times. This fact suggests classical diffusion $\perp$ to the field.

The influence of the magnetic field seems to be rather limited, since it only affects the asymptotic value of the *MSD*.

[25] This is true for the general Case II only. In Case III:

$$R_0(t) = \frac{v_\perp^0}{(\Omega^2 + \gamma_\perp^2)^{1/2}}[(1 - e^{-2\gamma_\perp t})^{1/2} \quad .$$

Chapter 11

Evolution equations for velocity moments

Summary

An analytical theory is suggested and briefly discussed, for the study of the evolution in time of observable local densities and, in particular, moments of the test-particle velocity.

When you follow two separate chains of thought, Watson,
you will find some point of intersection
which should approximate the truth.
(Sherlock Holmes)

`Sir Arthur Conan Doyle`
in *The Disappearance of Lady Francis Carfax*

11.1 Introduction

We have previously derived the explicit form of a Markovian Fokker-Planck- (*FP*-) type kinetic operator for magnetized plasma. In this chapter, we will present the set of equations describing the evolution in time of *velocity moments* (*fluid variables*) under the action of this operator.

The idea and general strategy relies on the formalism presented in reference [5][1], and may be viewed as an adaptation of that methodology to the test-particle formalism - as described by a (linear) *FPE*. Notation and terminology used therein will be adopted here, for convenience.

Throughout this chapter, we will use the notation q_r ($r = 1, 2, 3$) for the microscopic variable denoting particle position, in distinction from x_r denoting the macroscopic space variable where an observable quantity $B(\mathbf{x}, ...)$ is evaluated.

11.2 Prerequisites - structure of the kinetic equation

The kinetic equation(s) derived above describe the evolution of a probability distribution function $f = f(\mathbf{q}, \mathbf{v};\, t)$; such an equation obeys the generic form[2]:

$$\frac{\partial f}{\partial t} = \Phi^{\alpha}\{f\} + \mathcal{F}^{\alpha}\{f\} + \mathcal{K}^{\alpha}\{f\} \tag{11.1}$$

where the three terms in the *rhs* denote[3]:

- a *convective* (free flow) term:

$$\Phi^{\alpha}\{f\} = -v_m \, \frac{\partial}{\partial q_m}$$

- a *forcing* (external field) term:

$$\mathcal{F}^{\alpha}\{f\} = -\frac{e_\alpha}{m_\alpha} \left[E_r(\mathbf{x},\, t) + \frac{1}{c} \epsilon_{rmn} v_m B_n(\mathbf{x},\, t) \right] \frac{\partial}{\partial v_r}$$

where ϵ_{ijk} obviously denotes the Levi-Civita tensor (of rank 3)[4]; in magnetized plasma, in specific:

$$\mathcal{F}^{\alpha}\{f\} = -s_\alpha \, \Omega_\alpha \, \epsilon_{rm3} v_m \, \frac{\partial}{\partial v_r}$$

- the *collision term*: $\mathcal{K}^{\alpha}\{f\}$. In our case, $\mathcal{K}$ is a second-order parabolic differential operator, bearing the following structure:

$$\mathcal{K}^{*}\{\,\cdot\,\} \quad = \quad \frac{\partial^2}{\partial v_r \, \partial v_s} D^{(VV)}_{rs} \; \cdot \; - \frac{\partial}{\partial v_r} \mathcal{F}^{(V)}_{r} \; \cdot$$

[1] See Chapter 43 in reference [5].

[2] Actually, this is true for all equations mentioned in this text.

[3] Remember that $\mathcal{K}$ consist of a sum of terms over different species' α'; keep in mind that α denotes t.p. species, while α' denotes reservoir particle species, in this Chapter.

[4] $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ and all other elements are zero.

$$
\begin{aligned}
&+ \quad \frac{\partial^2}{\partial q_r\, \partial q_s} D^{(XX)}_{rs} \cdot - \frac{\partial}{\partial q_r} \mathcal{F}^{(X)}_r \cdot \\
&+ \quad \frac{\partial^2}{\partial v_r\, \partial q_s} D^{(VX)}_{rs} \cdot \\
&= \quad \mathcal{K}^*_{VV} + \mathcal{K}^*_{XX} + \mathcal{K}^*_{VX}
\end{aligned} \tag{11.2}
$$

where all definitions are obvious; the superscript '*' will be used to distinguish an operator from all others, i.e. $* = \Phi, \Theta, L$ (= Landau) etc. Obviously,

$$
\mathcal{K}\, f(\mathbf{v};\, t) = \mathcal{K}_{VV}\, f(\mathbf{v};\, t) \ , \qquad \mathcal{K}\, f(\mathbf{q};\, t) = \mathcal{K}_{XX}\, f(\mathbf{q};\, t)
$$

so a reduced collision term takes over e.g. in the homogeneous case (as discussed above).

Remember that all coefficients come out to be function of $\mathbf{v}$ (only, i.e. not $\mathbf{q}$). Therefore, integrating over position $\mathbf{q}$, one obtains:

$$
\int d\mathbf{q} \mathcal{K}\, f(\mathbf{q}, \mathbf{v};\, t) = \mathcal{K}_{VV} \int d\mathbf{q}\, f(\mathbf{q}, \mathbf{v};\, t) \equiv \mathcal{K}_{VV}\, f_{loc}(\mathbf{v};\, t)
$$

where we defined the local distribution function:

$$
f_{loc}(\mathbf{v};\, t) = \int d\mathbf{q}\, f(\mathbf{q}, \mathbf{v};\, t)\, . \tag{11.3}
$$

In the following, we will keep full generality in analytical expressions, in order to point out the particular aspects of this theory for velocity moments, based on the Fokker-Planck equation. The multiple-species notation will be kept, and general expressions will refer to an *em* field $\mathbf{E}$ and/or $\mathbf{B}$. Nevertheless, don't forget that our purpose is limited to the case: $E_r = 0$, $B_r = B\, \delta_{r3}$ ($B = const.$).

11.3 Evolution of observables

In kinetic-transport theory, one is interested in studying the evolution in time of *macroscopic* observable quantities like, say, $B(\mathbf{x},\, t)$ which are related to functions of microscopic variables, say $b(\mathbf{q}, \mathbf{v},\, t)$[5]. B is conventionally defined as the average value of b:

$$
B = \int d\mathbf{q} \int d\mathbf{v}\, b\, f \tag{11.4}
$$

so that their time evolution is described by an equation of the form:

$$
\frac{\partial B}{\partial t} = \frac{\partial}{\partial t} \int d\mathbf{q} \int d\mathbf{v}\, bf = \int d\mathbf{q} \int d\mathbf{v}\, \frac{\partial b}{\partial t} f = \int d\mathbf{q} \int d\mathbf{v}\, b\, \frac{\partial f}{\partial t} = \ldots \tag{11.5}
$$

and substituting from (11.1) we may compute the exact form of the evolution equation for a given quantity. A well-known postulate of Statistical Mechanics [4] was used to pass from the third to the fourth step, in this relation.

[5] No need to point out that B should not be mistaken for the norm of the magnetic field, taken to be constant throughout this text.

11.3.1 Local densities

According to a widely used scheme in macroscopic theories, one is interested in the evolution of *local densities*, i.e. macroscopic quantities defined locally at some point of real space $\mathbf{x}$; one thus considers microscopic variables of the form [5], [84]:

$$b = \beta\, \delta(\mathbf{q} - \mathbf{x}) \tag{11.6}$$

so the above evolution equation for B becomes:

$$\begin{aligned} \frac{\partial B}{\partial t} &= \int d\mathbf{v} \int d\mathbf{v}\, b\, \frac{\partial f}{\partial t} = \int d\mathbf{v}\, \beta\, \frac{\partial f_{loc}}{\partial t} \\ &= \int d\mathbf{v}\, \beta \left(\Phi\{f_{loc}\} + \mathcal{F}\{f_{loc}\} + \mathcal{K}\{f_{loc}\} \right) \end{aligned} \tag{11.7}$$

where we used definition (11.3). Of course, the operators need now to be redefined, by formally replacing q_r by x_r in all space gradients. The index *'loc'* will henceforth be dropped where obvious.

11.4 Moments - definitions

Setting β equal to 1, we obtain the *particle number density*:

$$\int d\mathbf{v}\, f(\mathbf{x}, \mathbf{v};\, t) \;=\; n_\alpha(\mathbf{x}, t) \tag{11.8}$$

Setting β equal to the velocity component v_r, we obtain the *average particle velocity* in the same direction:

$$\int d\mathbf{v}\, v_r\, f(\mathbf{x}, \mathbf{v};\, t) \;=\; n_\alpha(\mathbf{x}, t)\, u_r^\alpha(\mathbf{x}, t) \tag{11.9}$$

It is easy to show that

$$\int d\mathbf{v}\, (v_r - u_r)\, f \;=\; 0 \tag{11.10}$$

Now, setting β equal to $\frac{1}{3} m_\alpha\, (v_r - u_r)\, (v_m - u_m)$, we obtain the average excess kinetic energy, described by the *total pressure* tensor:

$$m_\alpha \int d\mathbf{v}\, (v_r - u_r)(v_m - u_m)\, f(\mathbf{x}, \mathbf{v};\, t) \;=\; P_{rm}^\alpha(\mathbf{x}, t) \tag{11.11}$$

Remember that P_{rm} can be decomposed as:

$$P_{rm}^\alpha = P_\alpha\, \delta_{rm} \;+\; \pi_{rm} \tag{11.12}$$

thus defining the *scalar pressure* (related to the diagonal part) and the (traceless symmetric) *dissipative pressure* tensor. The trace of $\mathbf{P}$ is related to the *internal energy density*:

$$\frac{1}{2} \sum_r P_{rr}^\alpha = n_\alpha(\mathbf{x}, t)\, \mathcal{E}_\alpha(\mathbf{x}, t) \tag{11.13}$$

i.e. to the *'local temperature'* $T_\alpha(\mathbf{x}, t)$:

$$\frac{1}{3} \sum_r P^\alpha_{rr} = P_\alpha = n_\alpha(\mathbf{x}, t)\, T_\alpha(\mathbf{x}, t) \tag{11.14}$$

(the constants were chosen so that the equation of state of a perfect gas is recovered: $\mathcal{E}_\alpha = \frac{3}{2}\, k_B\, T_\alpha$). Moments of higher order e.g. the *heat flux density*:

$$m_\alpha \int d\mathbf{v}\, (v_r - u_r)|\mathbf{v} - \mathbf{u}^\alpha|^2\, f(\mathbf{x}, \mathbf{v};\, t) \;=\; q^\alpha_{rm}(\mathbf{x}, t) \tag{11.15}$$

can be defined, yet will be of little importance in what will follow.

It should be underlined that this is *not* a truly macroscopic theory: quantities defined here do not refer to the bulk plasma, as a whole, but to a small population injected in it. As a matter of fact, the background plasma is taken to be in homogeneous equilibrium, so only this 'astray' population of a (or *a few*) test-particle(s) is distributed in a non-uniform manner. Terms introduced above (e.g. pressure, heat flux etc.) should therefore be used *'in quotation'*. Nevertheless, this formalism correctly predicts mean values of velocity moments and is adequate for the study of their dependence on the magnetic field via the modified collision term, which is our scope here.

11.5 Moment equations

Combining the above definitions with the general evolution equation (11.7) and explicitly substituting with definitions of all operators therein, we obtain a set of *coupled* temporal evolution equations for the above quantities.

11.5.1 Plasma-dynamical balance equations

The equation of particle number conservation reads:

$$\frac{\partial n_\alpha}{\partial t} \;=\; -\frac{\partial}{\partial x_r}(n_\alpha\, u^\alpha_r) \tag{11.16}$$

The equation for the mean velocity is:

$$\frac{\partial}{\partial t}(m_\alpha\, n_\alpha\, u^\alpha_r) \;=\; -\frac{\partial}{\partial x_m}(m_\alpha\, n_\alpha\, u^\alpha_r\, u^\alpha_m + P_{rm}) + e_\alpha\, n_\alpha\, (E_r + \frac{1}{c}\epsilon_{rmn}\, u^\alpha_m\, B_n) + \mathcal{R}^\alpha_r \tag{11.17}$$

In detail: in the $x-$ direction[6] we obtain:

$$\frac{\partial}{\partial t}(m_\alpha\, n_\alpha\, u^\alpha_x) \;=\; -\frac{\partial}{\partial x}(m_\alpha\, n_\alpha\, u^{\alpha\,2}_x + P_\alpha + \pi_{11}) - \frac{\partial}{\partial y}(m_\alpha\, n_\alpha\, u^\alpha_x\, u^\alpha_y + \pi_{12})$$
$$-\frac{\partial}{\partial z}(m_\alpha\, n_\alpha\, u^\alpha_x\, u^\alpha_z + \pi_{13}) + n_\alpha\, (e_\alpha\, E_x + m_\alpha\, s\, \Omega_\alpha\, u^\alpha_y) + \mathcal{R}^\alpha_x \tag{11.18}$$

[6]Do not forget that: $B_n = B\, \delta_{n3}$.

The analogous equation in the $y-$ direction is readily obtained by setting $x \leftrightarrow y$, $1 \to 2$ *and* $\Omega \to -\Omega$ (since $\epsilon_{213} = -\epsilon_{123} = -1$) in this equation; finally, in the $z-$ direction, the magnetic field disappears:

$$\frac{\partial}{\partial t}(m_\alpha \, n_\alpha \, u_z^\alpha) = -\frac{\partial}{\partial x}(m_\alpha \, n_\alpha \, u_x^\alpha \, u_z^\alpha + \pi_{31}) - \frac{\partial}{\partial y}(m_\alpha \, n_\alpha \, u_y^\alpha \, u_z^\alpha + \pi_{32}) - \frac{\partial}{\partial z}(m_\alpha \, n_\alpha \, u_z^{\alpha 2} + n_\alpha T_\alpha + \pi_{33}) + e_\alpha \, n_\alpha \, E_z + \mathcal{R}_3^\alpha \tag{11.19}$$

The mean *friction force* appearing in these equations is defined as:

$$\mathcal{R}_r^\alpha(\mathbf{x};\, t) = m_\alpha \int d\mathbf{v} \, v_r \, \mathcal{K}\{f(\mathbf{x}, \mathbf{v};\, t)\} \tag{11.20}$$

$(r = 1, 2, 3 \equiv x, y, z)$.

The evolution equation for the *local temperature* T reads:

$$n_\alpha \frac{\partial T_\alpha}{\partial t} = -n_\alpha u_r^\alpha \frac{\partial T_\alpha}{\partial x_r} - \frac{2}{3} n_\alpha T_\alpha \frac{\partial u_r^\alpha}{\partial x_r} - \frac{2}{3} \pi_{mn}^\alpha \frac{\partial u_n^\alpha}{\partial x_m} - \frac{2}{3} \frac{\partial q_m^\alpha}{\partial x_m} + \frac{2}{3} \mathcal{Q}^\alpha \tag{11.21}$$

The *collisional heat-exchange rate* appearing in this equation is given by:

$$\mathcal{Q}^\alpha = \frac{1}{2} m_\alpha \int d\mathbf{v} \, |\mathbf{v} - \mathbf{u}^\alpha|^2 \, \mathcal{K}\{f(\mathbf{x}, \mathbf{v};\, t)\} \tag{11.22}$$

These evolution equations seems somewhat standard. However, the new aspect lies in the appearance of the *collisional* terms $\mathcal{R}_r^\alpha$, $\mathcal{Q}^\alpha$; since these are related to our *new* (cylindrical-symmetric) collision term $\mathcal{K}^{(\Phi)}\{f\}$, they are now expected to give birth to *space diffusion* terms. We will come back to this subtle point in a little while.

11.5.2 'Hydrodynamical' balance equations

The average functions (moments) defined above referred to different species α, separately. An alternative way to study velocity moments is to define a set of 'hydrodynamical' quantities, taking into account all particles populations combined. For the sake of reference, we shall explicitly provide the corresponding equations; these results should form a concise starting point for future work on the computation of mean values as related to a test-particle problem.

Definitions

Let us define

- the *mass density* ρ:

$$\rho = \sum_\alpha m_\alpha \, n_\alpha \tag{11.23}$$

- the *charge density* σ:

$$\sigma = \sum_\alpha e_\alpha \, n_\alpha \tag{11.24}$$

- the *mean velocity* u_r:

$$u_r = \frac{1}{\rho} \sum_\alpha m_\alpha \, n_\alpha \, u_r^\alpha \tag{11.25}$$

- the *current density* j_r:

$$j_r = \sum_\alpha e_\alpha \, n_\alpha \, u_r^\alpha \tag{11.26}$$

If the case where *two* species e.g. $e - i$ are present, these expressions reduce to:

$$\begin{aligned} \rho &= m_e n_e + m_i n_i \\ \sigma &= -e \, n_e + Z_i \, e \, n_i \\ u_r &= \frac{1}{\rho} (m_e \, n_e \, u_r^e + m_i \, n_i \, u_r^i) \\ j_r &= -e \, n_e \, u_r^e + Z \, e \, n_i \, u_r^i \end{aligned} \tag{11.27}$$

Notice that the former two quantities are scalar ones, while the latter are vectors. The right-hand-sides of these equations show immediately how the 'new' variables are related to the 'old' ones in the previous section. In the next paragraph, we will see that these definitions provide a new set of equations in our (test-particle) case.

See that the first two equations form a 2×2 *Cramer* system:

$$\begin{aligned} \rho &= m_e n_e + m_i n_i \\ \sigma &= e_e n_e + e_i n_i \end{aligned}$$

Solving in terms of particle densities, we obtain:

$$\begin{aligned} n_e &= \frac{\rho \, e_i - \sigma \, m_i}{m_e \, e_i - e_e \, m_i} \approx \frac{\rho \, \frac{e_i}{m_i} - \sigma}{-e_e} = \frac{\rho \, \frac{+Ze}{m_i} - \sigma}{-(-e)} = \frac{\rho}{m_i} Z - \frac{\sigma}{e} \\ n_i &= -\frac{\rho \, e_e - \sigma \, m_e}{m_e \, e_i - e_e \, m_i} \approx \frac{\rho}{m_i} \end{aligned}$$

where we have used[7] the fact that $m_e \ll m_i$. Inspired by same idea, let us generalize these expressions by introducing the functions:

$$\begin{aligned} n_\alpha = g_\alpha(\sigma, \rho) &= \frac{\rho \, e_\beta - \sigma \, m_\beta}{m_\alpha \, e_\beta - m_\beta \, e_\alpha} \\ n_\beta = g_\beta(\sigma, \rho) &= \frac{-\rho \, e_\alpha + \sigma \, m_\alpha}{m_\alpha \, e_\beta - m_\beta \, e_\alpha} \end{aligned} \tag{11.28}$$

which will be used later on. These expressions refer to *any* two-species plasma[8].

[7] as suggested in [5], where the expressions finally obtained here appear just as they stand.

[8] Not necessarily $e - i$, that is; the idea is that a *t.p.* problem may refer to situations different from an electron moving against an ion background; see e.g. the discussion about α particle stopping in inertial fusion experiments in [10], [59], [79]. This latter picture is, in fact, closer to the original Brownian image of a heavy particle erratically moving against a light particle background.

In the same manner, the system of equations for **u**, **j** above gives:

$$\begin{aligned} n_\alpha\, u_r^\alpha = \tilde{g}_\alpha(\rho\, u_r,\, j_r) &= \frac{\rho\, e_\beta\, u_r - m_\beta\, j_r}{m_\alpha\, e_\beta - m_\beta\, e_\alpha} \\ n_\beta, u_r^\beta = \tilde{g}_\beta(\rho\, u_r,\, j_r) &= \frac{-\rho\, e_\alpha\, u_r + m_\alpha\, u_r}{m_\alpha\, e_\beta - m_\beta\, e_\alpha} \end{aligned} \tag{11.29}$$

or, as above:

$$\begin{aligned} n_e\, u_r^e = \tilde{g}_e &\approx \frac{Z}{m_i}\,\rho\, u_r - \frac{1}{e}\, j_r \\ n_i, u_r^i = \tilde{g}_i &\approx \frac{\rho}{m_i}\, j_r \end{aligned}$$

(so that: $u_r^i \approx \tilde{g}_r^i / g^i$, as in [5]).

Remember, once more, that α, β refer to *t.p.*- (and not R-plasma) species. In the simplest picture, of *one t.p.* species moving against a thermal background, cancelling β may seem appropriate in all relations in the two last sections.

Evolution equations

Combining the above, we obtain a set of balance equations for quantities defined here.

The evolution equation for the *mass density* is:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_r}(\rho\, u_r) \tag{11.30}$$

The equation for *momentum* is:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho\, u_r) = & -\frac{\partial}{\partial x_m}\Big\{\rho\, u_r\, u_m + \delta_{rm}\Big[g_\alpha(\sigma,\, \rho)\, T_\alpha + g_\beta(\sigma,\, \rho)\, T_\beta\Big] + \pi_{rm}^\alpha + \pi_{rm}^\beta\Big\} \\ & +\sigma\, E_r + \frac{1}{c} j_m\, \epsilon_{rmn}\, B_n) + \sum_\alpha \mathcal{R}_r^\alpha \end{aligned} \tag{11.31}$$

Of course, in our case: $E_r = 0$ and $j_m\, \epsilon_{rmn}\, B_n = j_m\, \epsilon_{rmn}\, B\, \delta_{n3} = \epsilon_{rm3}\, j_m\, B$, i.e. equal to $+j_2\, B$ if $r = 1$, $-j_1\, B$ if $r = 2$ and zero otherwise. Notice the last term:

$$\sum_\alpha \mathcal{R}_r^\alpha = \sum_\alpha m_\alpha \int d\mathbf{v}\, v_r\, \mathcal{K}_\alpha \tag{11.32}$$

This term cancels in 'traditional' transport theory, due to momentum conservation by the kinetic operator. As mentioned before, this is not of relevance in a test-particle problem.

The evolution equation for the *temperature* T reads:

$$g_\alpha\, \frac{\partial T_\alpha}{\partial t} = -\tilde{g}_r^\alpha \frac{\partial T_\alpha}{\partial x_r} - \frac{2}{3} g_\alpha T_\alpha \frac{\partial}{\partial x_r}\left(\frac{\tilde{g}_r^\alpha}{g^\alpha}\right) - \frac{2}{3}\pi_{mn}^\alpha \frac{\partial}{\partial x_m}\left(\frac{\tilde{g}_n^\alpha}{g^\alpha}\right) - \frac{2}{3}\frac{\partial q_m^\alpha}{\partial x_m} + \frac{2}{3}\mathcal{Q}^\alpha \tag{11.33}$$

and a similar equation holds for β instead of α.

Electrodynamic evolution equations

In addition to the above, we may obtain a balance equation for:

- the *charge density* σ:

$$\frac{\partial \sigma}{\partial t} = -\frac{\partial}{\partial x_r}(\rho\, j_r) \tag{11.34}$$

expressing conservation of electric charge;

- the electric current $\mathbf{j}$; by multiplying the two[9] eqs. (11.17) by e_β/m_β, e_α/m_α respectively, and summing up, we obtain a lengthy expression of the form:

$$\frac{\partial j_r}{\partial t} = \mathcal{H} + \mathcal{R}'_r \tag{11.35}$$

where the right-hand-side (*rhs*) consists of:
- $\mathcal{H}$: *exactly* equal to the *rhs* in (3.4.30) in [5] (actually the first 3 lines therein; we do not reproduce that lengthy expression here) and
- $\mathcal{R}'_r$: a new friction vector equal to:

$$\mathcal{R}'_r = \sum_\alpha \frac{e_\alpha}{m_\alpha}\mathcal{R}^\alpha_r = \sum_\alpha e_\alpha \int d\mathbf{v}\, v_r\, \mathcal{K}\{f(\mathbf{x},\mathbf{v};\, t)\} \tag{11.36}$$

(see (11.20)). Expression (11.35) corresponds to the *generalized Ohm's law*, as applied to a test-particle problem[10].

It is of course implied that these equations are assumed to be coupled with *Maxwell*'s equations:

$$\begin{aligned}
\frac{\partial E_r}{\partial x_r} &= 4\,\pi\,\sigma \\
\epsilon_{rms}\frac{\partial E_s}{\partial x_m} &= -\frac{1}{c}\,\frac{\partial B_r}{\partial t} \\
\frac{\partial B_r}{\partial x_r} &= 0 \\
\epsilon_{rms}\frac{\partial B_s}{\partial x_m} &= \frac{1}{c}\,\frac{\partial E_r}{\partial t} + \frac{4\,\pi}{c}\,j_r
\end{aligned} \tag{11.37}$$

Finally, expressions (3.4.32) through (3.4.35) derived in [5] for the *particle flux* Γ^α_r, the *heat flux* q^α_r and pressure, are valid *just as* they stand, so they will not be reproduced here. The difference we obtain in this case is in the form of the *collisional contribution*s appearing therein, which should now be computed by using the FPE collision term $\mathcal{K}^{(\Phi)}\{f\}$. For reference, they are:

$$\mathcal{R}^\alpha_r(\mathbf{x};\, t) \;=\; m_\alpha \int d\mathbf{v}\, v_r\, \mathcal{K}\{f(\mathbf{x},\mathbf{v};\, t)\}$$

---

[9] e.g. for species α and β.

[10] Again, if only electrons and heavy ions are present, $\mathcal{R}'_r$ reduces to $-\frac{e}{m_e}\,\mathcal{R}_r$, as in (3.4.30) in [5].

$$
\begin{aligned}
\mathcal{Q}^{\alpha}(\mathbf{x};\,t) &= \frac{1}{2}\,m_\alpha \int d\mathbf{v}\,|\mathbf{v}-\mathbf{u}^\alpha|^2\,\mathcal{K}\{f(\mathbf{x},\mathbf{v};\,t)\}\\
\mathcal{R}_r^{\alpha\,(3)}(\mathbf{x};\,t) &= \frac{1}{2}\,m_\alpha \int d\mathbf{v}\,|\mathbf{v}-\mathbf{u}^\alpha|^2\,(v_r-u_r^\alpha)\,\mathcal{K}\{f(\mathbf{x},\mathbf{v};\,t)\}\\
\mathcal{R}_{rs}^{\alpha\,(2)}(\mathbf{x};\,t) &= m_\alpha \int d\mathbf{v}\,(v_r-u_r^\alpha)\,(v_s-u_s^\alpha)\,\mathcal{K}\{f(\mathbf{x},\mathbf{v};\,t)\}\\
\mathcal{R}_{rs}^{\alpha\,(4)}(\mathbf{x};\,t) &= \frac{1}{2}\,m_\alpha \int d\mathbf{v}\,|\mathbf{v}-\mathbf{u}^\alpha|^2\,(v_r-u_r^\alpha)\,(v_s-u_s^\alpha)\,\mathcal{K}\{f(\mathbf{x},\mathbf{v};\,t)\}
\end{aligned}
\tag{11.38}
$$

11.6 The generalized frictions

As we said above, the new aspect in the above evolution equations, otherwise quite standard actually, lies in the appearance of the *collisional* terms $\mathcal{R}_r^\alpha$, $\mathcal{Q}^\alpha$, etc., summarized right above, which are related to our *new* (cylindrical-symmetric) collision term $\mathcal{K}^{(\Phi)}\{f\}$. Let us calculate the general form of these terms.

All these terms obey the general form (see (11.2)):

$$
\begin{aligned}
Q_\psi &= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\,\mathcal{K}\\
&= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\,\left[\mathcal{K}_{(VV)}+\mathcal{K}_{(XX)}+\mathcal{K}_{(VX)}\right]\\
&\equiv \left[Q_\psi^{(VV)}+Q_\psi^{(XX)}+Q_\psi^{(VX)}\right]
\end{aligned}
\tag{11.39}
$$

Recall (11.2) for definitions. Obviously ψ denotes a microscopic function, e.g. v_r for $Q_{v_r}=\mathcal{R}_r$ etc.

Now, let us explicitly combine (11.2) with the above expression, in order to calculate Q_ψ. The first ('$VV-$')part reads:

$$
\begin{aligned}
Q_\psi^{(VV)} &= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\,\mathcal{K}_{(VV)}\{f\}\\
&= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\left[\frac{\partial^2}{\partial v_i\,\partial v_j}D_{ij}^{(VV)}\,f\;-\;\frac{\partial}{\partial v_i}\mathcal{F}_i^{(V)}\,f\right]\\
&= -\frac{1}{n_\alpha}\int d\mathbf{v}\,\frac{\partial\psi}{\partial v_i}\left\{\frac{\partial}{\partial v_j}\left[D_{ij}^{(VV)}\,f\right]-\mathcal{F}_i^{(V)}\,f\right\}\\
&= \frac{1}{n_\alpha}\int d\mathbf{v}\,\frac{\partial^2\psi}{\partial v_i\,\partial v_j}\,D_{ij}^{(VV)}\,f\;+\;\frac{1}{n_\alpha}\int d\mathbf{v}\,\frac{\partial\psi}{\partial v_i}\,\mathcal{F}_i^{(V)}\,f
\end{aligned}
$$

The second ('$VX-$')part reads:

$$
\begin{aligned}
Q_\psi^{(VX)} &= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\,\mathcal{K}_{(VX)}\{f\}\\
&= \frac{1}{n_\alpha}\int d\mathbf{v}\,\psi\,\frac{\partial^2}{\partial v_i\,\partial x_j}D_{ij}^{(VX)}\,f
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial v_i} \frac{\partial}{\partial x_j} \left[D_{ij}^{(VX)} f \right] \\
&= \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial^2 \psi}{\partial v_i\, \partial x_j} D_{ij}^{(VX)} f - \frac{1}{n_\alpha} \frac{\partial}{\partial x_j} \int d\mathbf{v}\, \frac{\partial \psi}{\partial v_i} D_{ij}^{(VX)} f
\end{aligned}
$$

The final ('$XX-$')part reads:

$$
\begin{aligned}
Q_\psi^{(XX)} &= \frac{1}{n_\alpha} \int d\mathbf{v}\, \psi\, \mathcal{K}_{(XX)}\{f\} \\
&= \frac{1}{n_\alpha} \int d\mathbf{v}\, \psi \left[\frac{\partial^2}{\partial x_i\, \partial x_j} D_{ij}^{(XX)} f - \frac{\partial}{\partial x_i} \mathcal{F}_i^{(X)} f \right] \\
&= \frac{1}{n_\alpha} \frac{\partial}{\partial x_i} \int d\mathbf{v}\, \psi \left\{ \frac{\partial}{\partial x_j} \left[D_{ij}^{(XX)} f \right] - \mathcal{F}_i^{(X)} f \right\} \\
&\qquad - \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_i} \left\{ \frac{\partial}{\partial x_j} \left[D_{ij}^{(XX)} f \right] - \mathcal{F}_i^{(X)} f \right\} \\
&= \frac{1}{n_\alpha} \frac{\partial^2}{\partial x_i\, \partial x_j} \int d\mathbf{v}\, \psi\, D_{ij}^{(XX)} f - \frac{1}{n_\alpha} \frac{\partial}{\partial x_i} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_j} D_{ij}^{(XX)} f \\
&\qquad - \frac{1}{n_\alpha} \frac{\partial}{\partial x_i} \int d\mathbf{v}\, \psi\, \mathcal{F}_i^{(X)} f \\
&\quad - \frac{1}{n_\alpha} \frac{\partial}{\partial x_j} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_i} D_{ij}^{(XX)} f + \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial^2 \psi}{\partial x_i\, \partial x_j} D_{ij}^{(XX)} f \\
&\qquad + \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_i} \mathcal{F}_i^{(X)} f \\
&= \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial^2 \psi}{\partial x_i\, \partial x_j} D_{ij}^{(XX)} f + \frac{1}{n_\alpha} \frac{\partial^2}{\partial x_i\, \partial x_j} \int d\mathbf{v}\, \psi\, D_{ij}^{(XX)} f \\
&\quad - \frac{1}{n_\alpha} \frac{\partial}{\partial x_i} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_j} \left[D_{ij}^{(XX)} + D_{ji}^{(XX)} \right] f \\
&\quad - \frac{1}{n_\alpha} \frac{\partial}{\partial x_i} \int d\mathbf{v}\, \psi\, \mathcal{F}_i^{(X)} f + \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial x_i} \mathcal{F}_i^{(X)} f
\end{aligned}
\tag{11.40}
$$

Gathering all three of the last expressions, we obtain the final evolution equation for any quantity ψ. For instance, if $\psi = \psi(\mathbf{v})$ only (i.e. not $\mathbf{x}$, such as v_r, for instance), we have the reduced equation:

$$
\begin{aligned}
Q_{\psi(\mathbf{v})} &= \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial^2 \psi}{\partial v_i\, \partial v_j} D_{ij}^{(VV)} f + \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial v_i} \mathcal{F}_i^{(V)} f \\
&\quad - \frac{1}{n_\alpha} \int d\mathbf{v}\, \frac{\partial \psi}{\partial v_i} D_{ij}^{(VX)} \frac{\partial f}{\partial x_j} \\
&\quad + \frac{1}{n_\alpha} \int d\mathbf{v}\, \psi\, D_{ij}^{(XX)} \frac{\partial^2 f}{\partial x_i\, \partial x_j} - \frac{1}{n_\alpha} \int d\mathbf{v}\, \psi\, \mathcal{F}_i^{(X)} \frac{\partial f}{\partial x_i}
\end{aligned}
\tag{11.41}
$$

Notice that the 1st line coincides in the two ($\Theta-$ and $\Phi-$) operators discussed in the text, the 2nd line differs, and the third one is completely absent from the former.

If, on the other hand, $\psi = \psi(\mathbf{x})$ only (i.e. does not depend on $\mathbf{v}$), $Q_{\psi(\mathbf{x})}$ simply reduces to $Q_{\psi}^{(XX)}$ (see right above), in the case of the $\Phi-$ operator and to *zero* (!) in the case of Θ.

11.6.1 The friction vector $\mathcal{R}_r^\alpha$

Eq. (11.41) directly provides us with the evolution equation for $\psi(\mathbf{v}) = v_r$, upon setting: $\frac{\partial\psi}{\partial v_i} = \frac{\partial v_r}{\partial v_i} = \delta_{ir}$ and $\frac{\partial^2\psi}{\partial v_i\,\partial v_j} = 0$; we obtain:

$$\begin{aligned} u_r^\alpha \;=\; & \frac{m_\alpha}{n_\alpha}\int d\mathbf{v}\,\mathcal{F}_r^{(V)}\,f \\ & -\frac{m_\alpha}{n_\alpha}\int d\mathbf{v}\,D_{rj}^{(VX)}\,\frac{\partial f}{\partial x_j} \\ & +\frac{\partial^2}{\partial x_i\,\partial x_j}\left[\frac{m_\alpha}{n_\alpha}\int d\mathbf{v}\,v_r\,D_{ij}^{(XX)}\,f\right] - \frac{m_\alpha}{n_\alpha}\frac{\partial}{\partial x_i}\left[\int d\mathbf{v}\,v_r\,\mathcal{F}_i^{(X)}\,f\right] \end{aligned} \tag{11.42}$$

The third line would be completely absent in the $\Theta-$ case; note the space diffusion term therein.

Given the axial symmetry (induced by the magnetic field), one is tempted to derive the analogous formula for the velocity coordinates across ($v_\perp$) and along ($v_\parallel$) the field. In the first case, we have: $\psi = (v_x^2 + v_y^2)^{1/2} = v_\perp$, so

$$\frac{\partial\psi}{\partial v_i} = \frac{v_i}{v_\perp}\,(\delta_{i1}+\delta_{i2})\,, \qquad \frac{\partial^2\psi}{\partial v_i\,\partial v_j} = \frac{1}{v_\perp}\left[\delta_{ij} - \frac{v_i\,v_j}{v_\perp^2}\right](\delta_{i1}+\delta_{i2})$$

while in the second:

$$\psi = v_z = v_\parallel\,, \qquad \frac{\partial\psi}{\partial v_i} = \delta_{i3}\,, \qquad \frac{\partial^2\psi}{\partial v_i\,\partial v_j} = 0$$

(and obviously: $\frac{\partial\psi}{\partial x_i} = 0$ in both). Substituting in (11.41) for the former i.e. $v_\perp$ and evaluating integrals (in polar coordinates) as deep as possible (remember that $\partial D_{ij}^*/\partial\theta = 0$, and the same we assume for f), we see that:

- the first line in (11.41) simplifies to: $-\frac{1}{n_\alpha}\int_0^\infty dv_\perp\,v_\perp\int_{-\infty}^\infty dv_\parallel\,D_\perp\,\frac{\partial f}{\partial v_\perp}$ (we have taken $\mathcal{F}_\perp \sim \partial D_\perp/\partial v_\perp$, as in the single species case, for simplicity);

- the second line simply cancels (so no difference is made between the two operators so far)

and

- the third line reduces to: $\frac{1}{n_\alpha}\int_0^\infty dv_\perp\,v_\perp^2\int_{-\infty}^\infty dv_\parallel\,D_\perp^{(XX)}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f$

(remember that $D_\perp^{(XX)} = \Omega^{-2}\,(D_\perp + Q)$, see elsewhere).

We see that space diffusion is introduced by the $\Phi-$ operator; the same is true for $v_\parallel$ as well (check by substituting into (11.41)).

11.6.2 The heat flux $\mathcal{Q}^\alpha$

A similar calculation can be carried out for $\mathcal{Q}^\alpha$, defined in (11.22), by setting

$$\psi = \frac{1}{2} m_\alpha \, |\mathbf{v} - \mathbf{u}^\alpha|^2 \, , \qquad \frac{\partial \psi}{\partial v_i} = m_\alpha \, (v_i - u_i^\alpha) \, , \qquad \frac{\partial^2 \psi}{\partial v_i \, \partial v_j} = m_\alpha \, \delta_{ij}$$

$$\frac{\partial \psi}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{1}{2} \big(v_r - u_r^\alpha \big)^2 = m_\alpha \, (v_r - u_r^\alpha) \, \frac{\partial u_r}{\partial x_i} = \ldots \, , \qquad \frac{\partial^2 \psi}{\partial x_i \, \partial x_j} = \ldots$$

(a summation over r is understood) in the above formulae. The calculation - quite lengthy - leads to the same final conclusion: space diffusion effects are taken into account by the $\Phi-$ operator (only), to an order $\sim \Omega^2$.

11.7 Solution of the evolution equations

We may now attempt to apply a method often used in plasma transport theory, in order to cope with non-tractability of kinetic equations and gain insight to the behaviour of the solution. It consists in linearizing around a known equilibrium function f_{eq}. The idea is that the system studied may find itself not so far from (but not at) equilibrium, so its deviation from f_{eq} is treated as a perturbation. This treatment is rather subtle to manipulate in our case, for a variety of reasons:

1st, our *FP* kinetic equation *is* linear already; nevertheless, this fact does not necessarily imply that the *t.p.* is *close to* equilibrium (see the discussion in the Introduction);

2nd, the bulk plasma (reservoir) is homogeneous, in our case: only the 'perturbation' is taken to be inhomogeneous, so there is no need to consider a local Maxwellian (with space-dependent coefficients)[11], as is the standard way (see e.g. [5], [53]).

3rd, the methodology used here is largely simplified by making use of conservation laws, satisfied by the kinetic operator. However, as already discussed above, only matter is conserved by the *FP* operator (i.e. mass/charge/particle number), yet neither momentum nor energy.

Therefore, the method exposed here is only provided as an interesting *per se* tool accompanying the study of moments presented above. Nevertheless, even though the setting (and our notation) may resemble strongly to the method introduced for the Landau equation e.g. in [5], [53] the treatment that will follow, actually adapted to the test-particle problem, will soon depart from that standard image.

11.7.1 Step 1: Reference state

We need an equilibrium function f_{eq}^0, such that:

$$\partial f_{eq}^0 \partial t = 0$$

[11] Given the form of the *FPE*, this would not be an equilibrium function, anyway.

A standard choise is the Maxwellian function:

$$f_{eq}^{\alpha} = n_{\alpha}^{0} \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{3/2} (2\pi)^{3/2}\, e^{-m_{\alpha}|\mathbf{v}-\mathbf{u}_{\alpha}^{0}|^{2}/2T_{\alpha}^{0}}$$

Let us point out straightaway the difference between parameters $(n_{\alpha}^{0}, T_{\alpha}^{0}, \mathbf{u}_{\alpha}^{0})$ in this function (tagged by the superscript '0') and homologous mean values $(n_{\alpha}(\mathbf{x},t), T_{\alpha}(\mathbf{x},t), \mathbf{u}_{\alpha}(\mathbf{x},t))$ defined in the previous sections: remember that the equilibrium function defined here is both the reservoir steady state *and* the *final* minority species (test-particle) state[12], so parameters in it are NOT equal to the average values (referring to the *t.p.* population). Furthermore, the equilibrium condition:

$$\mathcal{K}\{f_{eq}^{\alpha}\} = 0$$

suggests the choise:

$$\frac{\partial n_{\alpha}^{0}}{\partial x_i} = 0\,, \qquad \frac{\partial T_{\alpha}^{0}}{\partial x_i}\,, \qquad \mathbf{u}_{\alpha}^{0} = \mathbf{0}$$

(cf. (11.2)). It is clear that this is *one* choise, not excluding the possibility of existence of other ones.

11.7.2 Step 2: Linearization.

In the following, we shall set:

$$f^{\alpha}(\mathbf{x},\mathbf{v};\,t) = f_{\alpha}^{0}(v)\big[1 + \chi^{\alpha}(\mathbf{x},\mathbf{v};\,t)\big] \tag{11.43}$$

Combining this expression with definitions of $n_{\alpha}(\mathbf{x},t), T_{\alpha}(\mathbf{x},t), \mathbf{u}_{\alpha}(\mathbf{x},t)$ in the beginning of the chapter, we obtain the conditions:

$$\begin{aligned} \int d\mathbf{v}\, f_{\alpha}^{0}\,\chi_{\alpha} &= n_{\alpha}(\mathbf{x},t) - n_{\alpha}^{0} \\ \int d\mathbf{v}\, v_r\, f_{\alpha}^{0}\,\chi_{\alpha} &= n_{\alpha}(\mathbf{x},t)\, u_r^{\alpha}(\mathbf{x},t) \\ \frac{1}{3}\int d\mathbf{v}\, |\mathbf{v}-\mathbf{u}_{\alpha}|^{2}\, f_{\alpha}^{0}\,\chi_{\alpha} &= \big[n_{\alpha}(\mathbf{x},t)\, T^{\alpha}(\mathbf{x},t) - n_{\alpha}^{0}\, T_{\alpha}^{0}\big] + \frac{1}{3}\, n_{\alpha}(\mathbf{x},t)\, u_{\alpha}^{2}(\mathbf{x},t) \end{aligned} \tag{11.44}$$

(the difference from the standard formalism, e.g. [5], [53] is substantial).

11.7.3 Step 3: Reduction to a non-dimensional form.

Let us define the non-dimensional 'velocities'[13]:

$$c_r \equiv \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{1/2} (v_r - u_r^{0}) = \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{1/2} v_r\,, \qquad \tilde{u}_r(\mathbf{x};\,t) \equiv \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{1/2} u_r(\mathbf{x};\,t) \tag{11.45}$$

[12] The number of test-particles (if not 1) is assumed to be negligible, compared to that of the reservoir, so the final equilibrium state is almost the same Maxwellian.

[13] Remember that $u_r^{0} = 0$.

The above Maxwellian becomes:

$$f_{eq}^{\alpha} = n_{\alpha}^{0} \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{3/2} (2\pi)^{3/2}\, e^{-c^2/2} \equiv n_{\alpha}^{0} \left(\frac{m_{\alpha}}{T_{\alpha}^{0}}\right)^{3/2} \phi^{0}(c) \tag{11.46}$$

and conditions (11.44) take the form:

$$\begin{aligned}
\int d\mathbf{v}\, \phi^{0}(c)\, \chi^{\alpha}(\mathbf{c}, \mathbf{x};\, t) &= \frac{n_{\alpha}(\mathbf{x}, t)}{n_{\alpha}^{0}} - 1 \\
\int d\mathbf{v}\, c_r\, \phi^{0}(c)\, \chi^{\alpha}(\mathbf{c}, \mathbf{x};\, t) &= \frac{n_{\alpha}(\mathbf{x}, t)}{n_{\alpha}^{0}}\, \tilde{u}_r \\
\frac{1}{3} \int d\mathbf{v}\, c^2\, \phi^{0}(c)\, \chi^{\alpha}(\mathbf{c}, \mathbf{x};\, t) &= \frac{n_{\alpha}(\mathbf{x}, t)}{n_{\alpha}^{0}} \frac{T_{\alpha}(\mathbf{x}, t)}{T_{\alpha}^{0}} - 1 \\
&\quad + \frac{1}{3}\, n_{\alpha}(\mathbf{x}, t)\, \frac{n_{\alpha}(\mathbf{x}, t)}{n_{\alpha}^{0}}\, \tilde{u}_{\alpha}^{2}(\mathbf{x}, t)
\end{aligned} \tag{11.47}$$

where all quantities are dimensionless.

As a familiar eye may have already noticed, in 'ordinary' transport theory, all the *right-hand-sides* are zero. This difference will lead to a different interpretation of moments in the following part of this text. In any case, check that all *rhs*s tend to zero as equilibrium is reached, since we then have:

$$\chi^{\alpha} \to 0\,, \qquad n_{\alpha} \to n_{\alpha}^{0}\,, \qquad T_{\alpha} \to T_{\alpha}^{0}\,, \qquad \mathbf{u}_{\alpha} \to \mathbf{u}_{\alpha}^{0} = 0\,.$$

11.7.4 Step 4: Expansion as a series of Hermite polynomials

We may develop $\chi^{\alpha}(\mathbf{c}, \mathbf{x};\, t)$ as follows:

$$\begin{aligned}
\chi^{\alpha}(\mathbf{c}, \mathbf{x};\, t) &= \sum_{n=0}^{\infty} h^{\alpha\,(2n)}(\mathbf{x};\, t)\, H^{(2n)}(\mathbf{c}) \\
&\quad + \sum_{n=0}^{\infty} h_{r}^{\alpha\,(2n+1)}(\mathbf{x};\, t)\, H_{r}^{(2n+1)}(\mathbf{c}) \\
&\quad + \sum_{n=1}^{\infty} h_{rs}^{\alpha\,(2n)}(\mathbf{x};\, t)\, H_{rs}^{(2n)}(\mathbf{c})
\end{aligned} \tag{11.48}$$

The quantities denoted by $H_{r_1,\ldots,r_q}^{(m)}(\mathbf{c})$ are *irreducible (tensorial) Hermite polynomials* of different rank: scalars, vectors or tensors, in the 1st, 2nd and 3rd lines, respectively; the corresponding moments are therefore naturally called *scalar*, *vector* and *tensor moments*.

Properties

The first polynomials are:

$$H^{(0)}(\mathbf{c}) = 1\,, \qquad H^{(2)}(\mathbf{c}) = \frac{1}{\sqrt{6}}(c^2 - 3)$$

$$H^{(1)}_r(\mathbf{c}) = c_r\,, \qquad H^{(3)}_r(\mathbf{c}) = \frac{1}{\sqrt{10}}\,c_r\,(c^2 - 10)$$

$$H^{(2)}_{rs}(\mathbf{c}) = \frac{1}{\sqrt{2}}\left(c_r\,c_s\, - \frac{1}{3}c^2\,\delta_{rs}\right) \tag{11.49}$$

etc. They obey a set of orthogonality/normalization conditions e.g.

$$\int d\mathbf{c}\,\phi_0(\mathbf{c})\,H^{(m)}_{r_1,\ldots,r_q}(\mathbf{c}) = \delta_{m0} \qquad \int d\mathbf{c}\,\phi_0(\mathbf{c})\,H^{(2n+1)}_r(\mathbf{c})\,H^{(2m+1)}_s(\mathbf{c}) = \delta_{nm}\,\delta_{rs} \tag{11.50}$$

The coefficients $h^{\alpha}_{r_1,\ldots,r_q}(m)(\mathbf{x};\,t)$ ("moments") are given by:

$$\begin{aligned} h^{\alpha}_{r_1,\ldots,r_q}(m)(\mathbf{x};\,t) &= \int d\mathbf{c}\,\phi_0(\mathbf{c})\,\left[1 + \chi(\mathbf{c},\mathbf{x};\,t)\right]H^{(m)}_{r_1,\ldots,r_q}(\mathbf{c}) \\ &= \int d\mathbf{c}\,\phi_0(\mathbf{c})\,\chi(\mathbf{c},\mathbf{x};\,t)\,H^{(m)}_{r_1,\ldots,r_q}(\mathbf{c}) \end{aligned} \tag{11.51}$$

the last step being a consequence of (11.50).

The properties of Hermite polynomials, as they influence the problem of plasma transport, are exhaustively studied in [5] (see ch. 4 therein), so going into many details would serve no real purpose here.

Truncation

Keeping only the lowest members in expression (11.48) we have:

$$\begin{aligned} \chi^{\alpha}(\mathbf{c},\mathbf{x};\,t) \;=\; & h^{\alpha\,(0)}(\mathbf{x};\,t)\,H^{(0)}(\mathbf{c}) + h^{\alpha\,(2)}(\mathbf{x};\,t)\,H^{(2)}(\mathbf{c}) + h^{\alpha\,(1)}_r(\mathbf{x};\,t)\,H^{(1)}_r(\mathbf{c}) \\ & + h^{\alpha\,(3)}_r(\mathbf{x};\,t)\,H^{(3)}_r + h^{\alpha\;(2)}_{rs}(\mathbf{x};\,t)\,H^{(2)}_{rs}(\mathbf{c}) \\ & + h^{\alpha\,(5)}_r(\mathbf{x};\,t)\,H^{(5)}_r + h^{\alpha\;(4)}_{rs}(\mathbf{x};\,t)\,H^{(4)}_{rs}(\mathbf{c}) \\ & + \ldots \end{aligned}$$

The *rhs* was arranged so as to distinguish:

- the terms which vanish in ordinary transport theory: 1st line in the *rhs*; remember that these terms are related to conditions (11.44) (whose *rhs* are *not* zero here).
- the terms usually kept in the lowest 13 moment (13M) approximation: 2nd line in the *rhs*; we now have to cope with a '18M approximation' (due to the first line)
- higher terms kept in the 21 moment (21M) approximation: 3rd line in the *rhs* (will be abandoned here, for simplicity).

Physical interpretation of moments

Combining the above with (11.47) we may obtain the physical meaning of hermitian moments defined above. We inevitably need to point out, once more, that results in this section differ strongly from the usual ones; only notation has been kept the same, basically.

$h^{(0)}$ is immediately seen to be equal to the *lhs* of (11.47a). We thus obtain:

$$n_\alpha(\mathbf{x},\, t) = n_\alpha^0 \left[1 + {h_\alpha}^{(0)}(\mathbf{x},\, t)\right] \tag{11.52}$$

so $h^{(0)}$ physically expresses the deviation from the equilibrium density.

In the same way, from (11.47b) we obtain:

$$h_r^{\alpha\,(1)}(\mathbf{x},\, t) = \frac{n_\alpha(\mathbf{x},\, t)}{n_\alpha^0} \left(\frac{m_\alpha}{T_\alpha^0}\right)^{1/2} u_r^\alpha(\mathbf{x},\, t) \tag{11.53}$$

so $h_r^{(1)}$ is related to the deviation of the mean velocity u_r^α from the equilibrium value $u_r^0 = 0$. Combining with the previous relation, we obtain:

$$u_r^\alpha(\mathbf{x},\, t) = \left(\frac{T_\alpha^0}{m_\alpha}\right)^{1/2} \frac{h_r^{\alpha\,(1)}}{1 + h_\alpha^{(0)}} \tag{11.54}$$

The same method, applied to (11.47c) gives:

$$h^{\alpha\,(2)}(\mathbf{x},\, t) = \sqrt{\frac{3}{2}} \frac{n_\alpha(\mathbf{x},\, t)}{n_\alpha^0} \left[\frac{T_\alpha(\mathbf{x},\, t)}{T_\alpha^0} - 1 + \frac{m_\alpha\, u_r^{\alpha\,2}(\mathbf{x},\, t)}{3\, T_\alpha^0}\right] \tag{11.55}$$

so $h^{(2)}$ is related to the deviation of the mean square velocity (temperature) T_α from the equilibrium value T_0. Combining with the previous relation, we obtain:

$$T_\alpha(\mathbf{x},\, t) = T_\alpha^0 \left[\sqrt{\frac{2}{3}}\, \frac{h^{\alpha\,(2)}}{1 + h_\alpha^{(0)}} + 1 - \frac{1}{3} \left(\frac{h_r^{\alpha\,(1)}}{1 + h_\alpha^{(0)}}\right)^2 \right] \tag{11.56}$$

A similar procedure relates ${h_{rs}^\alpha}^{(2)}$ to the dissipative pressure tensor:

$$\pi_{rs}^\alpha(\mathbf{x},\, t) = n_\alpha^0\, T_\alpha^0 \left\{ \sqrt{2} {h_{rs}^\alpha}^{(2)} + (1 - \sqrt{2}) \left[H_{rs}\left(\frac{h_r^{\alpha\,(1)}}{1 + h_\alpha^{(0)}}\right)\right] \left[1 + h_\alpha^{(0)}\right] \right\} \tag{11.57}$$

$h_r^{\alpha\,(3)}$ to the heat flux:

$$q_r^\alpha(\mathbf{x},\, t) = \sqrt{\frac{5}{2}} m_\alpha \left(\frac{T_\alpha^0}{m_\alpha}\right)^{3/2} h_r^{\alpha\,(3)}(\mathbf{x},\, t) \,+\, f\left(\left\{h^{\alpha\,(0)}, {h_{r'}^\alpha}^{(1)}, h^{\alpha\,(2)}, {h_{r's}^\alpha}^{(2)}\right\}\right) \tag{11.58}$$

(f is too complicated to provide here); notice the coupling between q_r^α and π_{rs}^α via ${h_{r's}^\alpha}^{(2)}$.

11.7.5 Step 5: Derivation of evolution equations for moments ${h^\alpha_{r_1,\dots,r_q}}^{(m)}$

We are now left with the task of finding a set of temporal evolution equations for the hermitian moments ${h^\alpha_{r_1,\dots,r_q}}^{(m)}$. Given the above definitions, complete knowledge of the evolution in time of ${h^\alpha_{r_1,\dots,r_q}}^{(m)}$ is equivalent to knowledge of the evolution in time of related physical quantities *and* of the test-particle *pdf close to equilibrium*, at least up to a certain approximation (due to the truncation invoked above).

One either proceeds by injecting the above expressions into the evolution equations derived in the beginning of the chapter, or by directly making use of (11.51); the two approaches are equivalent. We have thus obtained a set of coupled evolution equations for ${h^\alpha_{r_1,\dots,r_q}}^{(m)}$. In the following we shall only briefly state the first few of these equations.

The first member of the hierarchy reads:

$$\partial_t \, h^{\alpha\,(0)} \;=\; -\nabla_m \, {h^\alpha_m}^{(1)} \tag{11.59}$$

This expression relates, as expected, density evolution to the mean velocity.

The evolution equation for ${h^\alpha_m}^{(1)}$ reads:

$$\begin{aligned}
\partial_t \, h^{\alpha\,(1)}_r \;&=\\
=\;\; & -\left(\frac{T^0_\alpha}{m_\alpha}\right)^{1/2} \nabla_m \left[\frac{h^{\alpha\,(1)}_r \, {h^\alpha_m}^{(1)}}{1+h^{\alpha\,(0)}} + \delta_{rm} \times \left(\sqrt{\frac{2}{3}}\, h^{\alpha\,(2)} + 1 + h^{\alpha\,(0)} \right.\right.\\
& \qquad\qquad \left. - \frac{1}{3} \, \frac{(h^{\alpha\,(1)}_k)^2}{1+h^{\alpha\,(0)}} \right)\\
& \qquad\qquad \left. + \sqrt{2}\, {h^\alpha_{rm}}^{(1)} (1-\sqrt{2}) \, H_{rm} \left(\frac{h^{\alpha\,(1)}_r}{1+h^{(0)}_\alpha} \right) (1 + h^{(0)}_\alpha) \right]\\
& + \left(\frac{m_\alpha}{T^0_\alpha}\right)^{1/2} \frac{e_\alpha}{m_\alpha} \, E_r \, (1 + h^{(0)}_\alpha)\\
& + s_\alpha \, \Omega_\alpha \, \epsilon_{rm3} \, {h^\alpha_m}^{(1)}\\
& + \frac{1}{n_\alpha} \int d\mathbf{v} \, H^{(1)}_r(\mathbf{c}) \, \mathcal{K}^\alpha
\end{aligned} \tag{11.60}$$

The third line cancels in our case ($\mathbf{E} = \mathbf{0}$); in the fourth we have set $B_j = B \, \delta_{j3}$; notice the contribution of the collision term (fifth line), which we will formally denote by $Q^{\alpha\,(1)}_r(\mathbf{x}, t)$.

In a similar way, we have obtained evolution equations for $h^{\alpha\,(2)}$, $h^{\alpha\,(3)}_r$ and ${h^\alpha_{rs}}^{(2)}$. Their complicated form will not be given here, so as not to uselessly burden the flow of this text. Nevertheless, let us mention that:

- they are coupled: each equation involves a moment of superior order, so an appropriate truncation hypothesis need to be adopted, should one solve them analytically;

- they involve collisional (friction) terms of the general form:

$$Q^{\alpha}_{r_1,\ldots,r_q}{}^{(m)}(\mathbf{x},\, t) = \frac{1}{n^0_\alpha} \int d\mathbf{v}\, H^{\alpha}_{r_1,\ldots,r_q}{}^{(m)}(\mathbf{c})\, \mathcal{K} \qquad (11.61)$$

which may be evaluated according to the method exposed above (see §11.6). Notice that $Q^{\alpha\,(0)}$, due to density (*pdf* norm) conservation by $\mathcal{K}$.

11.8 Perspective

We prefer to stop this analytical investigation here. In principle, solving the evolution equations for moments would demand a specific truncation scheme, plus the detailed calculation of the collisional contribution appearing in them. As a qualitative conclusion, retain that the latter result in the appearance of space diffusion terms in the 'hydrodynamic' equations derived above, so one should carefully examine the result from both qualitative (prediction of particle lateral escape/random walk) and practical (ideally, comparison with experimental data) point of view. This goes beyond the scope of this thesis and is left for future work.

Let us add that this chapter is viewed as complementary to ch. 10, where a similar initiative has been undertaken from a different starting point. That investigation, yet somewhat restricted (since a simplifying assumption is imposed on coefficients), relied on fundamental principles of Statistical Mechanics (calculation of average values from the *pdf*) and thus appears to be more appropriate for *space*-related observable quantities (rather neglected in this chapter).

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Chapter 12

Concluding remarks

This is often the way it is in physics -
our mistake is not that we take our theories too seriously,
but that we do not take them seriously enough.
`Steven Weinberg` in *The First Three Minutes.*

12.1 Conclusions

Ending this thesis, let us summarize and briefly discuss our results.

In Part A, we have presented an analytical formalism allowing the derivation of a Fokker-Planck equation from the microscopic laws of motion. This method applies to any weakly-coupled system, provided an explicit solution of the free (single-) particle problem of motion is known. The features of this method include: (i) taking into account the external field(s) in the derivation of the kinetic equation (as compared to existing Landau-type equations), (ii) the explicit appearance of physical parameters in expressions for the force correlations and diffusion-friction coefficients (as compared to phenomenological stochastic, Langevin-type theories, based on *ad hoc* assumptions on the collision mechanism) and (iii) taking into account space inhomogeneity of the probability distribution function. We have shown that this is *correctly* done by considering a Markovian kinetic operator, the $\Phi-$*operator*, which was explicitly expressed in terms of the system physical features (interaction laws, field)[1].

In Part B, these results were applied to an electrostatic plasma in a uniform magnetic field. By explicitly constructing the $\Phi-$*operator* in this case, we have derived a new kinetic equation for a test-particle in magnetized plasma. In particular, we have shown that the positivity of the test-particle (probability) distribution function f was guaranteed by this modified collision operator; this

[1] Let us mention that we have also applied this formalism to a standard paradigm in Statistical Mechanics, namely the kinetic behaviour of one-dimensional non-uniform oscillator gases. The results, omitted here, appear to justify the motivation of this work; in specific, the construction of the markovian $\Phi-$ operator leads to a *modified* diffusion equation in 1 dimension, accounting for diffusion in both position and velocity spaces.

was not the case in the old one (Θ). A new element of this equation is the appearance of space-diffusion terms. We have thus suggested this plasma-*FPE* as a basis for further study of kinetic plasma properties. Explicit expressions were obtained for all diffusion-/friction coefficients, involving force correlations, time and - most important - the magnitude of the magnetic field. We have shown that the effect of taking into account gyrating motion between collisions, rather than considering free trajectories instead, is traced on a simple characteristic (numerical) parameter λ, roughly the ratio of the Larmor radius to the Debye length for the specific system studied (or the ratio of the plasma frequency to the cyclotron frequency).

In Part C, we have analyzed of the results of Part B. First, we have investigated the dependence of relaxation times (diffusion coefficients) on particle velocity and the magnitude of the magnetic field, via the parameter λ defined above. Different regimes are discussed, in both equal-species and multi-species cases, in relation to the *unmagnetized* plasma case. Considering particle interactions in the presence of the field as a multi-dimensional Ornstein-Uhlenbeck process, we have proceeded by explicitly solving the *FPE* for f as a function of time. Diffusion phenomena are thus naturally modeled by the $\Phi-$ collision operator. This could not have been possible via the 'old' (Θ) kinetic operator.

Finally, we have suggested an approximation scheme for the study of the evolution of velocity moments. This is only a starting point for further development of the formalism, which would allow for a systematic study of the influence of the collision term on the evolution of mean values. Retain the appearance of field-dependent diffusion terms in the moment evolution equations; this element was completely absent in previous treatments.

12.2 Relation to previous works

Let us make a few comments concerning our results as related to previous works in the field.

Our *FPE* is essentially similar to a kinetic equation derived in the past for a *homogeneous*[2] magnetized plasma by N. Rostoker [101], Silin and coworkers [64], [105], P. Schram [103], D. Montgomery and co-workers [85] (in chronological order); our results are in agreement with those works. Nevertheless, our work concerns a *test-particle* problem, i.e. a *'linearized'* version[3] of the full problem treated in those works, as we take the bulk plasma to be in equilibrium.

Attempts to generalize those works to a non-uniform system by obtaining equations in the form of eq. (4.4) (here identified as the *ill-defined* $\Theta-$ kinetic operator) include works carried out by P. Ghendrih [16], [65], A. Øien [88], [90], [94]. The mathematical properties of the 'Θ- *operator*' were not questioned therein, though, since the cross-V-X diffusion term (second term in the *rhs* of (4.4)) has always been neglected, or even plainly omitted, through physical arguments.

[2] i.e. for the *velocity df.*

[3] See discussion in §1.4.

Finally, let us point out that previous results have been confirmed and are incorporated in this work. Concerning the influence of the magnetic field **B**, in particular, we have recovered:
(i) the *'weak'* dependence of relaxation times on **B**, anticipated by Montgomery *et al.* in [85], [86] for *low* values of the field magnitude (see Ch. 9[4]) and
(ii) the dependence of diffusion rates on B as $D_\perp \sim 1/B^2$, for *high* values of the field magnitude B, obtained e.g. by Simon [106] and Rosenbluth & Kaufman [100] in the 1950's[5].

[4] See, in particular, figures 9.11, 12 in the region of high values of $\lambda \sim 1/B$.
[5] See figure 9.11b in the region of $\lambda \sim 1/B$ close to zero.

Appendix

Appendix A

Non - dimensional general form of a Fokker - Planck - type collision term

The Fokker-Planck equation (FPE) obtained to 2nd order in the (weak) interaction is a 2nd-order parabolic linear partial derivative equation (*pde*) of the form:

$$\partial_t f + \mathbf{v}\nabla f + \mathbf{a}\,\partial_{\mathbf{v}} f = \mathcal{K}$$

where $\mathbf{v}$, $\mathbf{a}$ denote the particle velocity and acceleration, respectively, in the presence of an external force field (yet in the absence of collisions). The (linear) *collision term* (*rhs*) obeys the general form:

$$\mathcal{K} = \frac{\partial^2}{\partial v_i \partial v_j}(D_{ij}^{(VV)} f) + \frac{\partial^2}{\partial v_i \partial x_j}(D_{ij}^{(VX)} f) + \frac{\partial^2}{\partial x_i \partial x_j}(D_{ij}^{(XX)} f) - \frac{\partial}{\partial v_i}(\mathcal{F}_i^{(V)} f) - \frac{\partial}{\partial x_i}(\mathcal{F}_i^{(X)} f)$$

The velocity diffusion matrix $D_{ij}^{(VV)} = A_{ij}$ was given by:

$$\begin{aligned}\mathbf{A} &= \frac{n_{\alpha'}}{m_\alpha^2}\int_0^{t\to\infty} d\tau \int d\mathbf{v_1}(2\pi)^3\ \phi_{eq}^{\alpha'}(v_1) \\ &\qquad \int d\mathbf{k}\,\tilde{V}_k^2\,\mathbf{k}\otimes\mathbf{k}\,e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}}\,e^{-i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}}\mathbf{N'}_\alpha^T(\tau) \\ &\equiv \frac{1}{m^2}\int_0^{t\to\infty} d\tau\,\mathbf{C}_{\alpha,\alpha'}(\mathbf{x},\mathbf{v};t,t-\tau)\mathbf{N'}_\alpha^T(\tau) \qquad \text{(A.1)}\end{aligned}$$

and similar relations hold for the other diffusion coefficients D_{ij}^{*} ($* = (VX), (XX)$) (all quantities are defined in the main text)[1].

[1] See Chapter 4 and forth; a summation over particle species α' is understood. In the single-species case, set $e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}}\,e^{-i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} = e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{g}}$, where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$; the drift

The external field appears in the dynamical matrices $\mathbf{N}(t)$, $\mathbf{N}'(t)$.

In the following, we will try to establish a non-dimensional form of the above FPE.

A.1 Space and time scaling

1. *Time scale:* First, let us define a time-scale, say τ_0, which is *a priori* inherent to the physical problem considered (e.g. $\tau_{coll} \sim \nu^{-1}$, $T_{plasma} \sim \omega_p^{-1}$, $T_{gyro} \sim \Omega_c^{-1}$ etc.). From their very definition (see (3.5)), a simple dimensional argument shows that the dynamical matrices $M_{ij}(t)$, $N'_{ij}(t)$ are dimensionless, whereas $N_{ij}(t)$, $(M'_{ij}(t))$ have the dimensions of time ($time^{-1}$ respectively); therefore, we have the scaling:

$$N_{ij}(t) = \tau_0 \, \tilde{N}_{ij}(t/\tau_0) \; , \qquad\qquad N'_{ij}(t) = \tilde{N}'_{ij}(t/\tau_0)$$

(plus expressions analogous to the latter one, for $M_{ij}(t)$, $M'_{ij}(t)$). Furthermore, time variables t, τ may now be expressed as[2]:

$$\tilde{t} = t/\tau_0 \quad , \qquad \tilde{\tau} = \tau/\tau_0$$

2. *Length scale:* Let r_0 be a typical length, which is characteristic of inter-particle interactions (e.g. radius a in a hard-sphere model, Debye length $\lambda_D = (k_B T_\alpha / 4\pi e_\alpha^2 n_\alpha)^{1/2}$ in electrostatic plasma, lattice constant D in a molecular chain etc.). The interaction potential can be scaled as:

$$V(r) = V_0 \, V(r/r_0)$$

implying, for its Fourier transform:

$$\tilde{V}(k) = \tilde{V}_0 \, \tilde{V}(k \, r_0) \equiv \tilde{V}_0 \, \tilde{V}(k/k_0)$$

terms $\mathcal{F}_i^{\dagger}$ ($\dagger = (V), (X)$) are defined through the former via a velocity derivative.

[2]Remember that this scaling is well-founded provided that τ_0 remains finite: $\tau_0 < \infty$ (e.g. the limit $\Omega \to 0$ should not be considered, in plasma, since the limit $\Omega t \to \infty$ is ill-defined then).

($k_0 = r_0^{-1}$; $\tilde{k} = k/k_0$)[3].
3. *Velocity scaling:* Particle velocity components can be rescaled over the thermal velocity $v_{th}^{\alpha} = T_\alpha/m\alpha$ by setting:

$$\tilde{v}_j = v_j/v_{th} \quad , \qquad \tilde{v}_{1,j} = v_{1j}/v_{th}$$

Therefore, velocity gradients are now expressed as:

$$\frac{\partial}{\partial v_i} \to v_{th}^{-1} \frac{\partial}{\partial \tilde{v}_i}$$

and the reservoir distribution function ϕ_{eq} becomes:

$$\phi_{eq} = v_{th}^{-3}\, \tilde{\phi}_{eq}$$

4. *(An additional) length scale:* Finally, a length-scale should be defined, say L, typical of the variation of the distribution function in space, so that space gradients are re-scaled as:

$$\nabla_i = \frac{\partial}{\partial x_i} \to L^{-1} \frac{\partial}{\partial \tilde{x}_i}$$

L may be characterize the length scale of phenomena one is interested in, so it *could* be assumed to be of the order of the mean-free-path $\lambda_{mfp} \sim n^{-1/3}$, the Larmor radius ρ_L in magnetized plasma etc. A *long* ('hydrodynamic') distance L_{hydro} is often chosen for L [4][4], implying a 'quiescent' (non-turbulent) macroscopic state, yet this is *not necessarily* explicitly assumed, in principle.

A.2 Coefficient scaling

Under these considerations, the velocity diffusion coefficient (of dimensions: $[D_{ij}^{(VV)}] = [velocity^2/time]$) is re-scaled as:

$$D_{ij}^{(VV)} = \tau_R^{-1}\, v_{th}^2\, \tilde{D}_{ij}^{(VV)}$$

[3] For example, for a Coulomb potential:

$$V(r) = \frac{e_\alpha e_{\alpha'}}{r} = \frac{e_\alpha e_{\alpha'}}{r_0} \frac{1}{r/r_0} \equiv V_0(r_0)\, V(r/r_0)$$

and

$$\tilde{V}(k) = \frac{e_\alpha e_{\alpha'}}{2\pi^2\, k^2} = \frac{e_\alpha e_{\alpha'}}{k_0^2} \frac{1}{2\pi^2\, (k/k_0)^2} \equiv \tilde{V}_0(k_0)\, \tilde{V}(k/k_0)$$

(definitions are obvious); for a Debye potential (setting $r_0 = r_D$):

$$V(r) = \frac{e_\alpha e_{\alpha'}}{r} e^{-r/r_D} = \frac{e_\alpha e_{\alpha'}}{r_D} \frac{1}{r/r_D} e^{-r/r_D} \equiv V_0(r_0)\, V(r/r_0)$$

and

$$\tilde{V}(k) = \frac{e_\alpha e_{\alpha'}}{2\pi^2\, (k^2 + k_D^2)} = \frac{e_\alpha e_{\alpha'}}{k_D^2} \frac{1}{2\pi^2\, [1 + (k/k_D)^2]} \equiv \tilde{V}_0(k_0)\, \tilde{V}(k/k_0)$$

and so forth.

[4] See the discussion about the hydrodynamic approximation in [4]; also in [5].

where τ_R, naturally arising as a combination of parameters, will be defined later on. In a similar way:

$$D_{ij}^{(VX)} = \tau_R^{-1}\, v_{th}^2\, \tau_0\, \tilde{D}_{ij}^{(VX)} \;\equiv\; \tau_R^{-1}\, \epsilon\, v_{th}\, L\, \tilde{D}_{ij}^{(VX)}$$

and

$$D_{ij}^{(XX)} = \tau_R^{-1}\, v_{th}^2\, \tau_0^2\, \tilde{D}_{ij}^{(XX)} \;\equiv\; \tau_R^{-1}\, \epsilon^2\, L^2\, \tilde{D}_{ij}^{(XX)}$$

(ϵ will be defined below). Finally, the drift terms may be ordered in the same way. The collision term finally becomes:

$$\mathcal{K} = \tau_R^{-1} \left[\frac{\partial^2}{\partial \tilde{v}_i \partial \tilde{v}_j} (\tilde{D}_{ij}^{(VV)} f) + \epsilon \frac{\partial^2}{\partial \tilde{v}_i \partial \tilde{x}_j} (\tilde{D}_{ij}^{(VX)} f) + \epsilon^2 \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j} (\tilde{D}_{ij}^{(XX)} f) \right.$$
$$\left. - \frac{\partial}{\partial \tilde{v}_i} (\tilde{\mathcal{F}}_i^{(V)} f) - \epsilon \frac{\partial}{\partial \tilde{x}_i} (\tilde{\mathcal{F}}_i^{(X)} f) \right] \qquad \text{(A.2)}$$

where, now,

$$\tilde{\mathbf{D}}^{(VV)} = \int_0^{\tilde{t}\to\infty} d\tilde{\tau} \int d\tilde{\mathbf{v}}_{\mathbf{1}} (2\pi)^3\; \tilde{\phi}_{eq}(\tilde{v}_1) \int d\tilde{\mathbf{k}}\, \tilde{V}_{\tilde{k}}^2\, \tilde{\mathbf{k}} \otimes \tilde{\mathbf{k}}\, e^{i\,\lambda\, \tilde{\mathbf{k}}\, \tilde{\mathbf{N}}(\tau)\, (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_{\mathbf{1}})}\, \tilde{\mathbf{N}}'^{T}(\tau) \qquad \text{(A.3)}$$

($\alpha = \alpha'$); the tilde will be dropped where obvious. Analogous expressions hold for the other coefficients.

A.3 Characteristic physical parameters

Regardless of the particular problem considered, three important quantities seem to arise naturally from this parameter sorting.

Relaxation time τ_R. *First*, the *relaxation time* τ_R is given by:

$$\tau_R \;=\; \frac{m_\alpha^2\, v_{th,\alpha}^2}{n_{\alpha'}\, \tilde{V}_0^2(k_0)\, k_0^5\, \tau_0} \qquad \text{(A.4)}$$

This is the most general expression for τ_R (given a specific problem of motion and an interaction law).

Plasma relaxation time $\tau_{R,pl}$. In the case of electrostatic interactions, e.g. of Coulomb or Debye type, we have: $\tilde{V}_0(k_0) = e_\alpha e_{\alpha'}/k_D^2$ (taking $k_0 = k_D$)[5], so

$$\tau_{R,plasma}^{\alpha,\alpha'} = \frac{m_\alpha^2 \, v_{th,\alpha}^2}{n_{\alpha'} \, e_\alpha^2 e_{\alpha'}^2 \, k_D \, \tau_0} = \cdots = \frac{n_\alpha}{n_{\alpha'}} \, (\frac{e_\alpha}{e_{\alpha'}})^2 \, \frac{m_\alpha \, (k_B T_\alpha)^{3/2}}{n_\alpha^{3/2} \, e_\alpha^5 \, (4\pi)^{1/2}} \frac{1}{\tau_0} \qquad \text{(A.5)}$$

It is interesting to see that the inverse of the relaxation time, and therefore the order of magnitude of the collision term (see above), is related to the Coulomb collision frequency $\nu_{Coulomb}$ and to a dimensionless parameter ξ (defined below) through:

$$(\tau_{R,plasma}^{\alpha,\alpha})^{-1} = \frac{n \, e^4 \, k_D \, \tau_0}{m^2 \, v_{th}^2} = \cdots \sim \frac{n \, e^4}{m^{1/2} \, T^{3/2}} \, (k_D \, v_{th} \, \tau_0) \sim \nu \, \xi \qquad \text{(A.6)}$$

ϵ: scaling parameter for the spatial-dependence of the collision term. We have defined the dimensionless 'ordering' parameter

$$\epsilon = \frac{v_{th}^\alpha \, \tau_0}{L} \equiv \frac{\tau_0}{\tau_{hydro}}$$

distinguishing the contributions of each term in the *rhs*. ϵ represents the ratio of the typical time scale of the dynamical problem to a time scale which is typical of the *spatial variation* of the d.f. f; since the latter is somehow arbitrary, the order of magnitude of ϵ is a priori not known[6]. Nevertheless, (as we have mentioned in the text) a number of works in plasma kinetic theory are based on the assumption that the cross-V-X-diffusion term (2nd term in the *rhs* in the collision term above) is negligible as compared to the V-V-diffusion term (1st term in the *rhs*); this hypothesis does not seem to be well-founded[7].

ξ: ratio of field-related to correlation scales. *Finally*, the dimensionless parameter ξ:

$$\xi_\alpha = k_0 \tau_0 v_{th}^\alpha \equiv \frac{\tau_0}{\tau_{cor}}$$

---

[5]In general, considering a central potential of the form: $V(r) \sim r^{-n}$ ($n \in \mathcal{N}^*$):

$$V(r) = V_0 \, (r/r_0)^{-n}$$

implying:

$$\tilde{V}(k) = \tilde{V}_0(k_0) \, \tilde{V}(k \, r_0) \sim \tilde{V}_0 \, (k/k_0)^{n-3}$$

where: $\tilde{V}_0(k_0) = \hat{V}_0/k_0^{n-3}$ ($\hat{V}_0 = const. \in \Re$), one has:

$$\tau_R = \frac{m_\alpha^2 \, v_{th,\alpha}^2}{n_{\alpha'} \, \hat{V}_0^2 \, k_0^{2n-1} \, \tau_0}$$

Setting $n = 1$, we obtain (A.5).

[6]For instance, we may choose $L = v_{th}^\alpha \, \tau_0$ in order to obtain $\epsilon = 1$; on the other hand, assuming $L \gg v_{th}^\alpha \, \tau_0$, we have $\epsilon \ll 1$.

[7]The choice of L needs to be made with respect to the typical scale of df spatial variation one expects to look into; for instance, $L = \rho_L$, $\tau_0 = \omega_p^{-1}$ would result in: $\epsilon = \frac{r_D}{\rho_L}$, which may be small or large. The order of magnitude of ϵ is *not necessarily small*!

appearing in the exponential (see the formula for the diffusion coefficient) measures the relative magnitude of the typical time scale of the dynamical problem (related to an external field e.g. $\tau_0 = \Omega_c^{-1}$, ...) and the corresponding *correlation* scale (related to particle interactions) e.g. the time $\tau_{cor} = r_D/v_{th}$ needed to cross a Debye sphere (or, equivalently, the ratio between the corresponding lengths e.g. the Larmor radius and the Debye length).

In the case of magnetized plasma, *in specific*, setting $k_0 = k_D$, $\tau_0 = \Omega_c^{-1}$:

$$\xi = \frac{\omega_p}{\Omega_c} \sim \frac{T_{gyro}}{r_D/v_{th}} \sim \frac{\rho_{Larmor}}{\lambda_{Debye}}$$

so ξ is essentially the dimensionless parameter λ defined in ch. 8.

Appendix B

Generalized gradients $\mathbf{D}_{\mathbf{V_i}}(t)$, $\mathbf{D}_{\mathbf{X_i}}(t)$ in Γ-space

Let $f = f(\mathbf{x}, \mathbf{v}) \equiv f(\mathbf{x}, \mathbf{v};\, 0)$ be a function of the phase-space variables $\{\mathbf{x}, \mathbf{v}\} \equiv \{\mathbf{x_j}(0), \mathbf{v_j}(0)\}$ corresponding to particle j^{α_j} (of species α_j); according to the standard notation of the text, $j = \sigma^\alpha$ $(1^{\alpha'})$ denotes the test- (reservoir-)particle (respectively).

As we saw in the text, the propagator $U^{(0)}(t) = e^{L_0\, t}$ is defined via the formal solution of the zeroth-order Liouville equation:

$$\partial_t\, f = L_0\, f$$

i.e.

$$f(t) = U^{(0)}(t)\, f(0)$$

According to Liouville's theorem, the action of $U(t)$ can be expressed as a position shift in phase-space:

$$f(t) \equiv f(\mathbf{x}, \mathbf{v};\, t) = U^{(0)}(t)\, f(\mathbf{x}, \mathbf{v};\, 0) = f(\mathbf{x}(-t), \mathbf{v}(-t);\, 0)$$

$(t \in \Re)$.

We have:

$$f(-t) = U^{(0)}(-t)\, f(\mathbf{x}, \mathbf{v};\, 0) = f(\mathbf{x}(t), \mathbf{v}(t);\, 0)$$

so

$$\begin{aligned}
\frac{\partial}{\partial v_r} U^{(0)}(-t)\, f &= \frac{\partial}{\partial v_r} f(\mathbf{x}, \mathbf{v};\, -t) = \frac{\partial}{\partial v_r} f(\mathbf{x}(t), \mathbf{v}(t);\, 0) \\
&= \left[\frac{\partial x_s(t)}{\partial v_r} \frac{\partial}{\partial x_s(t)} + \frac{\partial v_s(t)}{\partial v_r} \frac{\partial}{\partial v_s(t)} \right] f(\mathbf{x}(t), \mathbf{v}(t);\, 0) \\
&= \left[N_{sr}(t) \frac{\partial}{\partial x_s(t)} + N'_{sr}(t) \frac{\partial}{\partial v_s(t)} \right] f(\mathbf{x}(t), \mathbf{v}(t);\, 0)
\end{aligned}$$

where we used the explicit form of the solution of the zeroth-order problem of motion (see (3.6) in the text)[1]. Therefore:

$$U^{(0)}(t)\,\frac{\partial}{\partial v_r}\,U^{(0)}(-t)\,f(\mathbf{x},\mathbf{v}) \;=$$

$$\begin{aligned} &= U^{(0)}(t)\left[N_{sr}(t)\,\frac{\partial}{\partial x_s(t)} + N'_{sr}(t)\frac{\partial}{\partial v_s(t)}\right] f(\mathbf{x}(t),\mathbf{v}(t);\,0) \\ &= \left[N_{sr}(t)\,\frac{\partial}{\partial x_s} + N'_{sr}(t)\,\frac{\partial}{\partial v_s}\right] f(\mathbf{x},\mathbf{v}) \end{aligned}$$

A similar calculation may be carried out for the space gradient ∇ instead of $\frac{\partial}{\partial \mathbf{v}}$.

In conclusion:

$$\begin{aligned} \mathbf{D}_{\mathbf{V_i}}(t) \equiv U^{(0)}(t)\frac{\partial}{\partial \mathbf{v_i}}U^{(0)}(-t) &= \mathbf{N_i^T}(t)\,\frac{\partial}{\partial \mathbf{x_i}} + \mathbf{N'}_{\mathbf{i}}^{\mathbf{T}}(t)\,\frac{\partial}{\partial \mathbf{v_i}} \\ \mathbf{D}_{\mathbf{X_i}}(t) \equiv U^{(0)}(t)\frac{\partial}{\partial \mathbf{x_i}}U^{(0)}(-t) &= \mathbf{M_i^T}(t)\,\frac{\partial}{\partial \mathbf{x_i}} + \mathbf{M'}_{\mathbf{i}}^{\mathbf{T}}(t)\,\frac{\partial}{\partial \mathbf{v_i}} \qquad i=\sigma,\ 1^R \end{aligned}$$

Note added in proof. Let us check relations (B.1) with a Maxwellian function:

$$\phi_{Max}(v) = \phi_{Max}(0)\,e^{-\beta v^2}$$

which is a stationary state of the Lorentz propagator i.e.

$$U(t)\phi_{eq}(v) = \phi_{eq}(v(-t)) = \phi_{eq}$$

(since $v^2(t) = v^2(0)$) so the final result is known:

$$\mathbf{D}_{\mathbf{V_i}}(t)\phi_{Max}(v) \equiv U^{(0)}(t)\frac{\partial}{\partial \mathbf{v_i}}U^{(0)}(-t)\phi_{Max}(v) = U^{(0)}(t)\frac{\partial}{\partial \mathbf{v_i}}\phi_{Max}(v)$$

$$= U^{(0)}(t)[-2\,\beta\,\mathbf{v_i}\phi_{Max}(v)] = -2\,\beta\,v_j(-t)\,\phi_{Max}(v) \equiv -2\,\beta\,R_{ij}(-t)\,v_j\,\phi_{Max}(v)$$
$$= R_{ji}(t)\,v_j\,\phi_{Max}(v)$$

This is exactly the rhs of (B.1a), as applied to ϕ_{Max}.

Also, the second relation (B.1b) evaluated with the Lorentz propagator (i.e. $\mathbf{M} = \mathbf{I}$) plainly gives

$$\mathbf{D}_{\mathbf{X_i}}\,\phi_{Max}(v) = \frac{\partial}{\partial \mathbf{x_i}}\,\phi_{Max}(v) = 0 \qquad .$$

[1] i.e.

$$x_s(t) = M_{sr}\,x_r \;+\; N_{sr}\,v_r \qquad v_s(t) = M'_{sr}\,x_r \;+\; N'_{sr}\,v_r \qquad .$$

Appendix C

Evaluation of the diffusion matrix elements

In the case of equal-species collisions (a single component system, e.g. electron plasma, or the same-species term in the general case) we have $\alpha = \alpha'$, so all particles obey the same dynamics ($\mathbf{N_1}(t) = \mathbf{N_\Sigma}(t)$). We saw, in Chapter 3, that the force auto-correlation matrix for such a system is given by relation:

$$\mathbf{C}^{(\alpha=\alpha')}(\mathbf{v};\tau) = n \int d\mathbf{v_1} (2\pi)^3 \int d\mathbf{k}\ \phi_{eq}(v_1)\, \tilde{V}_k^2\, e^{i\mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v}-\mathbf{v_1})}\, \mathbf{k}\otimes\mathbf{k} \qquad \text{(C.1)}$$

This expression directly follows from (3.36bis) in §3.4.2. Notice that the integrand can be expressed as a function of $\mathbf{v} - \mathbf{v_1} \equiv \mathbf{g}$ only.

This expression for $\mathbf{C}$ has to be evaluated and then substituted in expression (4.10) for the velocity diffusion coefficient $\mathbf{A}$.

The method in the general (multiple-species) case is exactly analogous.

C.1 Method of evaluation

Let us evaluate the quantity:

$$\mathbf{k}\,\mathbf{\Delta r} \;=\; \mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v} - \mathbf{v_1}) \equiv\; \mathbf{k}\,\mathbf{N}(\tau)\,\mathbf{g}$$

which appears in the above expression. Consider a frame with the $z-$ axis along the magnetic field $\mathbf{B}$ [1]. In cylindrical coordinates, the velocity difference $\mathbf{g}$ is given by:

$$\mathbf{g} = (g_x, g_y, g_z) \equiv (g_\perp \cos\theta, g_\perp \sin\theta, g_{||})$$

where:

$$g_\perp \equiv \left(g_x^2 + g_y^2\right)^{1/2} \quad \geq 0\,, \qquad g_{||} \equiv g_z$$

[1] the $x-$, $y-$ axes do not have to be specified yet.

and θ is an appropriate angle, i.e.

$$\theta = \arctan(g_y/g_x)$$

In the same way, the wave-number $\mathbf{k}$ is expressed as :

$$\mathbf{k} = (k_x, k_y, k_z) \equiv (k_\perp \cos\alpha, k_\perp \sin\alpha, k_\parallel)$$

and the reservoir-particle velocity $\mathbf{v_1}$ reads:

$$\mathbf{v_1} = (v_{1x}, v_{1y}, v_{1z}) \equiv (v_{1\perp} \cos\beta, v_{1\perp} \sin\beta, v_{1\parallel})$$

where all definitions are obvious. Finally, the matrix $\mathbf{N}(\tau)$ was defined in the beginning of this chapter.

Collecting the above definitions we have:

$$\begin{aligned}
\mathbf{N}(\tau)\,\mathbf{g} &= \Omega^{-1} \begin{pmatrix} \sin\Omega\tau & s\,(1-\cos\Omega\tau) & 0 \\ s\,(\cos\Omega\tau - 1) & \sin\Omega\tau & 0 \\ 0 & 0 & \Omega\tau \end{pmatrix} \begin{pmatrix} g_\perp \cos\theta \\ g_\perp \sin\theta \\ g_\parallel \end{pmatrix} \\
&= s\,\Omega^{-1} \begin{pmatrix} \sin s\Omega\tau & (1-\cos s\Omega\tau) & 0 \\ (\cos\Omega\tau - 1) & \sin s\Omega\tau & 0 \\ 0 & 0 & s\Omega\tau \end{pmatrix} \begin{pmatrix} g_\perp \cos\theta \\ g_\perp \sin\theta \\ g_\parallel \end{pmatrix} \\
&= s\,\Omega^{-1} \begin{pmatrix} g_\perp\,[\sin\theta - \sin(\theta - s\Omega\tau)] \\ -g_\perp\,[\cos\theta - \cos(\theta - s\Omega\tau)] \\ s\Omega g_\parallel \tau \end{pmatrix}
\end{aligned}$$

so now,

$$\begin{aligned}
\mathbf{k}\,\mathbf{N}(\tau)\,\mathbf{g} &= \begin{pmatrix} k_\perp \cos\alpha \\ k_\perp \sin\alpha \\ k_\parallel \end{pmatrix} s\,\Omega^{-1} \begin{pmatrix} g_\perp\,[\sin\theta - \sin(\theta - s\Omega\tau)] \\ -g_\perp\,[\cos\theta - \cos(\theta - s\Omega\tau)] \\ s\Omega g_\parallel \tau \end{pmatrix} \\
&= \quad \cdots \\
&= s\,\frac{k_\perp\, g_\perp}{\Omega}\,[\sin(\theta-\alpha) - \sin(\theta - \alpha - s\Omega\tau)] + k_\parallel\, g_\parallel\, \tau \\
&\equiv Z'\,[\sin(\theta-\alpha) - \sin(\theta - \alpha - \Omega'\tau)] + k_\parallel\, g_\parallel\, \tau \qquad \text{(C.2)}
\end{aligned}$$

where we set:

$$Z' = s\,\frac{k_\perp\, g_\perp}{\Omega} \equiv s\,Z \qquad \Omega' = s\,\Omega$$

Therefore, finally:

$$e^{i\,\mathbf{k}\,\mathbf{N}(\tau)\,\mathbf{g}} = e^{i\,Z'\,\sin(\theta-\alpha)}\, e^{-i\,Z'\,\sin(\theta-\alpha-\Omega'\tau)}\, e^{i\,k_\parallel\, g_\parallel\, \tau}$$

Note that, in the limit of vanishing Ω, one recovers $e^{i\,\mathbf{k}\,\mathbf{g}\,\tau}$ as expected (see in J)[2].

Now, the tensor product $\mathbf{k}\otimes\mathbf{k}$ can be evaluated as:

$$\begin{aligned}
\mathbf{k}\otimes\mathbf{k} &= \begin{pmatrix} k_\perp \cos\alpha \\ k_\perp \sin\alpha \\ k_\| \end{pmatrix} \begin{pmatrix} k_\perp \cos\alpha \\ k_\perp \sin\alpha \\ k_\| \end{pmatrix}^T \\
&= \begin{pmatrix} k_\perp^2 \cos^2\alpha & k_\perp^2 \sin\alpha\cos\alpha & k_\perp k_\| \cos\alpha \\ SYM & k_\perp^2 \sin^2\alpha & k_\perp k_\| \sin\alpha \\ SYM & SYM & k_\|^2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} k_\perp^2 (1-\cos 2\alpha) & k_\perp^2 \sin 2\alpha & 2k_\perp k_\| \cos\alpha \\ SYM & k_\perp^2(1+\cos 2\alpha) & 2k_\perp k_\| \sin\alpha \\ SYM & SYM & 2k_\|^2 \end{pmatrix} \qquad \text{(C.3)}
\end{aligned}$$

We may now use *Euler*'s formulae:

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}), \qquad \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi})$$

and the Bessel function identity:

$$e^{i\,x\,\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\,\phi} \qquad \forall x, \phi \in \Re \qquad \text{(C.4)}$$

in order to evaluate the diffusion matrix elements.

C.2 Correlations $C_{ij}(\tau)$

Let us evaluate the matrix elements D_{ij} of the force auto-correlation matrix. We have seen that they are given by (C.1) (or (3.36bis)):

$$C_{ij}(\tau) = n \int d\mathbf{v_1} (2\pi)^3 \int d\mathbf{k}\ \phi_{eq}(v_1)\, \tilde{V}_k^2\, k_i\, k_j\, e^{i\mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v}-\mathbf{v_1})}$$

Remember that $\tilde{V}_k$ is a *real* function of k (see in §3.4.2). Combining with all the above relations, we have e.g.

$$\begin{aligned}
C_{11}(\tau) &= \frac{n}{m^2}(2\pi)^3 \int_0^\infty dv_{1\perp}\, v_{1\perp} \int_{-\infty}^{\infty} dv_{1\|} \int_0^{2\pi} d\beta\ \phi_{eq}(v_1) \\
&\qquad \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^{\infty} dk_\| \int_0^{2\pi} d\alpha\, \tilde{V}_k^2\, k_\perp^2 \cos^2\alpha \\
&\qquad\qquad e^{i\,Z'\,\sin(\theta-\alpha)}\, e^{-i\,Z'\,\sin(\theta-\alpha-\Omega'\tau)}\, e^{i\,k_\|\,g_\|\,\tau} \qquad \text{(C.5)}
\end{aligned}$$

[2] As $\Omega \to 0$, $\mathbf{N}(\tau) \to \tau\mathbf{I}$ so

$$\mathbf{k}\,\mathbf{\Delta r} = \mathbf{k}\,\mathbf{N}(\tau)\,\mathbf{g} \to \mathbf{k}\,\mathbf{g}\,\tau\,.$$

The angle integration in α can be carried out straightaway. Using the Bessel function identity (6.9), the exponentials can be expressed as:

$$e^{i\,sZ\,\sin(\theta-\alpha)} = \sum_{m=-\infty}^{\infty} J_m(Z)\, e^{ims(\theta-\alpha)}$$

$$e^{-i\,sZ\,\sin(\theta-\alpha-s\Omega\tau)} = \sum_{n=-\infty}^{\infty} J_n(Z)\, e^{-ins(\theta-\alpha)}\, e^{in\Omega\tau}$$

so that:

$$e^{i\,\mathbf{k}\,\mathbf{N}(\tau)\,\mathbf{g}} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_n(Z)\, J_m(Z)\, e^{i(m-n)s(\theta-\alpha)}\, e^{i(n\Omega\tau\,+\,k_\parallel\, g_\parallel\,\tau)} \qquad \text{(C.6)}$$

Let us choose the $x-$axis to be along $\mathbf{g}\perp$ (see figure 1); as a consequence, $\theta = 0$. Furthermore, collecting the quantities containing the angle variable α and using Euler's formulae and the relation:

$$\int_0^{2\pi} e^{ik\alpha}\, d\alpha \;=\; 2\pi\, \delta_{k,0}^{Kr}$$

($\delta_{k,0}^{Kr}$ denotes the Kronecker delta symbol, equal to 1 if and only if the integer k is equal to zero) we find a set of selection rules relating the dummy variables n, m:

$$\int_0^{2\pi} e^{i(n-m)s\alpha}\, \cos^2\alpha\, d\alpha \;=\; \cdots \;=\; \frac{1}{4}(2\pi)(2\delta_{n,m}^{Kr} + \delta_{m,n+2}^{Kr} + \delta_{m,n-2}^{Kr})$$

In the same way:

$$\int_0^{2\pi} e^{i(n-m)s\alpha}\, \sin^2\alpha\, d\alpha \;=\; \cdots \;=\; \frac{1}{4}(2\pi)(2\delta_{n,m}^{Kr} - \delta_{m,n+2}^{Kr} - \delta_{m,n-2}^{Kr})$$

and

$$\int_0^{2\pi} e^{i(n-m)s\alpha}\, \sin\alpha\, \cos\alpha\, d\alpha \;=\; \cdots \;=\; \frac{1}{4}(2\pi)\, i\, s(\delta_{m,n+2}^{Kr} - \delta_{m,n-2}^{Kr})$$

The last three expressions appear in the elements C_{11}, C_{22} and C_{12} respectively.

We thus end up with expression (6.10) (in the main text) for the force auto-correlations:

$$\underline{\underline{\mathbf{C}}}(\tau) = n \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1})\, (2\pi)^4 \int_0^{\infty} dk_\perp\, k_\perp \int_{-\infty}^{\infty} dk_\parallel\, \tilde{V}_k^2 \sum_{n=-\infty}^{\infty} e^{i\omega_n(\tau)}$$

$$J_n \begin{pmatrix} \frac{1}{4}\, k_\perp^2\, (2J_n + J_{n+2} + J_{n-2}) & i\, s\, \frac{1}{4}\, k_\perp^2\, (J_{n+2} - J_{n-2}) \\ i\, s\, \frac{1}{4}\, k_\perp^2\, (J_{n+2} - J_{n-2}) & \frac{1}{4}\, k_\perp^2\, (2J_n + J_{n+2} + J_{n-2}) \\ \frac{1}{2}\, k_\perp k_\parallel (J_{n-1} + J_{n+1}) & -s\frac{1}{2}\, k_\perp k_\parallel (J_{n-1} - J_{n+1}) \end{pmatrix}$$

$$\left.\begin{matrix} \frac{1}{2}\,k_\perp k_\| (J_{n-1}+J_{n+1}) \\ -s\frac{1}{2}\,k_\perp k_\| (J_{n-1}-J_{n+1}) \\ k_\|^2 J_n \end{matrix}\right) \qquad \text{(C.7)}$$

i.e. (6.10) (in frame 1). J_n stands for the Bessel function of the first kind $J_n(Z)$; Z was defined above; ω_n is defined as:

$$\omega_n = k_\| g_\| + n\Omega \qquad .$$

C.3 Diffusion coefficients A_{ij}

As an example, let us evaluate A_{11}. From the results of chapter 3 we see that it is given by:

$$A_{11} \;=\; \frac{1}{m^2}\int_0^\infty d\tau\, C_{1j}(\tau)\, R_{1j}(\tau)$$

(a summation over $j = 1, 2, 3$ is understood) so combining with all the above relations, we obtain:

$$\begin{aligned} A_{11} &= \frac{1}{m^2}\int_0^\infty d\tau\, n \int d\mathbf{v_1} (2\pi)^3 \int d\mathbf{k}\;\, \phi_{eq}(v_1)\, \tilde{V}_k^2\, k_1\, k_j\, e^{i\mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v}-\mathbf{v_1})}\, R_{1j}(\tau) \\ &= \frac{n}{m^2}(2\pi)^3 \int_0^\infty d\tau \int d\mathbf{v_1} \int d\mathbf{k}\;\, \phi_{eq}(v_1)\, \tilde{V}_k^2\, e^{i\mathbf{k}\,\mathbf{N}(\tau)\,(\mathbf{v}-\mathbf{v_1})} \\ &\qquad \left[k_x\, k_x\, R_{11}(\tau) \,+\, k_x\, k_y\, R_{12}(\tau) \,+\, k_x\, k_z\, R_{13}(\tau)\right] \\ &= \frac{n}{m^2}(2\pi)^3 \int_0^\infty d\tau \int_0^\infty dv_{1\perp}\, v_{1\perp} \int_{-\infty}^\infty dv_{1\|} \int_0^{2\pi} d\beta\;\, \phi_{eq}(v_1) \\ &\qquad \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^\infty dk_\| \int_0^{2\pi} d\alpha\, \tilde{V}_k^2 \\ &\quad e^{i\,Z'\;\sin(\theta-\alpha)}\; e^{-i\,Z'\;\sin(\theta-\alpha-\Omega\tau)}\; e^{i\,k_\|\;g_\|\;\tau} \\ &\qquad (k_\perp^2 \cos^2\alpha\, \cos\Omega'\tau \,+\, k_\perp^2 \sin\alpha\cos\alpha\, \sin\Omega'\tau) \end{aligned}$$

or, in more compact form:

$$A_{11} \quad = \quad \frac{1}{m^2}\int_0^\infty d\tau\, (C_{11}\, \cos\Omega\tau \,+ C_{12}\, s\, \sin\Omega\tau) \qquad \text{(C.8)}$$

Now, using the above result (6.10) for C_{ij}, as well as the definition of ω_n, along with Euler's formulae:

$$\begin{aligned} \cos\Omega\tau\, e^{i\omega_n\tau} &= \frac{1}{2}(e^{i\Omega\tau} + e^{-i\Omega\tau})\, e^{i\omega_n\tau} \;=\; \frac{1}{2}(e^{i\omega_{n+1}\tau} + e^{i\omega_{n-1}\tau}) \\ \sin\Omega\tau\, e^{i\omega_n\tau} &= \frac{1}{2i}(e^{i\Omega\tau} - e^{-i\Omega\tau})\, e^{i\omega_n\tau} \;=\; \frac{1}{2i}(e^{i\omega_{n+1}\tau} - e^{i\omega_{n-1}\tau}) \end{aligned}$$

the infinite series in the integrand in (C.8) becomes:

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \big[\frac{1}{4}\, J_n\,(2J_n + J_{n+2} + J_{n-2}) \cos\Omega\tau\, e^{i\omega_n \tau} \\
&\qquad\qquad + i\, s\, \frac{1}{4}\, J_n\,(J_{n+2} - J_{n-2})\, s\, \sin\Omega\tau\, e^{i\omega_n \tau} \,+\, 0\big] \\
= \;& \frac{1}{4} \sum_{n=-\infty}^{\infty} \big[J_n\,(2J_n + J_{n+2} + J_{n-2})\, \frac{1}{2}(e^{i\omega_{n+1}\tau} + e^{i\omega_{n-1}\tau}) \\
&\qquad\qquad + i\, J_n\,(J_{n+2} - J_{n-2})\, \frac{1}{2i}(e^{i\omega_{n+1}\tau} - e^{i\omega_{n-1}\tau})\big]
\end{aligned}$$

or[3], shifting the series from n to $n \pm 1$ appropriately:

$$\begin{aligned}
\cdots = \frac{1}{4} \sum_{n=-\infty}^{\infty} (J_{n+1} + J_{n-1})^2\, e^{i\omega_n \tau} \;&=\; \sum_{n=-\infty}^{\infty} \left(\frac{n\, J_n}{Z}\right)^2 e^{i\omega_n \tau} \\
&\equiv\; \sum_{n=-\infty}^{\infty} (M_1)_{11}^{(n)}\, e^{i\omega_n \tau}
\end{aligned}$$

where we have used a well-known Bessel function recursive relation (some properties of Bessel functions of the first kind are summarized in a separate Appendix).

We have thus obtained one of the elements appearing in the final expressions (6.14a), (6.15a) in the text. Iterating the same procedure for all the matrix elements A_{ij}, G_{ij} as defined by (6.14), we obtain the bulky expressions (6.15) for all the matrix elements $M_{1,2}{}_{ij}^{(n)}$.

C.4 Coefficients in frame 2

Consider a new reference frame $x'y'z'$ ('frame 2'). Once more, the x-axis lies along the field $\mathbf{B}$, yet the x-axis is now taken along $v_\perp$:

$$\{\hat{e}_1,\, \hat{e}_2,\, \hat{e}_3\} \,=\, \{\hat{v}_\perp,\, \hat{b} \times \hat{v}_\perp,\, \hat{b}\}$$

This frame is essentially obtained through a rotation by an angle $\beta = (\widehat{g_\perp, v_\perp})$ with respect to frame 1; cf. fig. 1, 2. Therefore, the form of the kinetic equation in this frame can be directly obtained by a rotational transformation of the linear operators in it.

Remember that the velocity-diffusion matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ -A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & -A_{yz} & A_{zz} \end{pmatrix}$$

[3]Remember that $s^2 = (\pm 1)^2 = 1$.

Now, we may shift to frame 2, by rotating the previous frame by $\beta = (\widehat{g_\perp, v_{1\perp}})$ (cf. fig. 1, 2) [4], and then carrying out the integration in β [5]. This results in a velocity-diffusion matrix of the elegant form:

$$
\begin{aligned}
\mathbf{D} &= \int_0^{2\pi} d\beta \, \mathbf{R}_3^{-1}(\beta) \, \mathbf{A} \, \mathbf{R}_3(\beta) \; = \; \cdots \\
&= \begin{pmatrix} \frac{1}{2}(A_{11} + A_{22}) & \frac{1}{2}(A_{12} - A_{21}) & 0 \\ \frac{1}{2}(A_{21} - A_{12}) & \frac{1}{2}(A_{11} + A_{22}) & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \\
&\equiv \begin{pmatrix} D_\perp & D_\angle & 0 \\ -D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix}
\end{aligned}
\tag{C.9}
$$

where $\mathbf{R}_3(\beta)$ is the rotation matrix around z:

$$
\mathbf{R}_3(\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Furthermore, one may easily show that applying the same transformation to (6.22) for the vector $\mathbf{a}$, that is evaluating $\mathbf{a}' = \mathbf{R}^{-1}(\beta)\,\mathbf{a}$ with respect to the *new* diffusion matrix elements (in frame 2), one obtains exactly expressions (6.26) for the vectors $\mathbf{a}'$ (we have dropped the primes in the text).

C.5 Correlations in frame 2

It is interesting to see that the correlation matrix calculated previously (see (6.10)) comes out to be diagonal in this frame. An easy way to see this, is by combining (**1.**40):

$$
\mathbf{A} = \frac{1}{m^2} \int_0^\infty d\tau \, \mathbf{C}(\tau) \, \mathbf{R}^T(\tau)
$$

with the above definition of $\mathbf{D} = \mathbf{A}'$. Since the dynamic matrix $\mathbf{R}(\tau)$ is a rotation matrix itself (rotating the particle velocity vector around the guiding center of the trajectory), its transpose $\mathbf{R}^T(\tau) = \mathbf{R}^{-1}(\tau) = \mathbf{R}(-\tau)$ permutes with $\mathbf{R}(\beta)$; we therefore have:

$$
\mathbf{A}' = \frac{1}{m^2} \int_0^\infty d\tau \, \mathbf{R}^{-1}(\beta) \left[\mathbf{C}(\tau) \, \mathbf{R}^T(\tau) \right] \mathbf{R}(\beta)
$$

[4] Under a rotational transformation: $\mathbf{v}' = \mathbf{R}(\beta)\,\mathbf{v}$ the gradient of a scalar u, i.e. $\frac{\partial u}{\partial v_i}$, becomes $R_{ij}(\beta)\,\frac{\partial u}{\partial v'_j}$, the divergence of a vector $\mathbf{a}$ $\frac{\partial}{\partial v_i}\,a_i$ becomes $\frac{\partial}{\partial v'_j}\,R^{-1}_{ji}(\beta)\,a_i$ and so forth.

[5] Remember the integration in $\mathbf{v_1}$:

$$
\int d^3\mathbf{v_1} \cdot \; = \int_0^\infty dv_{1,\perp}\, v_{1,\perp} \int_{-\infty}^\infty dv_{1,\parallel} \int_0^{2\pi} d\beta \;\; \cdot
$$

; also note that the (any) form of the equilibrium reservoir d.f. $\phi_{eq} = \phi(v_\perp, v_\parallel)$ is independent of β, so this integration can be straightforward carried out.

$$= \frac{1}{m^2} \int_0^\infty d\tau \left[\mathbf{R}^{-1}(\beta)\, \mathbf{C}(\tau)\, \mathbf{R}(\beta) \right] \mathbf{R}^T(\tau)$$
$$\equiv \frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{C}'(\tau)\ \mathbf{R}^T(\tau)$$

So, evaluating $\mathbf{R}^{-1}(\beta)\, \mathbf{C}(\tau)\, \mathbf{R}(\beta)$ from (6.10) we obtain the diagonal matrix:

$$\mathbf{C}'(t_1, t_2) =$$

$$\begin{aligned}
= \quad & n \int d\mathbf{v_1}\, \phi_{eq}(\mathbf{v_1})\, (2\pi)^3 \int_0^\infty dk_\perp\, k_\perp \int_{-\infty}^\infty dk_\parallel\, \tilde{V}_k^2 \sum_{n=-\infty}^{\infty} e^{i\omega_n (t_1 - t_2)} \\
& J_n \begin{pmatrix} \frac{1}{4}\, k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & 0 & 0 \\ 0 & \frac{1}{4}\, k_\perp^2 (2J_n + J_{n+2} + J_{n-2}) & 0 \\ 0 & 0 & k_\parallel^2 J_n \end{pmatrix} \\
\equiv \quad & \begin{pmatrix} C_\perp & 0 & 0 \\ 0 & C_\perp & 0 \\ 0 & 0 & C_\parallel \end{pmatrix} \qquad \text{(C.10)}
\end{aligned}$$

in frame 2.

Appendix D

Derivation of the markovian ($\Phi-$) collision term

Let us explain the derivation of the final result for the plasma *M-FP* equation (see ch. 7).

We have defined the new 6x6 *diffusion matrix* $\mathbf{D}^{(\Phi)}$ (according to (5.9))[1]:

$$\begin{aligned}
\mathbf{D}^{(\Phi)} &= \\
&= \frac{n}{m^2}\int_0^t d\tau\, \mathcal{A}_{t'} \begin{pmatrix} \mathbf{N}(t') \\ \mathbf{R}(t') \end{pmatrix} \mathbf{C}(t-t', t-t'-\tau) \begin{pmatrix} \mathbf{N}^{\mathbf{T}}(t'+\tau) & \mathbf{R}^{\mathbf{T}}(t'+\tau) \end{pmatrix} \\
&\equiv \begin{pmatrix} \mathbf{D_{11}}^{(\Phi)} & \mathbf{D_{12}}^{(\Phi)} \\ \mathbf{D_{21}}^{(\Phi)} & \mathbf{D_{22}}^{(\Phi)} \end{pmatrix} \qquad \text{(D.1)}
\end{aligned}$$

Remember that $\mathcal{A}_{t'}$ was defined in (5.4) (also see in Appendix E). The dynamic matrices $\mathbf{N}$, $\mathbf{R}$ were defined in the text (see (6.5)). For the sake of comparison, notice that setting $t' = 0$ we immediately recover the analogous expression for the Θ operator:

$$\begin{aligned}
\mathbf{D}^{(\Theta)} &= \mathbf{D}^{(\Theta)}(\mathbf{v}; t; \Omega) \\
&= \frac{n}{m^2}\int_0^t d\tau \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix} \mathbf{C}(t, t-\tau) \begin{pmatrix} \mathbf{N}^{\mathbf{T}}(\tau) & \mathbf{R}^{\mathbf{T}}(\tau) \end{pmatrix} \\
&= \frac{n}{m^2}\int_0^t d\tau\; \mathbf{C}(\tau) \begin{pmatrix} \mathbf{N}^{\mathbf{T}}(\tau) \\ \mathbf{R}^{\mathbf{T}}(\tau) \end{pmatrix} \\
&\equiv \begin{pmatrix} \mathbf{D_{11}}^{(\Theta)} & \mathbf{D_{12}}^{(\Theta)} \\ \mathbf{D_{21}}^{(\Theta)} & \mathbf{D_{22}}^{(\Theta)} \end{pmatrix} \qquad \text{(D.2)}
\end{aligned}$$

(remember that $\mathbf{N}(0) = \mathbf{I}$, $\mathbf{R}(0) = \mathbf{0}$). Obviously, $\mathbf{I}$ and $\mathbf{0}$ denote the *unit* and *zero* (square) matrix, respectively, of order 3.

[1] The asymptotic limit $t \to \infty$ defines, rigorously speaking, the two operators in the first two formulae.

D.1 Calculation of $\mathbf{D_{22}}^{(\Phi)}$

Let us first calculate the velocity-diffusion matrix $\mathbf{D_{22}}^{(\Phi)}$. The integrand (in the time-integral) in it will be:

$$\mathcal{A}_{t'}\, \mathbf{R}(t')\, \mathbf{C}(\tau)\, \mathbf{R^T}(t'+\tau)$$

Recalling the (diagonal) form of the correlation matrix [2]:

$$C_{ij} \;=\; C_\perp\, \delta_{ij}\, (1 - \delta_{i3}\delta_{j3}) \;+\; C_\parallel\, \delta_{i3}\, \delta_{j3} \tag{D.3}$$

we see that the integrand in $D_{lm}^{(22)} \equiv D_{lm}^{(VV)}$ will be:

$$\sum_{i,j=1,3} \mathcal{A}_{t'}\, R_{li}(t')\, C_{ij}(\tau)\, R_{mj}(t'+\tau) =$$

$$= C_\perp(\tau)\, \mathcal{A}_{t'} \sum_{i=1,2} R_{li}(t')\, R_{mi}(t'+\tau) \;+\; C_\parallel(\tau)\, \mathcal{A}_{t'}\, R_{l3}(t')\, R_{m3}(t'+\tau) \tag{D.4}$$

However, $R_{mj}(t'+\tau) = R_{jm}(-t'-\tau)$ (so $R_{lj}(t')\, R_{mj}(t'+\tau) = R_{lm}(-\tau) = R^T_{lm}(\tau)$); we thus obtain:

$$... \;=\; C_\perp(\tau)\, \mathcal{A}_{t'}\, R^T_{lm}(\tau) \;+\; C_\parallel(\tau)\, \mathcal{A}_{t'}\, R^T_{l3}(\tau)$$

$$... \;=\; C_\perp(\tau)\, R^T_{lm}(\tau) \;+\; C_\parallel(\tau)\, \mathcal{A}_{t'}\, R^T_{l3}(\tau)$$

which is *exactly* the expression appearing in the $\Theta-$ counterpart of the integrand inside the diffusion matrix:

$$\mathbf{I}\, \mathbf{C}(\tau)\, \mathbf{R^T}(\tau)$$

We have thus shown that the old velocity diffusion matrix is exactly recovered:

$$\mathbf{D_{22}}^{(\Phi)} = \mathbf{D_{22}}^{(\Theta)}$$

The form of $\mathbf{D_{22}}$ was presented in the text (and evaluated in detail in the Appendix)[3]. The final result therefore reads:

$$\mathbf{D_{22}} = \mathbf{A}(v_\perp, v_\parallel) = \begin{pmatrix} D_\perp & D_\angle & 0 \\ -D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \equiv D_\perp\, \mathbf{I}_\perp \;+\; D_\angle\, \mathbf{I}_\angle \;+ D_\parallel\, \mathbf{I}_\parallel \tag{D.5}$$

The definition of matrices $\mathbf{I}_\perp$, $\mathbf{I}_\angle$, $\mathbf{I}_\parallel$ is obvious.

[2] Remember that $\mathbf{C}(t-t', t-t'-\tau) = \mathbf{C}(\tau)$ since this is a stationary process; literally speaking, the calculation provided here is valid in frame 2, i.e. for a diagonal correlation matrix: otherwise, it is replaced by a *longer* calculation, giving *the same* result (see in Appendix I).

[3] Of course, this result is confirmed by carefully evaluating all matrix elements one by one.

Remember that only the *symmetric part* of the matrix will survive in the diffusive term $D_{ij}\frac{\partial^2 f}{\partial v_i \partial v_j}$, so we finally obtain:

$$\mathbf{D_{VV}} = \mathbf{D_{22}}^{(SYM)} = \frac{1}{2}\big(\mathbf{A} + \mathbf{A}^T\big) = \begin{pmatrix} D_\perp & 0 & 0 \\ 0 & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \equiv D_\perp \, \mathbf{I}_\perp \, + D_\parallel \, \mathbf{I}_\parallel \tag{D.6}$$

Relations (D.3, 4) are identical to (I.25, 26) obtained in Appendix without making use of the (diagonal) structure of C_{ij} (in frame 2). As said previously (see footnote 1), the same comment holds for the expressions that will be derived in the following: a set of identical expressions have been derived in Appendix I (§I.3) via a different (less concise, more lengthy, yet more general) method.

D.2 Calculation of $\mathbf{D_{21}}^{(\Phi)}$

Let us now calculate $\mathbf{D_{21}}^{(\Phi)}$ (see (D.1)). Following the same method as in the previous paragraph, and taking into account (D.3), the integrand in it will be given by an expression analogous to (D.4):

$$\sum_{i,j=1,3} \mathcal{A}_{t'} \, R_{li}(t') \, C_{ij}(\tau) \, N_{mj}(t'+\tau) =$$

$$= \, C_\perp(\tau) \, \mathcal{A}_{t'} \sum_{i=1,2} R_{li}(t') \, N_{mi}(t'+\tau) \; + \; C_\parallel(\tau) \, \mathcal{A}_{t'} \, R_{l3}(t') \, N_{m3}(t'+\tau) \tag{D.7}$$

i.e. in compact form (thanks to (D.3)):

$$\mathcal{A}_{t'} \, \mathbf{R}(t') \, \mathbf{C}(\tau) \, \mathbf{N^T}(t'+\tau) \; = \; \mathbf{C}(\tau) : \Big[\mathcal{A}_{t'} \, \mathbf{R}(t') \, \mathbf{N^T}(t'+\tau)\Big]$$

where $\mathbf{A}{:}\mathbf{B} = \mathbf{M}$ denotes the tensor product of matrices $\mathbf{A}$ and $\mathbf{B}$, so $M_{ij} = A_{ij} \, B_{ij}$. We may now substitute from the definition of the $\mathbf{N}$, $\mathbf{R}$ matrices and then evaluate the action of the averaging operator $\mathcal{A}_{t'}$. This procedure is exact, though quite lengthy. However, the above quantity can be evaluated in a very concise (and elegant!) manner, by making use of the group properties established in §3.1.1. The result found this way confirms exactly the previous one.

We have:

$$\begin{aligned} \mathcal{A}_{t'} \, \mathbf{R}(t') \, \mathbf{N}^T(t'+\tau) \quad &= \quad \mathcal{A}_{t'} \Big[\mathbf{N}(t'+\tau) \, \mathbf{R}^T(t')\Big]^T \\ &= \quad \mathcal{A}_{t'} \Big[-\mathbf{N}(-t'-\tau) \, \mathbf{R}(t'+\tau) \, \mathbf{R}(-t')\Big]^T \\ &= \quad \mathcal{A}_{t'} \Big[-\mathbf{N}(-t'-\tau) \, \mathbf{R}(\tau)\Big]^T \end{aligned}$$

$$= \; -\mathcal{A}_{t'} \left[\mathbf{R}^T(\tau)\, \mathbf{N}^T(-t'-\tau) \right]$$
$$= \; \mathbf{R}^T(\tau) \left[-\mathcal{A}_{t'}\, \mathbf{N}^T(-t'-\tau) \right]$$

Now, using the properties listed in Appendix E, we can immediately show that:

$$-\mathcal{A}_{t'}\, \mathbf{N}^T(-t'-\tau) \;=\; s\,\Omega^{-1}\, \mathbf{I}_{\angle} \;+\; \mathcal{A}_{t'}\, \tau\, \mathbf{I}_{\|}$$

(where $\mathbf{I}_{\angle} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{I}_{\|} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, as in the end of the previous paragraph) and the integrand finally becomes:

$$s\,\Omega^{-1}\, \mathbf{R}^T(\tau)\, \mathbf{I}_{\angle} \;=\; \ldots \;=\; s\,\Omega^{-1}\, \cos\Omega\tau\, \mathbf{I}_{\angle} \;-\; \Omega^{-1}\, \sin\Omega\tau\, \mathbf{I}_{\perp}$$

Finally, recalling that $\cos\Omega\tau = R_{11}(\tau)$, $\sin\Omega\tau = R_{12}(\tau)$ appear in the expression which defines $\mathbf{D_{22}} = \mathbf{A}$ (cf. (D.2, D.1), (D.5)) we find:

$$\begin{aligned} \mathbf{D_{21}}^{(\Phi)} &= s\,\Omega^{-1} \begin{pmatrix} -D_{\angle} & D_{\perp} & 0 \\ -D_{\perp} & -D_{\angle} & 0 \\ 0 & 0 & 0 \end{pmatrix} + G_{33} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= s\,\Omega^{-1}\, D_{\perp}\, \mathbf{I}_{\angle} \;-\; s\,\Omega^{-1}\, D_{\angle}\, \mathbf{I}_{\perp} \;+\; G_{33}\, \mathbf{I}_{\|} \end{aligned} \tag{D.8}$$

All quantities were defined previously[4].

D.3 Calculation of $\mathbf{D_{12}}^{(\Phi)}$

Let us calculate $\mathbf{D_{12}}^{(\Phi)}$. The integrand in it will be given by an expression analogous to (D.4):

$$\sum_{i,j=1,3} \mathcal{A}_{t'}\, N_{li}(t')\, C_{ij}(\tau)\, R_{mj}(t'+\tau) =$$

$$= C_{\perp}(\tau)\, \mathcal{A}_{t'} \sum_{i=1,2} N_{li}(t')\, R_{mi}(t'+\tau) \;+\; C_{\|}(\tau)\, \mathcal{A}_{t'}\, N_{l3}(t')\, R_{m3}(t'+\tau) \tag{D.9}$$

i.e. in compact form (due to (D.3), and only):

$$\mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{C}(\tau)\, \mathbf{R}^T(t'+\tau) \;=\; \mathbf{C}(\tau) : \left[\mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{R}^{\mathbf{T}}(t'+\tau) \right]$$

As in the previous paragraph, we may either:
(a) substitute from the definition of the $\mathbf{N}$, $\mathbf{R}$ matrices and then evaluate the action of the averaging operator $\mathcal{A}_{t'}$, or
(b) evaluate the above quantity by making use of the properties relating these

[4]Note, in particular, that G_{33} is just as defined in Ch. 6; see (6.14, 15).

matrices (see in §3.1.1).
Both ways should (and *do* in fact) give the same result.

Let us adopt method (b) once again (more compact and elegant). Remember that $\mathbf{R}^T(t) = \mathbf{R}(-t) = \mathbf{R}^{-1}(t)$; also $\mathbf{R}(t_1 + t_2) = \mathbf{R}(t_1)\mathbf{R}(t_2)$ and $\mathbf{N}^T(t) = -\mathbf{N}(-t)$ $(t, t_1, t_2 \in \Re)$. Therefore, we have:

$$\begin{aligned}
\mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{R}^T(t'+\tau) &= \mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{R}(-t'-\tau) \\
&= \mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{R}(-t')\mathbf{R}(-\tau) \\
&= \mathcal{A}_{t'}\, \left[-\mathbf{N}(-t')\right] \mathbf{R}(-\tau) \\
&= \mathcal{A}_{t'}\, \mathbf{N}^T(t')\, \mathbf{R}^T(\tau)
\end{aligned}$$

Now, using the properties listed in Appendix E, we can immediately show that:

$$\mathcal{A}_{t'}\, \mathbf{N}^T(t') = -s\,\Omega^{-1} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv -s\,\Omega^{-1}\, \mathbf{I}_{\angle}$$

The integrand finally becomes:

$$s\,\Omega^{-1}\, \mathbf{R}^T(\tau)\, \mathbf{I}_{\angle} = \ldots = -s\,\Omega^{-1}\, \cos\Omega\tau\, \mathbf{I}_{\angle} + \Omega^{-1}\, \sin\Omega\tau\, \mathbf{I}_{\perp}$$

Finally, recalling that $\cos\Omega\tau = R_{11}(\tau)$, $\sin\Omega\tau = R_{12}(\tau)$ appear in the expression which defines $\mathbf{D_{22}} = \mathbf{A}$ (cf. (D.2, D.1), (D.5)) we find:

$$\begin{aligned}
\mathbf{D_{12}}^{(\Phi)} &= s\,\Omega^{-1} \begin{pmatrix} D_{\angle} & -D_{\perp} & 0 \\ D_{\perp} & D_{\angle} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= -s\,\Omega^{-1}\, D_{\perp}\, \mathbf{I}_{\angle} + s\,\Omega^{-1}\, D_{\angle}\, \mathbf{I}_{\perp}
\end{aligned} \tag{D.10}$$

All quantities were defined previously.

Notice that this expression, along with its counterpart derived in the end of the previous paragraph, finally give a cross-velocity-position diffusion matrix (cf. Chapter 5):

$$\mathbf{D_{VX}} \equiv \frac{1}{2}(\mathbf{D_{21}} + \mathbf{D_{12}}^T) = \cdots = s\,\Omega^{-1}\, D_{\angle}\, \mathbf{I}_{\angle} + \frac{1}{2} G_{33}\, \mathbf{I}_{\parallel}$$

D.4 Calculation of $\mathbf{D_{11}}^{(\Phi)}$

Let us calculate $\mathbf{D_{11}}^{(\Phi)}$. The integrand in it can be expressed as:

$$\mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{C}(\tau)\, \mathbf{N}^T(t'+\tau) = \mathbf{C}(\tau) : \left[\mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{N}^{\mathbf{T}}(t'+\tau)\right]$$

(taking into account (D.3)). Unlike previously, we will substitute from the definition of the $\mathbf{N}$ matrices and then evaluate the action of the averaging operator $\mathcal{A}_{t'}$. We have:

$$\mathcal{A}_{t'}\, \mathbf{N}_\alpha(t')\mathbf{N}_\alpha^{\mathbf{T}}(t'+\tau) =$$

$$\mathcal{A}_{t'}\, {\Omega_\alpha}^{-1} \begin{pmatrix} \sin\Omega_\alpha t' & s\,(1-\cos\Omega_\alpha t') & 0 \\ s\,(\cos\Omega_\alpha t' - 1) & \sin\Omega_\alpha t' & 0 \\ 0 & 0 & \Omega t' \end{pmatrix} \times$$

$$\qquad {\Omega_\alpha}^{-1} \begin{pmatrix} \sin\Omega_\alpha (t'+\tau) & s\,(1-\cos\Omega_\alpha (t'+\tau)) & 0 \\ s\,(\cos\Omega_\alpha (t'+\tau) - 1) & \sin\Omega_\alpha (t'+\tau) & 0 \\ 0 & 0 & \Omega (t'+\tau) \end{pmatrix}$$

$$= \; ...$$

$$= \; {\Omega_\alpha}^{-2} \begin{pmatrix} 1+\cos\Omega_\alpha\tau & -s\sin\Omega_\alpha\tau & 0 \\ s\sin\Omega_\alpha\tau & 1+\cos\Omega_\alpha\tau & 0 \\ 0 & 0 & \Omega_\alpha^2 \mathcal{A}_{t'} {t'}^2 \end{pmatrix} \qquad \text{(D.11)}$$

In order to evaluate the action of $\mathcal{A}_{t'}$ on the matrix elements, we have used the properties listed in the next section of this Appendix (see Appendix E). In brief, the averaging operator yields zero when acting on $\sin\Omega t'$, $\cos\Omega t'$, $1/2$ on $\sin^2\Omega t'$, $\cos^2\Omega t'$ and 1 when acting on 1.

Upon direct inspection from the old $(\Theta-)$ case (see in the beginning of this Appendix), we see that the above matrix results in a space-diffusion matrix of the form:

$$\mathbf{D_{11}}^{(\Phi)} = \Omega^{-2} \begin{pmatrix} Q+D_\perp & -D_\angle & 0 \\ D_\angle & Q+D_\perp & 0 \\ 0 & 0 & (\Omega^2 D_\parallel)\times\infty \end{pmatrix}$$

$$= \Omega^{-2}\,(Q+D_\perp)\,\mathbf{I}_\perp \,-\, \Omega^{-2}\,D_\angle\,\mathbf{I}_\angle \qquad \text{(D.12)}$$

The 33$-$ element $(D_\parallel^{(XX)} = D_\parallel\,\mathcal{A}_{t'}{t'}^2 = \infty)$ will be omitted, as discussed in the text, since neither D_{ij} nor f (according to our assumption) depend on z. All quantities were defined previously.

Note that all matrix elements are only functions of velocity $\mathbf{v}$, and not position $\mathbf{x}$, so the off-diagonal elements disappear; we thus remain with the diagonal matrix:

$$\mathbf{D_{XX}} = \Omega^{-2}\,(Q+D_\perp)\,\mathbf{I}_\perp$$

The same expression has been obtained, in a different way, in Appendix I; see (I.31, 32).

D.5 Final form of $\mathbf{D}^{(\Phi)}$, $\mathbf{a}^{(\Phi)}$

Collecting the final results of the last paragraphs, we obtain the matrix:

$$\mathbf{D}^{(\Phi)} = \begin{pmatrix} \Omega^{-2}(D_\perp+Q) & 0 & s\,\Omega^{-1}D_\angle & -s\,\Omega^{-1}D_\perp & 0 \\ 0 & \Omega^{-2}(D_\perp+Q) & s\,\Omega^{-1}D_\perp & s\,\Omega^{-1}D_\angle & 0 \\ -s\,\Omega^{-1}D_\angle & s\,\Omega^{-1}D_\perp & D_\perp & D_\angle & 0 \\ -s\,\Omega^{-1}D_\perp & -s\,\Omega^{-1}D_\angle & -D_\angle & D_\perp & 0 \\ 0 & 0 & 0 & 0 & D_\parallel \end{pmatrix} \qquad \text{(D.13)}$$

or, taking the symmetric part:

$$\mathbf{D}^{(\Phi)\,(SYM)} = \begin{pmatrix} \Omega^{-2}(D_\perp + Q) & 0 & 0 & -s\,\Omega^{-1}D_\perp & 0 \\ 0 & \Omega^{-2}(D_\perp + Q) & s\,\Omega^{-1}D_\perp & 0 & 0 \\ 0 & s\,\Omega^{-1}D_\perp & D_\perp & 0 & 0 \\ -s\,\Omega^{-1}D_\perp & 0 & 0 & D_\perp & 0 \\ 0 & 0 & 0 & 0 & D_\parallel \end{pmatrix} \tag{D.14}$$

For the sake of comparison, the corresponding matrix in the $\Theta-$ FPE reads:

$$\mathbf{D}^{(\Theta)} = \begin{pmatrix} 0 & 0 & \frac{1}{2}D_\angle^{(VX)} & -\frac{1}{2}D_\perp^{(VX)} & 0 \\ 0 & 0 & \frac{1}{2}D_\perp^{(VX)} & \frac{1}{2}D_\angle^{(VX)} & 0 \\ \frac{1}{2}D_\angle^{(VX)} & \frac{1}{2}D_\perp^{(VX)} & D_\perp & D_\angle & 0 \\ -\frac{1}{2}D_\perp^{(VX)} & \frac{1}{2}D_\angle^{(VX)} & -D_\angle & D_\perp & 0 \\ 0 & 0 & 0 & 0 & D_\parallel \end{pmatrix} \tag{D.15}$$

(the symmetric part is simply obtained by cancelling $D_\angle$ in the $VV-$ part).

The calculation for the drift vector **a** (see 5.10) is exactly analogous. It finally gives:

$$\mathbf{a} = (\mathbf{a}_{(\mathbf{x})}, \mathbf{a}_{(\mathbf{v})}) = (-s\,\Omega^{-1}\left(a_{(v)}\right)_x, s\,\Omega^{-1}\left(a_{(v)}\right)_y, 0;\, \left(a_{(v)}\right)_x, \left(a_{(v)}\right)_y, \left(a_{(v)}\right)_z) \tag{D.16}$$

Notice that the familiar relations:

$$a_i = -\frac{\partial D_{ij}^{(\Phi)}}{\partial q_j} \tag{D.17}$$

($\mathbf{q} = (\mathbf{x}, \mathbf{v})$) are exactly recovered by the new quantities (in 6d).

Appendix E

The time-averaging operator $\mathcal{A}_{t'}$

In order to construct the $\Phi-$ operator (see Appendix D), we have evaluated the explicit action of the averaging operator

$$\mathcal{A}_{t'} \cdot \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \cdot$$

on certain functions of t'. A list is provided below.

- Any *odd* function of t' gives zero:

$$f(-t') = f(t') \qquad \forall t' \in \Re \qquad \Longrightarrow \qquad \mathcal{A}_{t'} \, f(t') \, = \, 0$$

- *Constant* function:

$$\mathcal{A}_{t'} \, c \, = \, c$$

- Polynomial of t':

$$\mathcal{A}_{t'} \, {t'}^{2n+1} \, = \, 0 \qquad\qquad n = 0, \pm 1, \pm 2 \ldots$$

$$\mathcal{A}_{t'} \, {t'}^{2n} \, = \, +\infty \qquad\qquad n = \pm 1, \pm 2 \ldots$$

- The operator gives a *finite* exact result on periodic functions; using:

$$\mathcal{A}_{t'} \, e^{i\lambda t'} \, = \, \delta_{\lambda,0}$$

one may use Euler's formulae in order to prove the properties:

$$\mathcal{A}_{t'} \, \sin \lambda t' \, = \, 0$$

$$\mathcal{A}_{t'} \, \cos \lambda t' \, = \, \delta_{\lambda,0}$$

$$\mathcal{A}_{t'} \, \sin(\lambda t' + \mu) \, = \, \delta_{\lambda,0} \sin \mu$$

$$\mathcal{A}_{t'} \; \cos(\lambda t' + \mu) \;=\; \delta_{\lambda,0} \cos\mu$$

$$\mathcal{A}_{t'} \; \sin^2 \lambda t' \;=\; \frac{1}{2}\,(1 - \delta_{\lambda,0})$$

$$\mathcal{A}_{t'} \; \cos^2 \lambda t' \;=\; \frac{1}{2}\,(1 + \delta_{\lambda,0})$$

$$\mathcal{A}_{t'} \; \sin \lambda t' \cos \lambda t' \;=0$$

$(\lambda, \mu \in \Re)$.

- Let us study the case of products of periodic functions and polynomials. For instance,

$$\mathcal{A}_{t'} \, t' \; \cos \lambda t' \;=0$$

(due to parity). However, the operator gives an *ill-defined* (or, rather, inexistent) result in the following case:

$$\begin{aligned}
\mathcal{A}_{t'} \, t' \; \sin \lambda t' \quad &= \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt' \, (t' \; \sin \lambda t') \\
&= \quad \lim_{T\to\infty} \frac{1}{2T} \left(\frac{\sin \lambda t'}{\lambda^2} - \frac{t' \; \cos \lambda t'}{\lambda} \right|_{-T}^{T} \\
&= \quad \lim_{T\to\infty} \frac{1}{2T} \left(2\frac{\sin \lambda T}{\lambda^2} - 2\frac{T \cos \lambda T}{\lambda} \right) \\
&= \quad \ldots \\
&= \quad \lim_{T\to\infty} \frac{\sin \lambda T - \lambda T \; \cos \lambda T}{\lambda^2 T} \\
&= \quad \lim_{T\to\infty} \, (T \; \sin \lambda T)
\end{aligned}$$

$(\lambda \in \Re)$. The integration in the first line was carried out by an integration by parts. The final result was obtained by *De l'Hôpital*'s rule.

Appendix F

Mathematical appendix

F.1 Bessel functions of the first kind: Summary of properties

1. Recursive relations. Here are some relations between Bessel functions of the first order:

$$
\begin{aligned}
J_{n+1}(x) &= \frac{2n}{x} J_n(x) - J_{n-1}(x) \\
J_n'(x) &= \frac{1}{2} [J_{n-1}(x) - [J_{n+1}(x)] \\
x J_n'(x) &= x J_{n-1}(x) - n J_n(x) \\
x J_n'(x) &= n J_n(x) - x J_{n+1}(x) \\
\frac{d}{dx}[x^n J_n(x)] &= x^n J_{n-1}(x) \\
\frac{d}{dx}[x^{-n} J_n(x)] &= -x^{-n} J_{n+1}(x)
\end{aligned}
\tag{F.1}
$$

(for a more complete list, see in [1], [19]). Combining these relations, we can express any linear combination of Bessel functions $\sum_m c_m J_m(x)$ in terms of $J_n(x)$, $J_n'(x)$. For instance, from the first two equations, we have:

$$
\begin{aligned}
J_{n+1}(x) &= \frac{n}{x} J_n(x) - J_n'(x) \\
J_{n-1}(x) &= \frac{n}{x} J_n(x) + J_n'(x)
\end{aligned}
\tag{F.2}
$$

Then, setting $n \to n \pm 1$, respectively, and combining with these relations again, we have:

$$
\begin{aligned}
J_{n+2}(x) &= \left[\frac{2n(n+1)}{x^2} - 1\right] J_n(x) - \frac{2(n+1)}{x} J_n'(x) \\
J_{n-2}(x) &= \left[\frac{2n(n-1)}{x^2} - 1\right] J_n(x) + \frac{2(n-1)}{x} J_n'(x)
\end{aligned}
\tag{F.3}
$$

Therefore, quantities like: $J_{n+2}(x) + J_{n-2}(x) \pm 2J_n(x)$ are given by:

$$J_{n+2}(x) + J_{n-2}(x) - 2J_n(x) = \left[\left(\frac{2n}{x}\right)^2 - 4\right] J_n(x) - \frac{4}{x} J'_n(x)$$
$$J_{n+2}(x) + J_{n-2}(x) + 2J_n(x) = \left(\frac{2n}{x}\right)^2 J_n(x) - \frac{4}{x} J'_n(x) \qquad \text{(F.4)}$$

and so forth.

2. Remember the parity relation:

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{(F.5)}$$

implying:

$$J^2_{-n}(x) = J^2_n(x) \qquad \text{(F.6)}$$

3. Summation. Note the important identity:

$$\sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\,\phi} = e^{i\,x\,\sin\phi} \qquad \forall x, \phi \in \Re \qquad \text{(F.7)}$$

and the derived relations:

$$\sum_{n=-\infty}^{\infty} J_n(x)\, e^{\pm in\,\pi/2} = e^{\pm i\,x} \quad , \qquad \sum_{n=-\infty}^{\infty} J_n(x) \;= 1 \qquad \forall x, \phi \in \Re \qquad \text{(F.8)}$$

4. Addition theorems. We have often used the theorems (for *integer* orders n, p, k):

Neumann's addition theorem:

$$\sum_{k=-\infty}^{\infty} J_{n\pm k}(u)\, J_k(v) \;= J_n(u \mp v) \qquad \text{(F.9)}$$

(see 9.1.75, p. 363 in [1]).

Gegenbauer's addition theorem (see 9.1.79 in [1]):

$$\sum_{k=-\infty}^{\infty} \mathcal{G}_{p+k}(u)\, J_k(v)\, e^{i\,k\,\alpha} \;= \mathcal{G}_p(w)\, e^{i\,p\,\chi}$$
$$w^2 = u^2 + v^2 - 2uv\cos\alpha$$
$$w\cos\chi = u - v\cos\alpha\,, \quad w\sin\chi = v\sin\alpha \qquad \text{(F.10)}$$

($\mathcal{G}_n(z)$ may denote either $J_n(z)$ or $I_n(z)$ here). In particular, if $u = v$, then $w = 2\,|u\,\sin\frac{\alpha}{2}|$, $\chi = \frac{\pi}{2} - \frac{\alpha}{2}$,so it reduces to:

$$\sum_{k=-\infty}^{\infty} \mathcal{G}_{p+k}(u)\, J_k(u)\, e^{i\,k\,\alpha} \;= \mathcal{G}_p(2\,|u\,\sin\frac{\alpha}{2}|)\, e^{i\,p\,(\pi-\alpha)/2} \qquad \text{(F.11)}$$

also, for $p = 0$, $\mathcal{G}_n = J_n$:

$$\sum_{k=-\infty}^{\infty} J_k^2(u)\, e^{i\,k\,\alpha} = J_0(2\,u\,\sin\frac{\alpha}{2}) \tag{F.12}$$

F.2 About δ, δ_+ functions

1. δ_+ function. Our primary reference is [3] (App. 2 therein), [5]; also: [41] (p. 226).

We have used the relation:

$$\begin{aligned}\int_0^\infty dk\, e^{i\,kx} &= \lim_{\epsilon\to 0^+}\frac{\epsilon}{x^2+\epsilon^2} + i\lim_{\epsilon\to 0^+}\frac{x}{x^2+\epsilon^2}\\ &\equiv \pi\,\delta(x) + i\,\mathcal{P}(\frac{1}{x})\\ &\equiv \pi\,\delta_+(x)\end{aligned} \tag{F.13}$$

where $\mathcal{P}(\frac{1}{x})$ denotes the principal value of the Cauchy integral. Remember that $\delta(x)$ is an *even* function of x, while $\mathcal{P}(\frac{1}{x})$ is an *odd* one.

Also,

$$\int_0^\infty dk\,k\,e^{i\,kx} = -i\,\frac{d}{dx}\int_0^\infty dk\,e^{i\,kx} = -i\frac{d}{dx}\left[\pi\,\delta(x) + \mathcal{P}(\frac{1}{x})\right] \equiv -i\,\pi\,\delta'_+(x) \tag{F.14}$$

2. δ and δ' function. Infinite integrals of a real function f involving a δ function are evaluated by using:

$$\int_{-\infty}^\infty f(x)\,\delta(x-a) = f(a) \tag{F.15}$$

In consequence:

$$\int_{-\infty}^\infty f(x)\,\delta(ax+b) = \frac{1}{|a|}\int_{-\infty}^\infty f\left(\frac{u-b}{|a|}\right)\delta(u) = \frac{1}{|a|}\,f\left(\frac{-b}{|a|}\right) \tag{F.16}$$

For integrals involving a $\delta'(x) = \frac{d}{dx}\delta(x)$ function:

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,\delta'(x-a) &= \int_{-\infty}^\infty f(x)\,\frac{d}{dx}\delta(x-a) = -\int_{-\infty}^\infty \frac{df(x)}{dx}\,\delta(x-a)\\ &= -\left.\frac{df(x)}{dx}\right|_{x=a}\end{aligned} \tag{F.17}$$

i.e.

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,\delta'(ax+b) &= \frac{1}{|a|}\int_{-\infty}^\infty f\left(\frac{u-b}{a}\right)\frac{d}{du}\delta(u) = -\frac{1}{|a|}\,\frac{d}{du}f\left(\frac{u-b}{a}\right)\Big|_{u=0}\\ &= -\frac{1}{a|a|}\,f'\left(\frac{-b}{a}\right)\end{aligned} \tag{F.18}$$

F.3 Some integrals involving Gaussian and/or Bessel functions

1. First, recall the general integral:

$$\int_{-\infty}^{\infty} e^{-(Ax-iB)^2}\, dx \;=\; \frac{\sqrt{\pi}}{|A|} \tag{F.19}$$

according to which:

$$\int_{-\infty}^{\infty} e^{-iax}\, e^{-bx^2}\, dx \;=\; e^{-\frac{a^2}{4b}} \int_{-\infty}^{\infty} e^{-b(x+i\frac{a}{2b})^2}\, dx \;=\; e^{-a^2/4b}\, \frac{\sqrt{\pi}}{\sqrt{b}} \tag{F.20}$$

for $b > 0$. For $b < 0$ the integral diverges.

2.

$$\int_0^{\infty} e^{-\rho^2\, x^2}\, J_p(ax)\, J_p(bx)\, x\, dx \;=\; \frac{1}{2\rho^2}\, e^{-\frac{a^2+b^2}{4\rho^2}}\, I_p\left(\frac{ab}{2\rho^2}\right) \tag{F.21}$$

(see 6.633.2 in [19]). For $b = 0$ we obtain:

$$\int_0^{\infty} e^{-\rho^2\, x^2}\, J_p(ax)\, x\, dx \;=\; \frac{1}{2\rho^2}\, e^{-\frac{a^2}{4\rho^2}} \tag{F.22}$$

3.

$$\int_0^{\infty} e^{-\rho^2\, x^2}\, I_p(ax)\, J_p(bx)\, x\, dx \;=\; \frac{1}{2\rho^2}\, e^{\frac{a^2-b^2}{4\rho^2}}\, J_p\left(\frac{ab}{2\rho^2}\right) \tag{F.23}$$

(see 6.633.4 in [19]) where $I_p(x)$ is the *modified* Bessel function: $I_p(x) = J_p(ix)$. For $b = 0$ we obtain:

$$\int_0^{\infty} e^{-\rho^2\, x^2}\, I_p(ax)\, x\, dx \;=\; \frac{1}{2\rho^2}\, e^{\frac{a^2}{4\rho^2}} \tag{F.24}$$

4. Moments of a Gaussian distribution.

$$\int_0^{\infty} dx\, x^n\, e^{-bx^2} = \begin{cases} \frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{2^{k+1}}\, \frac{\sqrt{\pi}}{b^{k+1/2}} & n = 2k \\ \frac{k!}{2b^{k+1}} & n = 2k+1 \end{cases} \tag{F.25}$$

F.4 Complementary error function

The complementary error function $Erfc(x)$ is defined as:

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{F.26}$$

(see figure F.1).

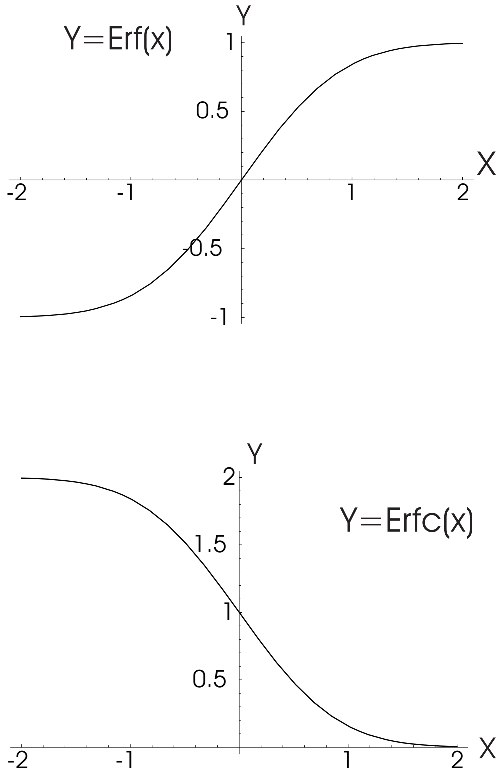

Figure F.1: The *error function* $Erf(x)$ and the *complementary error function* $Erfc(x)$ plotted against their argument x. Notice that $Erfc(x)$ converges to zero very rapidly; its value lies below 10^{-3} for $x > 2.3$, and even goes below 10^{-5} for $x > 3.1$.

Appendix G

Proof of the relation: $\mathbf{Q}\left(\mathbf{v}-\mathbf{v_1}\right)=\mathbf{0}$

Let us demonstrate the proof of relation (6.41)[1]. As we saw, this relation is of importance in establishing the mathematical properties of the kinetic operator (see chs. 6 and 7)[2].

Let us recall the original form of the collision term:

$$\mathcal{K}\{f\} = \frac{\partial J_i}{\partial v_i} \tag{G.1}$$

where the *probability current* J_i has the structure:

$$\begin{aligned} J_i &= \int d\mathbf{v_1}\, Q_{ij} \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_{1,j}} \right) \phi_{eq}(\mathbf{v_1})\, f(\mathbf{x}, \mathbf{v};\, t) \\ &= \dots \\ &\equiv D_{ij}\, \frac{\partial f}{\partial v_j} + a_i\, f \end{aligned} \tag{G.2}$$

The definition of $\mathbf{Q}$ is obvious, upon inspection (see formulae in the text):

$$Q_{ij} = n\,(2\pi)^3 \int_0^\infty d\tau \int d\mathbf{k}\, \tilde{V}_{\mathbf{k}}^2\, k_i\, k_l\, N'_{jl}(\tau)\, e^{ik_l N_{lj}(\tau)(v_j - v_{1\,j})} \tag{G.3}$$

(cf. (3.36) in the text). A summation upon all indices (except i) is assumed.

We shall show that:

$$Q_{ij}\,(v_j - v_{1\,j}) = n\,(2\pi)^3 \int_0^\infty d\tau \int d\mathbf{k}\, \tilde{V}_{\mathbf{k}}^2\, k_i\, k_l\, N'_{jl}(\tau)\, e^{ik_l N_{lj}(\tau)(v_j - v_{1\,j})}\,(v_j - v_{1\,j}) = 0 \tag{G.4}$$

---

[1] We take $\mu = 1$ throughout this Appendix. The general case can be treated in the same manner.

[2] This relation is valid in the free-motion (field-free) case [3], [5], and has also been discussed in the complete (i.e. nonlinear) magnetized Landau case [85] (though the proof provided therein seems a little heuristic). Presumably, this relation holds in general, yet no such proof has been given, to our knowledge.

1st method. Here is one sketch of a proof, which is quite brief and easy to follow[3].

First, as it is well known that force correlations (evaluated in the text) are even in the time argument:

$$C_{ij}(-\tau) = C_{ij}(\tau)$$

nothing changes if we substitute $N_{lj}(\tau)$ in the exponential in (G.3) with[4]:

$$N_{lj}(-\tau) = -N_{jl}(\tau)$$

(where we used the matrix properties in chapter 3).

Now, let us now isolate the quantity (see (G.3)):

$$\int_0^\infty d\tau\, k_l\, N'_{jl}(\tau)\,(v_j - v_{1\,j})\, e^{-ik_l N_{jl}(\tau)(v_j - v_{1\,j})} = \int_0^\infty d\tau\, i\, \frac{d}{d\tau}\, e^{-ik_l N_{jl}(\tau)(v_j - v_{1\,j}) - \epsilon\,\tau} = 1 \tag{G.5}$$

where ϵ was inserted to ensure convergence at infinity (the limit $\epsilon \to 0$ is understood).

The remaining integral:

$$Q_{ij}\,(v_j - v_{1\,j}) \sim \int d\mathbf{k}\, \tilde{V}_{\mathbf{k}}^2\, k_i = 0 \tag{G.6}$$

now cancels, for reasons of symmetry.

2nd Method. A more rigorous method consists in substituting in (G.4) with the exact expressions for the matrices.

For reference, let us see what happens in the absence of external field. We have:

$$i\, k_l\, N_{lj}(\tau)(v_j - v_{1\,j}) = i\, k_l\, \tau\, \delta_{lj}(v_j - v_{1\,j}) \equiv i\, \mathbf{k}\cdot\mathbf{g}\,\tau$$

in the exponent, while: $N'_{jl}(\tau) = \delta_{lj}$. (G.4) becomes:

$$Q_{ij}\,(v_j - v_{1\,j}) = n\,(2\pi)^3 \int d\mathbf{k}\;\tilde{V}_{\mathbf{k}}^2\, k_i\, k_j\, g_j \int_0^\infty d\tau\, e^{i\,\mathbf{k}\cdot\mathbf{g}\,\tau} \tag{G.7}$$

The time integration gives a δ_+ function (see (6.13) in the text), and the Fourier-integral then cancels[5].

For an external magnetic field, we have:

$$i\, k_l\, N_{lj}(\tau)(v_j - v_{1\,j}) = \ldots$$

[3] This method (actually less rigorous than the one that will follow) is inspired by a similar (yet not as generic) proof provided in [85].

[4] We have analytically verified this statement, yet there is no point in providing the calculation here.

[5] Actually, the δ-part gives an integral of the form: $\int dx\, x\, \delta(x) = 0$, while the corresponding principal part cancels for reasons of parity.

$$= \Omega^{-1}\, k_x\, [g_x\, \sin\Omega\tau + s\, g_y\, (1-\cos\Omega\tau)] + \Omega^{-1}\, k_y\, [g_y\, \sin\Omega\tau - s\, g_x\, (1-\cos\Omega\tau)] + k_z\, g_z\, \tau$$

$$= ... = \rho\, \sin(\Omega\tau+\theta) + k_z\, g_z\, \tau$$

in the exponent, where:

$$\rho = \Omega^{-1}\, (k_x^2 + k_y^2)^{1/2}\, (g_x^2+g_y^2)^{1/2} \;\equiv\; \frac{k_\perp g_\perp}{\Omega}$$

and

$$\theta = s\, \frac{\mathbf{g}_\perp \cdot \mathbf{k}_\perp}{g_\perp\, k_\perp} \equiv s\, \alpha$$

(see figure 6.1); in the same way, we have:

$$k_i\, k_j\, g_j\, N'_{jl}(\tau) \;=\; k_i\, k_j\, g_j\, R_{jl}(\tau) \;=\; ... =\; k_i\, \rho\, \cos(\Omega\tau - \theta) + k_z\, g_z\, \tau$$

(same definitions for ρ and θ) so (G.7) becomes:

$$Q_{ij}\, (v_j - v_{1\,j}) = n\, (2\pi)^3 \int d\mathbf{k} \int_0^\infty d\tau\;\, \tilde{V}_{\mathbf{k}}^2\, k_i\, [\rho(\mathbf{k})\, \cos(\Omega\tau-\theta) + k_z\, g_z\, \tau]\, e^{i\,[\rho(\mathbf{k})\, \sin(\Omega\tau+\theta)+k_z\, g_z\, \tau]} \tag{G.8}$$

The $i=1$ and $i=2$ (x,y) components now cancel (for reasons of symmetry, since the integrand is *even* in $k_{x,y}$), while the z component cancels for the same reason as in the free-motion case.

QED

.

Appendix H

Evaluation of the Fourier integrals in (8.22)

The definite integrals in $k_\|$ appearing in §8.4.4 (see (8.22)) are of the general form:

$$\begin{aligned} I_{\{1,2\}}(A,B,C) &= \int_{-\infty}^{\infty} dx \, \frac{e^{-A(x-iB)^2}}{(x^2+C^2)^2} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\} \\ &= e^{AB^2} \int_{-\infty}^{\infty} dx \, \frac{e^{-Ax^2}\cos 2ABx}{(x^2+C^2)^2} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\} \end{aligned} \tag{H.1}$$

where the upper (lower) quantity in brackets corresponds to the index 1 (2) i.e. to the '$\perp -$' ('$\| -$') integral in the main text. Note that the imaginary part of the integral cancels for reasons of symmetry, as the integrand in it is an *odd* function of x. In the following we will derive exact expressions for the above integrals.

H.1 Limit cases

Notice that these integrals are *even* with respect to B, C, so $B, C \in \Re_+$ may be assumed with no loss of generality (otherwise the result will be a function of their absolute values). Furthermore $A < 0$ yields a *divergent* (non-analytic) result, as one may readily check. In the following, we will assume that parameters A, B, C are *positive* quantities.

Let us briefly recall what happens if one of these parameters cancels.
(i) If $A = 0$, upon substitution with $\tan u = x/C$ in (H.1) we obtain:

$$I_{\{1,2\}}(0,B,C) = \int_{-\infty}^{\infty} dx \, \frac{1}{(x^2+C^2)^2} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\} = \frac{\pi}{2} \, C^{\{3,1\}} \tag{H.2}$$

(ii) Let $B = 0$; the integrals $I_{\{1,2\}}(A, 0, C)$ may be obtained from the known result:

$$\int_{-\infty}^{\infty} dx \, \frac{e^{-Ax^2}}{x^2 + C^2} = \frac{\pi}{C} \, e^{AC^2} \, Erfc(C\sqrt{A}) \tag{H.3}$$

(see §3.466, p. 338 in [19]) via a simple derivation with respect to C^2.
(ii) If $C = 0$; the integrals $I_{\{1,2\}}(A, B, 0)$ do not converge[1].

H.2 The integral $\tilde{I}_0 = \int_{-\infty}^{\infty} \frac{e^{-a^2x^2} \cos x \, dx}{x^2+b^2}$

Consider the function:

$$f(x) = \frac{e^{-a^2x^2}}{x^2 + b^2}$$

$f(x)$ is an *even* function of x, i.e.

$$f(-x) = f(x) \qquad \forall x \in \Re$$

The Fourier transform (F.T.) of an even function is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \, e^{ikx} = 2 \int_{0}^{\infty} f(x) \, \cos kx$$

so integrals of the form:

$$\int_{-\infty}^{\infty} f(x) \, \cos x \qquad\qquad (f : even)$$

can be evaluated in terms of the F.T. of the function $f(x)$ as follows:

$$\int_{-\infty}^{\infty} f(x) \, \cos x = \int_{-\infty}^{\infty} f(x) \, e^{ix} = \tilde{f}(1)$$

(the imaginary part of the integral is an *odd* function of x and thus disappears) that is, the F.T. evaluated at $k = 1$. Let us apply this principle to the above function f, in order to evaluate the integral

$$\tilde{I}_0 = \int_{-\infty}^{\infty} \frac{e^{-a^2x^2} \cos x \, dx}{x^2 + b^2}$$

The F.T. of the above function $f(x)$ is given by [39]:

$$\begin{aligned} \tilde{f}(k; a, b) &= \frac{\pi}{2b} \, e^{a^2b^2} \left[e^{-bk} Erfc(ab - \frac{k}{2a}) + e^{bk} Erfc(ab + \frac{k}{2a}) \right] \\ &= \frac{\pi}{2b} \, e^{a^2b^2} \sum_{s=\pm 1} e^{sbk} Erfc(ab + s\frac{k}{2a}) \end{aligned} \tag{H.4}$$

[1] Remember that this divergence for $k_{Debye} = 0$, say, is precisely the divergence of the Coulomb integral discussed in the text.

where $Erfc(x)$ denotes the *complementary error function*:

$$Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(see figure 1) so $\tilde{I}_0(a,b) = \tilde{f}(1;a,b)$ i.e. finally

$$\begin{aligned} \tilde{I}_0(a,b) &= \int_{-\infty}^{\infty} \frac{e^{-a^2x^2} \cos x\, dx}{x^2+b^2} \\ &= \frac{\pi}{2b}\, e^{a^2b^2} \sum_{s=\pm1} e^{sb} Erfc(ab + s\frac{1}{2a}) \end{aligned} \tag{H.5}$$

H.3 The integrals $\tilde{I}_{\{1,2\}} = \int_{-\infty}^{\infty} \frac{e^{-a^2x^2}\cos x\, dx}{(x^2+b^2)^2} \left\{1, x^2\right\}$

The integral $\tilde{I}_0$ defined above may now be used as a starting point for the evaluation of the integral:

$$\tilde{I}_1 = \int_{-\infty}^{\infty} \frac{e^{-a^2x^2} \cos x\, dx}{(x^2+b^2)^2}$$

Noting that:

$$\frac{1}{(x^2+b^2)^2} = -\frac{\partial}{\partial b^2} \frac{1}{x^2+b^2} = -\frac{1}{2b} \frac{\partial}{\partial b} \frac{1}{x^2+b^2}$$

we have:

$$\tilde{I}_1 = -\frac{\partial}{\partial b^2} \tilde{I}_0 = -\frac{1}{2b}\frac{\partial}{\partial b} rhs(H.5) = ...$$

One finally obtains:

$$\begin{aligned} \tilde{I}_1(a,b) &= \int_{-\infty}^{\infty} \frac{e^{-a^2x^2} \cos x\, dx}{(x^2+b^2)^2} \\ &= \sqrt{\pi}\frac{a}{b^2}\, e^{-1/4a^2} + \frac{\pi}{4b^3}\, e^{a^2b^2} \sum_{s=\pm1} (1 - 2a^2b^2 - sb)\, e^{sb}\, Erfc(ab + \frac{s}{2a}) \end{aligned} \tag{H.6}$$

Now, noting that:

$$\frac{x^2}{(x^2+b^2)^2} = -\frac{\partial}{\partial b^2} \frac{x^2}{x^2+b^2} = -\frac{\partial}{\partial b^2} \left(1 - \frac{b^2}{x^2+b^2}\right) = \frac{\partial}{\partial b^2} \frac{b^2}{x^2+b^2}$$

we have:

$$\tilde{I}_2 = \frac{\partial}{\partial b^2}(b^2 \tilde{I}_0) = \tilde{I}_0 + b^2 \frac{\partial}{\partial b^2} \tilde{I}_0 = \tilde{I}_0 - b^2 \tilde{I}_1 = rhs(H.5) - b^2 rhs(H.6) = ...$$

One thus obtains:

$$\begin{aligned}\tilde{I}_2(a,b) &= \int_{-\infty}^{\infty} \frac{e^{-a^2x^2}\ \cos x\, x^2\, dx}{(x^2+b^2)^2} \\ &= -\sqrt{\pi}a\, e^{-1/4a^2} + \frac{\pi}{4b}\, e^{a^2b^2} \sum_{s=\pm 1} (1+2a^2b^2+sb)\, e^{sb}\, Erfc(ab+\frac{s}{2a})\end{aligned} \tag{H.7}$$

The final result of this paragraph, i.e. relations (H.6) and (H.7), appears in a more compact form as:

$$\tilde{I}_{\{1,2\}}(a,b) = \int_{-\infty}^{\infty} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\} \frac{e^{-a^2x^2}\ \cos x\, dx}{(x^2+b^2)^2}$$

$$\left\{ \begin{array}{c} 1 \\ b^2 \end{array} \right\} \left[\pm\sqrt{\pi}\frac{a}{b^2}\, e^{-1/4a^2} + \frac{\pi}{4b^3}\, e^{a^2b^2} \sum_{s=\pm 1} (1 \mp 2a^2b^2 \mp sb)\, e^{sb}\, Erfc(ab+\frac{s}{2a}) \right] \tag{H.8}$$

H.4 The general integrals (H.1)

The (two) integrals defined in the beginning of this Appendix are now readily computed from (H.8) by setting:

$$x \to 2ABx \qquad dx \to 2ABdx$$

$$a = \sqrt{A}/2AB \qquad b = 2ABC$$

therein (i.e. in (H.6), (H.7)) and then multiplying by $e^{AB^2}\,(2AB)^{\{3,1\}}$ (one thus obtains exactly (H.1)). We thus finally obtain:

$$\begin{aligned}I_{\{1,2\}}(A,B,C) &= \int_{-\infty}^{\infty} dx\, \frac{e^{-A(x-iB)^2}}{(x^2+C^2)^2} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\} \\ &= e^{AB^2} \int_{-\infty}^{\infty} dx\, \frac{e^{-Ax^2}\cos 2ABx}{(x^2+C^2)^2} \left\{ \begin{array}{c} 1 \\ x^2 \end{array} \right\}\end{aligned}$$

$$= \left\{ \begin{array}{c} 1 \\ C^2 \end{array} \right\} \Bigg[\pm\sqrt{\pi}\frac{\sqrt{A}}{C^2} +$$

$$\frac{\pi}{4C^3}\, e^{A(B^2+C^2)} \sum_{s=\pm 1} (1 \mp 2AC^2 \mp 2sABC)\, e^{s2ABC}\, Erfc(\sqrt{A}C + s\sqrt{A}B) \Bigg] \tag{H.9}$$

where the upper (lower) quantity in brackets corresponds, once more, to the index 1 (2) (i.e. to the '$\perp -$' ('$\| -$') integral in the main text).

For the sake of reference, having carried out the above substitution in (H.5) would have given the integral:

$$\begin{aligned} I_0(A,B,C) &= \int_{-\infty}^{\infty} dx\, \frac{e^{-A(x-iB)^2}}{x^2+C^2} \\ &= e^{AB^2} \int_{-\infty}^{\infty} dx\, \frac{e^{-Ax^2}\cos 2ABx}{x^2+C^2} \\ &= \frac{\pi}{2C}\, e^{A(B^2+C^2)} \sum_{s=\pm 1} e^{s2ABC}\, Erfc(\sqrt{A}C + s\sqrt{A}B) \end{aligned} \tag{H.10}$$

Setting $B = 0$ in this integral, we recover the F.T. of $\frac{e^{-Ax^2}}{x^2+C^2}$ as provided in tables

$$I_0(A,0,C) = \int_{-\infty}^{\infty} dx\, \frac{e^{-Ax^2}}{x^2+C^2} = \frac{\pi}{C}\, e^{AC^2}\, Erfc(\sqrt{A}C) \tag{H.11}$$

(cf. (H.3) above).

H.5 The final integrals in $k_\parallel$

Now we are ready to calculate the (two) integrals in $k_\parallel$ defined in the text (in brackets in (8.21)), say $I_{k_\parallel}^{(\perp,\parallel)}$ [2]:

$$\begin{aligned} I_{k_\parallel}^{\{\perp,\parallel\}} &= \int_{-\infty}^{\infty} dk_\parallel\; e^{-\sigma_\parallel \left(k_\parallel \tau - i\frac{2v_\parallel}{\sigma_\parallel}\right)^2/4}\, \frac{1}{(k_\perp^2+k_\parallel^2+k_D^2)^2} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \\ &= e^{v_\parallel^2/\sigma_\parallel} \int_{-\infty}^{\infty} dk_\parallel\; e^{-\sigma_\parallel\, k_\parallel^2\, \tau^2/4}\, \frac{\cos(k_\parallel\, v_\parallel\, \tau)}{(k_\perp^2+k_\parallel^2+k_D^2)^2} \left\{ \begin{array}{c} 1 \\ k_\parallel^2 \end{array} \right\} \end{aligned} \tag{H.12}$$

Comparing to (H.1), we immediately see that they are directly recovered upon setting:

$$x = k_\parallel\,, \qquad A = \sigma\tau^2/4\,, \qquad B = 2v_\parallel/\sigma\tau\,, \qquad C = \tilde{k}_\perp \equiv (k_\perp^2 + k_D^2)^{1/2}$$

Carrying out the same substitution in (H.9), one immediately obtains:

$$I_{k_\parallel}^{\{\perp,\parallel\}} = \frac{1}{\tilde{k}_\perp^{\{3,1\}}} \Bigg\{ \pm\frac{\sqrt{\pi}}{2}\, \sqrt{\sigma_\parallel}\, \tilde{k}_\perp\, \tau + \frac{\pi}{4} e^{v_\parallel^2/\sigma_\parallel}\, e^{\sigma_\parallel \tilde{k}_\perp^2 \tau^2/4}$$

$$\sum_{s=+1,-1} \left[e^{s\tilde{k}_\perp v_\parallel \tau}\, \left(1 \mp \sigma_\parallel \tilde{k}_\perp^2 \tau^2/2 \mp s\tilde{k}_\perp v_\parallel \tau\right) Erfc\Big(\frac{1}{2}\sqrt{\sigma_\parallel}\tilde{k}_\perp \tau + s\frac{v_\parallel}{\sqrt{\sigma_\parallel}}\Big) \right] \Bigg\} \tag{H.13}$$

[2] Note, once more, that the imaginary part of the integral cancels for reasons of symmetry, as the integrand in it is an *odd* function of $k_\parallel$.

where the upper (lower) sign holds for $I^{(\perp)}$ $(I^{(\|)})$. The complementary error function $Erfc(x)$ was defined above.

Note that the integrals $I_{k_\|}^{(\perp,\|)}$:
(i) converge to zero at both $k_\perp \to \infty$ and $\tau \to \infty$, as - more or less - expected.
(ii) give a *finite* limit at $\tau \to 0$:

$$\lim_{\tau \to 0} I_{k_\|}^{\{\perp,\|\}} = \frac{\pi}{2} e^{v_\|^2/\sigma_\|}$$

(note that $Erfc(x) + Erfc(-x) = 2$) and
(iii) do *not* diverge at $k_\perp \to 0$, i.e. $\tilde{k}_\perp \to 1$!

It is interesting to see that the quantity in brackets in the rhs of (H.13), can be re-arranged as:

$$rhs(H.13) =$$

$$\frac{1}{\tilde{k}_\perp^{\{3,1\}}} \left\{ \pm\sqrt{\pi}\,\phi + \frac{\pi}{4}\, e^{-\tilde{v}_\|^2}\, e^{\phi^2} \sum_{s=+1,-1} \left[e^{s2\,\phi\,\tilde{v}_\|} \left(1 \mp 2\,\phi^2 \mp s2\,\phi\,\tilde{v}_\|\right) Erfc\left(\phi + s\,\tilde{v}_\|\right) \right] \right. \tag{H.14}$$

(the upper (lower) signs corresponding to the $\perp$ $(\|)$- parts respectively) where:

$$\phi = \frac{1}{2}\sqrt{\sigma_\|}\,\tilde{k}_\perp\,\tau\,, \qquad \tilde{v}_\| \equiv v_\|/\sqrt{\sigma_\|}$$

where all variables are non-dimensional; remember that $\tilde{k}_\perp = (k_\perp^2 + k_D^2)^{1/2}$.

Note. One last word seems to be appropriate. Given the rather complicated analytic form of the above integrals, one is tempted to verify these results by comparing them to the exact ones (obtained by direct numerical integration). Our analytical results have been tested and absolutely confirmed; details are omitted for brevity[3].

[3]A useful tool for this purpose is symbolic computing software. One may formally define all the above quantities, say $\{Q\}$, as consisting of two sets, e.g. $\{Q_{exact}\}$ and $\{Q_{analytic}\}$, and then compare the latter to the former numerically. An appropriate MATHEMATICA code was initially meant to be listed in the remaining part of this Appendix, yet was finally omitted for brevity.

Appendix I

QM-FPE & *M-FPE* in comparison to the unmagnetized case

In the text, we have derived explicit expressions for the second-order kinetic operators defined as Θ and Φ, as applied in magnetized plasma (see Chapters 6 and 7 respectively). As a matter of fact, the final expressions obtained in the former, in the form of an infinite series, were not explicitly evaluated till the end, in an arbitrary frame[1]. Furthermore, expressions derived in the latter were presented in a simplified manner[2] which, even though correct, may somewhat appear to lack generality. For these reasons we present, in brief, the derivation of both operators in a general (*any*) frame, with the aim of
(i) comparing them to one-another and also
(ii) elucidating their relation to the *unmagnetized* case.

The analytical method presented in Chapter 8 will be exactly followed; unnecessary details in the calculation will therefore be omitted.

I.1 Θ operator in an arbitrary frame

The Fokker-Planck equation (FPE) obtained to 2nd order in the (weak) interaction is a 2nd-order parabolic linear partial-derivative-equation (*pde*) of the form:

$$\partial_t f + \mathbf{v}\nabla f + \mathbf{a}\,\partial_{\mathbf{v}} f = \mathcal{K}$$

where $\mathbf{v}$, $\mathbf{a}$ denote the particle velocity and acceleration, respectively, in the presence of an external force field (yet in the absence of collisions). The (linear)

[1] The frames introduced therein were meant to facilitate the integrations in Fourier space.
[2] See that the derivation in Ch. 7 was based on frame 2, defined in Ch. 6.

collision term (rhs) obeys the general form:

$$\mathcal{K} = \frac{\partial^2}{\partial v_i \partial v_j}(D_{ij}^{(VV)} f) + \frac{\partial^2}{\partial v_i \partial x_j}(D_{ij}^{(VX)} f) + \frac{\partial^2}{\partial x_i \partial x_j}(D_{ij}^{(XX)} f) - \frac{\partial}{\partial v_i}(\mathcal{F}_i^{(V)} f) - \frac{\partial}{\partial x_i}(\mathcal{F}_i^{(X)} f)$$

The form of the coefficients - for a given physical problem - depends, as we saw, on the operator considered (i.e. Θ, Φ or else).

The velocity diffusion matrix $D_{ij}^{(VV)} = A_{ij}$ is given by:

$$\begin{aligned}\mathbf{A} &= \frac{n_{\alpha'}}{m_\alpha^2} \int_0^t d\tau \int d\mathbf{v_1} (2\pi)^3 \; \phi_{eq}^{\alpha'}(v_1) \\ &\qquad \int d\mathbf{k}\, \tilde{V}_k^2 \, \mathbf{k} \otimes \mathbf{k} \, e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}} \, e^{-i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} \mathbf{N'}_\alpha^T(\tau) \\ &\equiv \frac{1}{m^2} \int_0^\infty d\tau \, \mathbf{C}_{\alpha,\alpha'}(\mathbf{x}, \mathbf{v}; t, t-\tau) \mathbf{N'}_\alpha^T(\tau)\end{aligned} \tag{I.1}$$

(see Chapter 4 and forth); a summation over particle species α' is understood[3]. A set of similar relations hold for the other diffusion coefficients D_{ij}^* ($* = (VX), (XX)$); the drift terms $\mathcal{F}_i^\dagger$ ($\dagger = (V), (X)$) are defined through the former via a velocity derivative.

The external field appears in the dynamical matrices $\mathbf{N}(t)$, $\mathbf{N}'(t)$, defined in Chapter 6 (see (6.5)).

Let us re-arrange this expression, for clarity, as:

$$\mathbf{A} = (2\pi)^3 \, \frac{n_{\alpha'}}{m_\alpha^2} \int_0^\infty d\tau \left\{ \int d\mathbf{k}\, \tilde{V}_k^2 \, \mathbf{k} \otimes \mathbf{k} \, e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}} \left[\int d\mathbf{v_1} \phi_{eq}^{\alpha'}(v_1) \, e^{-i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} \right] \right\} \mathbf{N'}_\alpha^T(\tau) \tag{I.2}$$

We see that the velocity integral in brackets, say $I_{\mathbf{v_1}}$, can be carried out straightaway and the integration in Fourier space may follow.

Following the method given in detail in Chapter 8 (see §8.3.1), we finally find that:

$$I_{\mathbf{v_1}} = e^{-\sigma\, p^2/4} \tag{I.3}$$

where the vector $\mathbf{p}$ was defined in (8.6):

$$p_m^{\alpha'}(\mathbf{k};\tau) = \sum_{n=1}^3 k_n N_{nm}^{\alpha'}(\tau) = \sum_{n=1}^3 k_n \int_0^\tau R_{nm}^{\alpha'}(t')dt'.$$

The $\mathbf{k}-$integral, say $I_\mathbf{k}$, now becomes:

$$I_\mathbf{k} = \int d\mathbf{k}\, \tilde{V}_k^2 \, \mathbf{k} \otimes \mathbf{k} \, e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}} \, e^{-\sigma\, p^2/4}$$

[3] In the single-species case, set $e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}} \, e^{-i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} = e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{g}}$, where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$; see in Chapter 4.

The exponential may be re-arranged (see §8.4.1) as[4]:

$$e^{i\mathbf{p}_\alpha \cdot \mathbf{v}}\, e^{-\sigma\, p_{\alpha'}^2/4} = e^{i\mathbf{p}_\perp{}^\alpha \cdot \mathbf{v}_\perp - \sigma\, p_\perp^{\alpha'\,2}/4}\, e^{i p_\parallel^\alpha v_\parallel - \sigma\, p_\parallel^{\alpha'\,2}/4}$$

Working in cylindrical coordinates, we can assume that:

$$\mathbf{k} = (k_\perp \cos\theta,\, k_\perp \sin\theta,\, k_\parallel)\,, \qquad \mathbf{v} = (v_\perp \cos\alpha,\, v_\perp \sin\alpha,\, v_\parallel)$$

The Fourier integral now becomes:

$$\begin{aligned} I_{\mathbf{k}} &= \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma p_\perp^{\alpha\,2}/4} \left\{ \int_{-\infty}^\infty dk_\parallel\, \tilde{V}_k^2\, e^{i p_\parallel^\alpha v_\parallel - \sigma\, p_\parallel^{\alpha'\,2}/4} \right. \\ &\qquad \left. \left[\int_0^{2\pi} d\theta\, e^{i\mathbf{p}_\perp{}^\alpha \cdot \mathbf{v}_\perp} \begin{pmatrix} k_\perp^2 \cos\theta \\ k_\perp^2 \sin\theta \\ k_\parallel \end{pmatrix} \begin{pmatrix} k_\perp^2 \cos\theta \\ k_\perp^2 \sin\theta \\ k_\parallel \end{pmatrix} \right] \right\} \end{aligned} \tag{I.4}$$

In the following we shall limit the description to the case $\alpha = \alpha'$, so superscripts denoting particle species will be dropped.

This (symmetric) expression involves 9 matrix elements: it may be split in three parts, say (i) the $\perp$ −part (11, 12, 21, 22 elements), (ii) the $\parallel$ −part (33) and (iii) the $\perp$ / $\parallel$ −part (13, 23, 31, 32). Let us treat them separately, pointing out how integrals get de-coupled in each case.

(i) The $\perp$ −part. This part reads:

$$\begin{aligned} I_{\mathbf{k}}^{(\perp)} &= \int_0^\infty dk_\perp\, k_\perp^3\, e^{-\sigma p_\perp{}^2/4} \left[\int_{-\infty}^\infty dk_\parallel\, \tilde{V}_k^2\, e^{i p_\parallel v_\parallel - \sigma\, p_\parallel{}^2/4} \right] \\ &\qquad \left[\int_0^{2\pi} d\theta\, e^{i\mathbf{p}_\perp \cdot \mathbf{v}_\perp} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \right] \\ &\equiv \int_0^\infty dk_\perp\, k_\perp^3\, e^{-\sigma p_\perp{}^2/4} \quad \tilde{I}_{k_\parallel}^{(\perp)} \quad I_\theta^{(\perp)} \end{aligned} \tag{I.5}$$

First, notice that the first quantity in brackets, say $\tilde{I}_{k_\parallel}^{(\perp)}$ is completely independent from the rest. It was computed in Ch. 8 (see details in the Appendix) and was found to be given by expression (8.24)[5]: a complex yet exact expression, to be henceforth denoted by $\tilde{I}_{k_\parallel}^{(\perp)}$ here, for brevity.

Now, the angle integral can be evaluated by substituting the expression:

$$\begin{aligned} \mathbf{p}_\perp \cdot \mathbf{v}_\perp &= \left(2\, \frac{k_\perp v_\perp}{\Omega}\, \sin\frac{\Omega\tau}{2} \right) \cos(\theta - \alpha - s\frac{\Omega\tau}{2}) \\ &\equiv Z' \sin\left(\frac{\pi}{2} - \theta + \alpha + s\frac{\Omega\tau}{2} \right) \end{aligned} \tag{I.6}$$

---

[4]The $\perp$ − and $\parallel$ − parts can be separated in our case (see in §8.4.1) thanks to the (block-diagonal) structure of $\mathbf{N}$.

[5]In fact, $\tilde{I}_{k_\parallel}^{(\perp)}$ defined here is equal to $\tilde{I}_{k_\parallel}^{(\perp)} = \tilde{V}_0^2\, e^{-v_\parallel^2/\sigma}\, I_{k_\parallel}^{(\perp)}$ therein.

(derived previously; see in §8.4.1) in the exponential and using Bessel function identities. The method is explained in §8.4.2; here, it finally gives:

$$I_\theta^{(\perp)} = \pi \begin{pmatrix} J_0(Z') - J_2(Z')\cos(2\alpha - s\Omega\tau) & -J_2(Z')\sin(2\alpha - s\Omega\tau) \\ -J_2(Z')\sin(2\alpha - s\Omega\tau) & J_0(Z') + J_2(Z')\cos(2\alpha - s\Omega\tau) \end{pmatrix} \tag{I.7}$$

We repeat the definition:

$$Z' = 2\,\frac{k_\perp v_\perp}{\Omega}\,\sin\frac{\Omega\tau}{2} \tag{I.8}$$

Furthermore, the exponential in the $k_\perp$−integral reduces to the simple expression:

$$e^{-\frac{\sigma p_\perp^2}{4}} = e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}} \tag{I.9}$$

(see in §8.4.1, 8.4.2). Collecting all these results, (I.5) becomes:

$$\begin{aligned} I_{\mathbf{k}}^{(\perp)} &= \int_0^\infty dk_\perp\, k_\perp^3\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}}\; I_{k_\parallel}^{(\perp)} \\ &\quad \pi \begin{pmatrix} J_0(Z') - J_2(Z')\cos(2\alpha - s\Omega\tau) & -J_2(Z')\sin(2\alpha - s\Omega\tau) \\ -J_2(Z')\sin(2\alpha - s\Omega\tau) & J_0(Z') + J_2(Z')\cos(2\alpha - s\Omega\tau) \end{pmatrix} \end{aligned} \tag{I.10}$$

(ii) The $\|$ −part. This part reads:

$$\begin{aligned} I_{\mathbf{k}}^{(\|)} &= \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma {p_\perp}^2/4} \left[\int_{-\infty}^\infty dk_\parallel\, \tilde{V}_k^2\, k_\parallel^2\, e^{ip_\parallel v_\parallel - \sigma\, {p_\parallel}^2/4}\right] \\ &\qquad \left[\int_0^{2\pi} d\theta\, e^{i\mathbf{p}_\perp\cdot\mathbf{v}_\perp} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] \\ &\equiv \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma {p_\perp}^2/4} \quad I_{k_\parallel}^{(\|)} \quad I_\theta^{(\|)} \end{aligned} \tag{I.11}$$

where $I_{k_\parallel}^{(\|)}$ is, again, defined and computed in Ch. 8 (see (8.24)) and $I_\theta^{(\|)}$ can be calculated as above. The result is:

$$I_{\mathbf{k}}^{(\|)} = \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}} \quad I_{k_\parallel}^{(\|)}\, \pi\, J_0(Z')\, \hat{e}_3\hat{e}_3 \tag{I.12}$$

(iii) The $\perp$ / $\|$ −part. Finally, this part consists of the elements:

$$\begin{aligned} \begin{Bmatrix} 13, 31 \\ 23, 32 \end{Bmatrix} &= \int_0^\infty dk_\perp\, k_\perp^2\, e^{-\sigma {p_\perp}^2/4} \left[\int_{-\infty}^\infty dk_\parallel\, \tilde{V}_k^2\, k_\parallel\, e^{ip_\parallel v_\parallel - \sigma\, {p_\parallel}^2/4}\right] \\ &\qquad \left[\int_0^{2\pi} d\theta\, e^{i\mathbf{p}_\perp\cdot\mathbf{v}_\perp} \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}\right] \\ &\equiv \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma {p_\perp}^2/4} \quad I_{k_\parallel}^{(\perp/\|)} \quad I_\theta^{(\perp/\|)} \end{aligned} \tag{I.13}$$

The calculation here is slightly different. The first quantity in brackets, say $I^{(\perp/\|)}_{k_\|}$, needs to be evaluated first. Notice that only the imaginary part of the exponential contributes, for reasons of symmetry; the remaining contribution can be evaluated using the trick:

$$\begin{aligned}
I^{(\perp/\|)}_{k_\|} &= i\int_{-\infty}^{\infty} dk_\| \, \tilde{V}_k^2 \, k_\| \, e^{-\sigma \, p_\|{}^2/4} \, \sin(p_\| v_\|) \\
&= i\,\frac{\partial}{\partial p_\|}\int_{-\infty}^{\infty} dk_\| \, \tilde{V}_k^2 \, k_\| \, e^{-\sigma \, p_\|{}^2/4} \, \cos(p_\| v_\|) \\
&= i\,\frac{1}{\tau}\,\frac{\partial}{\partial v_\|}\int_{-\infty}^{\infty} dk_\| \, \tilde{V}_k^2 \, e^{-\sigma \, k_\|{}^2 \, \tau^2/4} \, \cos(k_\| v_\| \tau) \\
&= i\,\frac{1}{\tau}\,\frac{\partial}{\partial v_\|} Re\int_{-\infty}^{\infty} dk_\| \, \tilde{V}_k^2 \, e^{-\sigma \, k_\|{}^2 \, \tau^2/4} \, e^{i k_\| v_\| \tau} \\
&= i\,\frac{1}{\tau}\,\frac{\partial}{\partial v_\|} I^{(\perp)}_{k_\|}
\end{aligned} \tag{I.14}$$

(remember that $p_\| = k_\| \, \tau$ by definition).

The angle integral(s) can be calculated as described above; the result is:

$$I^{(\perp/\|)}_{\theta} = \int_0^{2\pi} d\theta \, e^{i\mathbf{p}_\perp\cdot\mathbf{v}_\perp} \left\{ \begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right\} = i\,\pi\,2\,J_1(Z') \left\{ \begin{array}{c} \cos\alpha \\ \sin\alpha \end{array} \right\}$$

so combining these results, we obtain:

$$\left\{ \begin{array}{c} 13,31 \\ 23,32 \end{array} \right\} = -\,2\pi\,\frac{1}{\tau}\,\frac{\partial}{\partial v_\|}\int_0^{\infty} dk_\perp \, k_\perp^2 \, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2\frac{\Omega\tau}{2}} \, I^{(\perp)}_{k_\|} J_1(Z') \left\{ \begin{array}{c} \cos\alpha \\ \sin\alpha \end{array} \right\} \tag{I.15}$$

Final form for $I_{\mathbf{k}}$. Combining all the above, we may substitute the Fourier integral (see (I.4)) into the expression for the force correlation (see in the beginning of this chapter) which now becomes:

$$\mathbf{C} = n(2\pi)^3 \, \pi \int_0^{\infty} dk_\perp \, k_\perp \, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2\frac{\Omega\tau}{2}}$$

$$\begin{pmatrix}
k_\perp^2 \, I^{(\perp)}_{k_\|} \left[J_0(Z') - J_2(Z')\cos(2\alpha - s\Omega\tau)\right] & -k_\perp^2 \, I^{(\perp)}_{k_\|} J_2(Z') \, \sin(2\alpha - s\Omega\tau) & -2\,k_\perp \, \frac{1}{\tau}\,\frac{\partial}{\partial v_\|} I^{(\perp)}_{k_\|} \, J_1(Z') \, \cos\alpha \\
-k_\perp^2 \, I^{(\perp)}_{k_\|} \, J_2(Z') \sin(2\alpha - s\Omega\tau) & k_\perp^2 \, I^{(\perp)}_{k_\|} \left[J_0(Z') + J_2(Z')\cos(2\alpha - s\Omega\tau)\right] & -2\,k_\perp \, \frac{1}{\tau}\,\frac{\partial}{\partial v_\|} I^{(\perp)}_{k_\|} \, J_1(Z') \, \sin\alpha \\
-2\,k_\perp \, \frac{1}{\tau}\,\frac{\partial}{\partial v_\|} I^{(\perp)}_{k_\|} \, J_1(Z') \, \cos\alpha & -2\,k_\perp \, \frac{1}{\tau}\,\frac{\partial}{\partial v_\|} I^{(\perp)}_{k_\|} \, J_1(Z') \, \sin\alpha & I^{(\|)}_{k_\|} \, J_0(Z')
\end{pmatrix} \tag{I.16}$$

Notice that, in the infinite Ω limit, only $J_0(Z')$ survives, so the matrix becomes diagonal: $\lim_{\Omega\to\infty}\mathbf{A} = A_\perp\,(\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2) + A_\parallel\,\hat{e}_3\hat{e}_3$.

For brevity in the following, we will denote the matrix in the above expression as, say, $\begin{pmatrix} a - b\cos(2\alpha - s\Omega\tau) & -b\sin(2\alpha - s\Omega\tau) & d\cos\alpha \\ -b\sin(2\alpha - s\Omega\tau) & a + b\cos(2\alpha - s\Omega\tau) & d\sin\alpha \\ d\cos\alpha & d\sin\alpha & c \end{pmatrix}$; where all definitions are obvious. The resulting expression for the diffusion matrix $\mathbf{A}$ now reads:

$$\begin{aligned}
\mathbf{A} &= \frac{n_{\alpha'}}{m_\alpha^2}(2\pi)^3\,\pi\int_0^t d\tau\int_0^\infty dk_\perp\,k_\perp\,e^{-\sigma_\perp\frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}} \\
&\quad \begin{pmatrix} a - b\cos(2\alpha - s\Omega\tau) & -b\sin(2\alpha - s\Omega\tau) & d\cos\alpha \\ -b\sin(2\alpha - s\Omega\tau) & a + b\cos(2\alpha - s\Omega\tau) & d\sin\alpha \\ d\cos\alpha & d\sin\alpha & c \end{pmatrix} \mathbf{N'}_\alpha^T(\tau) \\
&= \ldots \\
&= \frac{n_{\alpha'}}{m_\alpha^2}(2\pi)^3\,\pi\int_0^t d\tau\int_0^\infty dk_\perp\,k_\perp\,e^{-\sigma_\perp\frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}}
\end{aligned}$$

$$\begin{pmatrix} a\cos\Omega\tau - b\cos 2(\alpha - s\Omega\tau) & -a\sin\Omega\tau - b\sin 2(\alpha - s\Omega\tau) & d\cos\alpha \\ a\sin\Omega\tau - b\sin(2\alpha - s\Omega\tau) & a\cos\Omega\tau + b\cos 2(\alpha - s\Omega\tau) & d\sin\alpha \\ d\cos(\alpha - s\,\Omega\tau) & d\sin(\alpha - s\,\Omega\tau) & c \end{pmatrix} \tag{I.17}$$

where we have used the definition (6.5) of the dynamical matrix $\mathbf{N}'$. Remember that:

- α is the velocity gyro-phase.

- Z' in the Bessel function argument was defined in (I.8).

- $\tilde{I}_{k_\parallel}^{(\dagger)}$ ($\dagger$ = either $\perp$ or $\parallel$) are functions of $\{k_\perp, v_\parallel, \tau\}$ given by: $\tilde{I}_{k_\parallel}^{(\dagger)} = \tilde{V}_0^2\,e^{-v_\parallel^2/\sigma}\,I_{k_\parallel}^{(\dagger)}$, where $I_{k_\parallel}^{(\dagger)}$ were computed in Ch. 8 (see expression (8.24); also see in the Appendix for details).

Note the limits:

$$\lim_{\Omega\to 0}\mathbf{N}'(\tau) = \mathbf{I}$$

$$\lim_{\Omega\to 0} Z' = k_\perp\,v_\perp\,\tau$$

$$\lim_{\Omega\to 0}\left(\sigma_\perp\,\frac{k_\perp^2}{\Omega^2}\sin^2\frac{\Omega\tau}{2}\right) = \sigma_\perp\,\frac{k_\perp^2\,\tau^2}{4}$$

which will be used in the following.

A similar calculation can be carried out for the $\mathbf{D_{VX}}$ matrix ($= \mathbf{G}^{(\Theta)}$ the text), appearing in the beginning of this chapter, yet it is of no importance in the following.

Retain the (symmetric) form of the correlation matrix $\mathbf{C}$:

$$\mathbf{C} = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \tag{I.18}$$

as well as the matrix $\mathbf{C}\,\mathbf{N'}^T(\tau)$ in the time integral in expressions above:

$$\mathbf{D_{22}}^{(\Theta)} \sim \mathbf{C}\,\mathbf{N'}^T(\tau) = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \begin{pmatrix} \cos\Omega\tau & -s\ \sin\Omega\tau & 0 \\ s\ \sin\Omega\tau & \cos\Omega\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} A\cos\Omega\tau + sD\sin\Omega\tau & -sA\sin\Omega\tau + D\cos\Omega\tau & E \\ D\cos\Omega\tau + sB\sin\Omega\tau & -sD\sin\Omega\tau + B\cos\Omega\tau & F \\ E\cos\Omega\tau + sF\sin\Omega\tau & -sE\sin\Omega\tau + F\cos\Omega\tau & C \end{pmatrix} \quad \text{(I.19)}$$

The corresponding quantity in the time-integral in the expression for the cross-V-X diffusion matrix $\mathbf{D_{21}}$ matrix reads:

$$\mathbf{D_{21}}^{(\Theta)} \sim \mathbf{C}\,\mathbf{N}^T(\tau) =$$

$$\begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \Omega^{-1} \begin{pmatrix} \sin\Omega\tau & -s\ (1-\cos\Omega\tau) & 0 \\ -s\ (\cos\Omega\tau - 1) & \sin\Omega\tau & 0 \\ 0 & 0 & \Omega\tau \end{pmatrix}$$

$$= \Omega^{-1} \begin{pmatrix} A\sin\Omega\tau + sD(1-\cos\Omega\tau) & -sA(1-\cos\Omega\tau) + D\sin\Omega\tau & E\Omega\tau \\ D\sin\Omega\tau + sB(1-\cos\Omega\tau) & -sD(1-\cos\Omega\tau) + B\sin\Omega\tau & F\Omega\tau \\ E\sin\Omega\tau + sF(1-\cos\Omega\tau) & -sE(1-\cos\Omega\tau) + F\sin\Omega\tau & C\Omega\tau \end{pmatrix} \quad \text{(I.20)}$$

See what happens if one takes $\Omega = 0$ (cf. next section): these expressions plainly reduce to $\mathbf{C}$ and $\tau\,\mathbf{C}$, respectively, as expected from the matrices' form corresponding to the *unmagnetized* limit.

In conclusion, expressions for the diffusion coefficients appearing in the end of Chapter 6 and forth, are exactly recovered, therefore; they correspond to:

$$\begin{aligned} D_\perp &= \frac{1}{2}\big[(D_{22})_{11} + (D_{22})_{22}\big] \sim \frac{1}{2}(A+B)\ \cos\Omega\tau \\ D_\angle &= \frac{1}{2}\big[(D_{22})_{12} - (D_{22})_{21}\big] \sim \frac{-s}{2}(A+B)\ \sin\Omega\tau \\ D_\parallel &= (D_{22})_{12} \sim C \end{aligned} \quad \text{(I.21)}$$

(see (I.19) above).

I.2 Vanishing magnetic field limit

The calculation for $\Omega = 0$ can be carried out in *exactly* the same way as in the previous section. We have carried it out completely from scratch, for the sake of confirmation, nevertheless reproducing it here would serve no purpose in the following. We shall state the final result, also obtained from the above formulae by setting $\Omega \to 0$ everywhere:

$$\lim_{\Omega\to 0} \mathbf{A} = \frac{n_{\alpha'}}{m_\alpha^2}(2\pi)^3\ \pi \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp\, e^{-\sigma_\perp k_\perp^2 \tau^2/4}\ \times$$

$$\begin{pmatrix} a - b\cos 2\alpha & -b\sin 2\alpha & d\cos\alpha \\ -b\sin 2\alpha & a + b\cos 2\alpha & d\sin\alpha \\ d\cos\alpha & d\sin\alpha & c \end{pmatrix} \tag{I.22}$$

Remember the definitions (see in the previous section):

$$\begin{aligned} a &= k_\perp^2 \, I_{k_\parallel}^{(\perp)} \, J_0(Z)\,, \qquad b = k_\perp^2 \, I_{k_\parallel}^{(\perp)} \, J_2(Z) \\ c &= I_{k_\parallel}^{(\parallel)} \, J_0(Z)\,, \qquad d = -2\,k_\perp \, \frac{1}{\tau} \frac{\partial}{\partial v_\parallel} I_{k_\parallel}^{(\perp)} \, J_1(Z) \end{aligned} \tag{I.23}$$

Note the difference in the argument of the Bessel functions, which now is:

$$Z = k_\perp \, v_\perp \, \tau \tag{I.24}$$

(cf. remark in the end of the previous section).

We see that the vanishing-field limit is correctly recovered from our expressions. Furthermore, going back to the original expressions in the absence of magnetic field, one immediately sees that the integrand therein is *spherical* symmetric, so all diagonal elements in expressions above (i.e. for $\lim_{\Omega\to 0} \mathbf{C}$ and $\lim_{\Omega\to 0} \mathbf{A}$) may be proved to be equal[6]: check (in the above relation) that:

$$D_\perp \equiv \frac{1}{2}(A_{11} + A_{22}) \;=\; A_{33} = D_\parallel \qquad .$$

I.3 Φ operator in an arbitrary frame

Equipped with expressions (I.19), (I.20) as reference, we may now use the general relations presented in Chapter 5 (see (5.9) precisely) in order to compute the coefficients in the markovian ($\Phi-$) FPE[7]. Details will be omitted where obvious. All that needs to be recalled in the following is:

- the reduced definitions (I.19), (I.20), (I.21) for the coefficients,
- expression (5.9) for the new ($\Phi-$) coefficients.
- expression (5.4) for the averaging operator $\mathcal{A}_{t'}$ (together with information provided in Appendix E).

The velocity-diffusion matrix $\mathbf{D_{22}}^{(\Phi)}$ reads:

$$\begin{aligned} \mathbf{D_{22}}^{(\Phi)} &\sim \mathcal{A}_{t'} \, \mathbf{N}'(t') \, \mathbf{C}(\tau) \, \mathbf{N}'^{T}(t' + \tau) = \ldots \\ &= \begin{pmatrix} \frac{1}{2}(A+B)\,\cos\Omega\tau & \frac{-s}{2}(A+B)\,\sin\Omega\tau & 0 \\ \frac{-s}{2}(A+B)\,\sin\Omega\tau & \frac{1}{2}(A+B)\,\cos\Omega\tau & 0 \\ 0 & 0 & C \end{pmatrix} \end{aligned}$$

[6]Cf. Appendix J.

[7]The hypothesis of a diagonal correlation matrix (in 'frame 2', see in previous chapters) is not made here, so this method, valid in *any* frame, is definitely more general than the one presented in Ch. 7.

$$= \begin{pmatrix} D_\perp & D_\angle & 0 \\ -D_\angle & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \tag{I.25}$$

so the symmetric part defining $\mathbf{D_{VV}}^{(\Phi)}$ reads:

$$\mathbf{D_{VV}}^{(\Phi)} = (\mathbf{D_{22}}^{(\Phi)})^{SYM} = \begin{pmatrix} D_\perp & 0 & 0 \\ 0 & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \tag{I.26}$$

The cross-velocity-position diffusion matrices read:

$$\begin{aligned} \mathbf{D_{21}}^{(\Phi)} &\sim \mathcal{A}_{t'}\, \mathbf{N}'(t')\, \mathbf{C}(\tau)\, \mathbf{N}^T(t'+\tau) = ... \\ &= \Omega^{-1} \begin{pmatrix} \frac{1}{2}(A+B)\,\sin\Omega\tau & \frac{s}{2}(A+B)\,\cos\Omega\tau & s\Omega\,F\,\tilde{X} \\ \frac{-s}{2}(A+B)\,\cos\Omega\tau & \frac{1}{2}(A+B)\,\sin\Omega\tau & -s\Omega\,E\,\tilde{X} \\ s\,F & -s\,E & C\,\Omega\tau \end{pmatrix} \\ &= -s\,\Omega^{-1} \begin{pmatrix} D_\angle & -D_\perp & 0 \\ D_\perp & D_\angle & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & sF\,\tilde{X} \\ 0 & 0 & -sE\,\tilde{X} \\ s\Omega^{-1}\,F & -s\Omega^{-1}\,E & D_\parallel \end{pmatrix} \end{aligned} \tag{I.27}$$

and

$$\begin{aligned} \mathbf{D_{12}}^{(\Phi)} &\sim \mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{C}(\tau)\, \mathbf{N}'^T(t'+\tau) = ... = \\ &= \Omega^{-1} \begin{pmatrix} -\frac{1}{2}(A+B)\,\sin\Omega\tau & \frac{-s}{2}(A+B)\,\cos\Omega\tau & sF \\ \frac{s}{2}(A+B)\,\cos\Omega\tau & -\frac{1}{2}(A+B)\,\sin\Omega\tau & -sE \\ \Omega\,(-E\sin\Omega\tau + s\,F\cos\Omega\tau) & \Omega\,(-Es\cos\Omega\tau - F\sin\Omega\tau) & 0 \end{pmatrix} \\ &= -s\,\Omega^{-1} \begin{pmatrix} -D_\angle & D_\perp & 0 \\ -D_\perp & -D_\angle & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & 0 & s\Omega^{-1}\,F \\ 0 & 0 & -s\Omega^{-1}\,E \\ s(D_{22}^{(\theta)})_{32}\,\tilde{X} & -s(D_{22}^{(\theta)})_{31}\,\tilde{X} & 0 \end{pmatrix} \end{aligned} \tag{I.28}$$

where $\tilde{X}$ denotes the ill-defined quantity (see discussion previously; also in E):

$$\tilde{X} = \mathcal{A}_{t'}\, t'\, \sin\lambda\, t'\,.$$

The final cross-V-X diffusion matrix $\mathbf{D_{VX}}^{(\Phi)}$ reads:

$$\mathbf{D_{VV}}^{(\Phi)} = (\mathbf{D_{21}}^{(\Phi)}) + (\mathbf{D_{12}}^{(\Phi)})^T = -s\,\Omega^{-1} \begin{pmatrix} 0 & -D_\perp & 0 \\ +D_\perp & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + ... \tag{I.29}$$

where, as said in Chapter 7, contributions involving $z-$ (and, inevitably, $\tilde{X}$) were cancelled and contributions involving v_z disappear once the averaging discussed in §6.5.4 (in frame 2) is carried out.

Finally, the space-diffusion matrices read:

$$\mathbf{D_{11}}^{(\Phi)} \sim \mathcal{A}_{t'}\, \mathbf{N}(t')\, \mathbf{C}(\tau)\, \mathbf{N}^T(t'+\tau) = ...$$

$$= \Omega^{-2} \begin{pmatrix} \frac{1}{2}(A+B)\,\cos\Omega\tau + B & \frac{-s}{2}(A+B)\,\sin\Omega\tau - D & 0 \\ \frac{s}{2}(A+B)\,\sin\Omega\tau - D & \frac{1}{2}(A+B)\,\cos\Omega\tau + A & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{X}(...) + \hat{X}(...) \tag{I.30}$$

where we omitted, as previously, ill-defined quantities (in the 3rd column and line only) involving $\tilde{X}$ and $\hat{X} = \mathcal{A}_{t'}\, {t'}^2 \sim \infty$. The final X- diffusion matrix $\mathbf{D_{XX}}^{(\Phi)}$ is:

$$\mathbf{D_{XX}}^{(\Phi)} = (\mathbf{D_{11}}^{(SYM)}) = -s\,\Omega^{-2} \begin{pmatrix} D_\perp + Q & 0 & 0 \\ 0 & D_\perp + Q & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{I.31}$$

where, once more, we have carried out the averaging over particle gyro-phase ('frame 2'). The new quantity Q is related to $\frac{1}{2}(A+B) = a = k_\perp^2\, I_{k_\parallel}^{(\perp)}\, J_0(Z)$ (recall previous definitions):

$$Q = \frac{n_{\alpha'}}{m_\alpha^2}(2\pi)^3\; \pi \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp^3\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, I_{k_\parallel}^{(\perp)}\, J_0(Z) \tag{I.32}$$

(remember that Z' was defined in (I.8)), so the total $\perp -$ space diffusion coefficient is equal to:

$$D_\perp^{(X)} = \frac{n_{\alpha'}}{m_\alpha^2}(2\pi)^3\; \pi \int_0^t d\tau \int_0^\infty dk_\perp\, k_\perp^3\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, I_{k_\parallel}^{(\perp)}\, J_0(Z)\,(1+\cos\Omega\tau) \tag{I.33}$$

Appendix J

The free-motion limit - Fokker-Planck equation for unmagnetized plasma

Let us consider the derivation of our F.P. equation in the unmagnetized limit[1]. This calculation will be carried out in spherical coordinates, in *no* external field, *independently* from the one presented in the text.

J.1 Introduction

Remember the equations obtained in the text, for an electrostatic plasma. The velocity diffusion matrix A_{rs} (in all those equations) and the cross-velocity-position diffusion matrix G_{rs} (see (4.4)) were given in Chapter 4:

$$\begin{aligned} \left\{ \begin{array}{c} \mathbf{A} \\ \mathbf{G} \end{array} \right\} &= \frac{n_{\alpha'}}{m_\alpha^2} \int_0^\infty d\tau \int d\mathbf{v_1} (2\pi)^3 \; \phi_{eq}^{\alpha'}(v_1) \int d\mathbf{k}\, \tilde{V}_k^2\, \mathbf{k} \otimes \mathbf{k} \\ & \qquad\qquad e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}}\; e^{i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} \left\{ \begin{array}{c} \mathbf{N'}_\alpha^T(\tau) \\ \mathbf{N}_\alpha^T(\tau) \end{array} \right\} \\ &\equiv \frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{C}_{\alpha,\alpha'}(\mathbf{x}, \mathbf{v}; t, t-\tau) \left\{ \begin{array}{c} \mathbf{N'}_\alpha^T(\tau) \\ \mathbf{N}_\alpha^T(\tau) \end{array} \right\} \end{aligned} \tag{J.1}$$

A summation over particle species α' is understood[2]. The external field appears in the dynamical matrices $\mathbf{N}(t)$, $\mathbf{N}'(t)$.

Let us consider an electrostatic plasma in *no* external field. The solution of the problem of motion:

$$\{x_i(t), v_i(t)\} = \{x_i + v_i t, v_i\} \qquad i = 1, \cdots, d$$

[1] i.e. the *RMJ* (*Rosenbluth-McDonald-Judd*) limit.

[2] In the single-species case, set $e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{v}}\, e^{i\mathbf{k}\,\mathbf{N}_{\alpha'}(\tau)\,\mathbf{v_1}} = e^{i\mathbf{k}\,\mathbf{N}_\alpha(\tau)\,\mathbf{g}}$, where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$; see in the text.

(with the initial condition $\{x, v\} \equiv \{x(0), v(0)\}$) implies

$$N_{ij} = \delta_{ij}\, t\,, \qquad N'_{ij} = \delta_{ij}$$

so the above relations become:

$$\begin{aligned} \left\{ \begin{array}{c} \mathbf{A} \\ \mathbf{G} \end{array} \right\} &= \frac{n_{\alpha'}}{m_\alpha^2} \int_0^\infty d\tau \int d\mathbf{v_1} (2\pi)^3\, \phi_{eq}^{\alpha'}(v_1) \int d\mathbf{k}\, \tilde{V}_k^2\, \mathbf{k} \otimes \mathbf{k}\, e^{i\mathbf{k}\,\mathbf{g}\,\tau} \left\{ \begin{array}{c} 1 \\ \tau \end{array} \right\} \\ &\equiv \frac{1}{m^2} \int_0^\infty d\tau\, \mathbf{C}(\mathbf{v}; t, t-\tau) \left\{ \begin{array}{c} 1 \\ \tau \end{array} \right\} \end{aligned} \tag{J.2}$$

where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$.

J.2 Evaluation of the general formulae

The above expressions can be evaluated by carrying out the time ($\tau-$) integration first, and then proceeding to an evaluation of the Fourier ($\mathbf{k}-$) integral in spherical coordinates. This calculation is actually valid in *any* weakly-coupled system of particles equipped with a long-range interaction potential, provided that *no* external field is present. The evaluation of the velocity diffusion matrix $\mathbf{A}$ is identical to the one evoked in the derivation of the Landau collision term (see [5]); furthermore, the detailed evaluation of the cross-velocity-diffusion matrix $\mathbf{G}$ can be found in [46], [68]. Therefore, details will be omitted in the following, and only new results will be stated (as they will serve as a basis for further analytical evaluation of the coefficients).

J.2.1 Final result

Choosing a frame where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$ lies along $\hat{z}$, the above diffusion matrices are given by the expressions:

$$\begin{aligned} \mathbf{A} &= \frac{n}{m^2}\, 8\pi^5 C_k \int d\mathbf{v_1}\, \phi_{eq}(v_1)\, \frac{1}{g} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \mathbf{G} &= \frac{n}{m^2}\, 32\pi^4 C'_k \int d\mathbf{v_1}\, \phi_{eq}(v_1)\, \frac{1}{g^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{J.3}$$

where $\mathbf{g} \equiv \mathbf{v} - \mathbf{v_1}$ (obviously: $g = |\mathbf{g}| \in \Re$).

The matrices appearing in these expressions are, respectively, of the form:

$$\mathbf{M_1} = \mathbf{I} - \hat{e}_3\, \hat{e}_3\,, \qquad \mathbf{M_2} = \mathbf{I} - 2\, \hat{e}_3\, \hat{e}_3$$

In an arbitrary frame, $\hat{e}_3$ should be replaced by the unit vector $\hat{g} = \frac{1}{g}\mathbf{g}$; the above expressions finally become:

$$\begin{aligned} A_{rs} &= \frac{n}{m^2}\, 8\pi^5 C_k \int d\mathbf{v_1}\, \phi_{eq}(v_1)\, \frac{g^2\, \delta_{rs} - g_r g_s}{g^3} \\ G_{rs} &= \frac{n}{m^2}\, 32\pi^4 C'_k \int d\mathbf{v_1}\, \phi_{eq}(v_1)\, \frac{g^2\, \delta_{rs} - 2 g_r g_s}{g^4} \end{aligned} \tag{J.4}$$

The first of these expressions has precisely the form of the Landau tensor [5] (upon substitution with a Maxwellian distribution function for the reservoir).

J.2.2 Fourier integrals

The quantities C_k, C'_k denote the Fourier integrals:

$$\begin{aligned} C_k &= \int_0^\infty dk\, k^3\, \tilde{V}_k^2 \\ C'_k &= \int_0^\infty dk\, k^2\, \tilde{V}_k^2 \end{aligned} \tag{J.5}$$

The integral C_k is known to diverge at both ends, i.e. at $k = 0$ and $k \to \infty$, for a Coulomb potential. This fact actually reflects the failure of the assumptions underlying this description for long- and short-distance encounters, respectively (remember that only binary interactions were retained in this picture). The first nuisance (for long-distance encounters) is due to the long-range character of the interactions, since large inter-particle distances do not exclude interactions. It can essentially be removed by considering charge screening through an appropriate Debye-type potential:

$$V(r) = \frac{e_\alpha\, e_{\alpha'}}{r}\, e^{-r/\lambda_D} \equiv V_0\, \frac{e^{-k_D\, r}}{r}$$

i.e.

$$\tilde{V}_k = \frac{e_\alpha\, e_{\alpha'}}{2\,\pi^2} \frac{1}{k^2 + k_D^2} \equiv \frac{\tilde{V}_0}{k^2 + k_D^2} \tag{J.6}$$

[3]. The second problem (at infinite k) implies failure of the weak-coupling approximation at short distances. It is traditionally removed by considering an upper cutoff, say k_{max}, instead of infinity (as the upper limit of integration). We thus obtain:

$$C_k = \frac{\tilde{V}_0^2}{2} \ln\big(1 + \frac{k_{max}^2}{k_D^2}\big) - \frac{\tilde{V}_0^2}{2} \frac{k_{max}^2}{k_{max}^2 + k_D^2} \approx \tilde{V}_0^2\, \ln \frac{k_{max}}{k_D} \equiv \tilde{V}_0^2 \ln \Lambda_{Coulomb} \tag{J.7}$$

($k_{max} \gg k_D$) giving rise to the celebrated *Coulomb logarithm*. We shall not go into more detail here, since the relevant discussion is very common in literature, and can be found for instance in a detailed study in [5] (see Appendix 2A.1

[3] $V_0 = e_\alpha\, e_{\alpha'}$, $\tilde{V}_0 = \frac{e_\alpha\, e_{\alpha'}}{2\pi^2}$ are constant quantities; λ_D is the Debye length:

$$\lambda_D \equiv k_D^{-1} = \left(\frac{4\pi e^2 n}{k_B T}\right)^{-1/2}$$

or, in general:

$$\lambda_D \equiv k_D^{-1} = min\Big\{ \left(\frac{4\pi e_\alpha^2 n_\alpha}{k_B T_\alpha}\right)^{-1/2} \Big\}.$$

therein); also see [3]. Let us point out that the same type of divergence reappears in the second matrix defined above, i.e. B_{rs} containing C', as remarked in [68]); thus, for a Debye potential:

$$C'_k = \frac{\tilde{V}_0^2}{2k_D}\left[\arctan\left(\frac{k_{max}}{k_D}\right) - \frac{k_{max}k_D}{k_{max}^2 + k_D^2}\right] \approx \frac{\tilde{V}_0^2}{2k_D}\arctan\left(\frac{k_{max}}{k_D}\right) \quad \to \quad \frac{\pi\tilde{V}_0^2}{4k_D} \tag{J.8}$$

(in the last step we have taken the limit $k_{max} \to \infty$).

Our final expressions for C_k, C'_k are in agreement with the expressions given in [68]). It should be stressed that, as argued therein, the difference in order of magnitude between G_{rs} and A_{rs} (typically of the order of C'_k, C_k respectively), which is often implied in literature (and leads to neglecting the former!), is *not* always well founded. As a matter of fact, once all quantities are expressed in a non-dimensional form, the ratio C'_k/C_k (of dimensions of $[1/k] = [length]$ above) is, according to the above expressions, roughly equal to:

$$\frac{C'_k}{C_k} \sim \frac{1}{k_D l \ln \Lambda_{Coulomb}}$$

where l represents some length scale, typical of the problem, e.g. the Debye radius or the Larmor radius in plasma. We see that the ratio is not always negligible, with respect to unity, as often suggested in the past (see discussion elsewhere in this text).

J.3 Evaluation of the formulae for a test-particle in a Maxwellian background

We may now evaluate expressions (J.4) for a Maxwellian reservoir d.f.:

$$\phi_{eq}(v_1) = \phi^{\alpha'}_{Max}(v_1) = \phi^{\alpha'}_0 \, e^{-v_1^2/\sigma_{\alpha'}} \tag{J.9}$$

where ϕ_0 is the normalization constant:

$$\phi_0 = \left(\frac{m_{\alpha'}}{2\pi T_{\alpha'}}\right)^{3/2} \equiv \frac{1}{(2\pi)^{3/2}\, {v^{\alpha'}_{th}}^3} \equiv \frac{1}{(\pi\sigma)^{3/2}}$$

and σ is related to the plasma temperature:

$$\sigma \equiv 2\, {v^{\alpha'}_{th}}^2 \equiv \frac{2T_{\alpha'}}{m_{\alpha'}} \qquad \forall i \in \{1,2,3\} \equiv \{x,y,z\}$$

Let us choose a reference frame where the $z-$axis lies along the velocity $\mathbf{v}$: $\hat{z} = \hat{v} = \frac{1}{v}\mathbf{v}$ (see figure (J.1) and express all quantities in spherical coordinates. The velocity integral becomes:

$$\int d\mathbf{v_1} \,\cdot\, = \int_0^\infty dv_1\, v_1^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi \,\cdot$$

The reservoir-particle velocity $\mathbf{v_1}$ and the velocity difference $\mathbf{g} = \mathbf{v} - \mathbf{v_1}$ are now expressed as:

$$\mathbf{v_1} = (v_1 \, \sin\theta \, \cos\phi, \, v_1 \, \sin\theta \, \sin\phi, \, v_1 \, \cos\theta)$$

and

$$\mathbf{g} = (g \, \sin\theta' \, \cos\phi', \, g \, \sin\theta' \, \sin\phi', \, g \, \cos\theta')$$

Notice that (see figure (J.1)):

$$v_1 \sin\theta = g \sin\theta' \quad \Rightarrow \quad \sin\theta' = \frac{v_1}{g} \sin\theta$$

and

$$\phi' = \pi + \phi$$

(remember that $\mathbf{v}$, $\mathbf{v_1}$, $\mathbf{g}$ lie on the same plane).

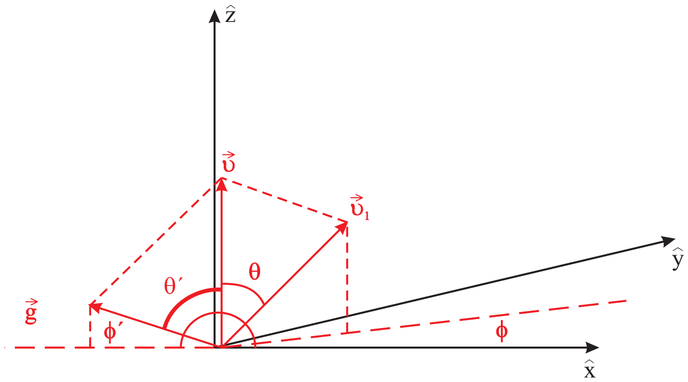

Figure J.1: An appropriate frame for the evaluation of the $\mathbf{v_1}$-integral: the z-axis is taken along particle velocity $\mathbf{v}$.

J.3.1 Diffusion coefficients A_{ij}

First, the off-diagonal elements in both (J.4a, b) are readily seen to disappear once the integration in ϕ is carried out, since they contain one of the integrals:

$$\int_0^{2\pi} d\phi \, \sin\phi = \int_0^{2\pi} d\phi \, \cos\phi = \int_0^{2\pi} d\phi \, \sin\phi \cos\phi = 0$$

The diffusion matrices thus come out to be diagonal (in this frame).

Let us evaluate the diagonal elements A_{ii} $(i = 1, 2, 3)$; first:

$$\begin{aligned} A_{11} &= \frac{n}{m^2} 8\pi^5 C_k \int d\mathbf{v_1} \, \phi_{eq}(v_1) \left(\frac{1}{g} - \frac{g_x^2}{g^3} \right) \\ &= \frac{n}{m^2} 8\pi^5 C_k \int_0^\infty dv_1 \, v_1^2 \, \phi_{eq}(v_1) \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\phi \, \frac{1}{g} \left(1 - \sin^2\theta' \cos^2\phi' \right) \end{aligned}$$

The integration in ϕ can be trivially carried out:

$$\int_0^{2\pi} d\phi = 2\pi \quad , \qquad \int_0^{2\pi} d\phi\, \cos^2\phi' = \int_0^{2\pi} d\phi\, \cos^2(\pi+\phi) = \int_0^{2\pi} d\phi\, \cos^2\phi' = \pi$$

In the same way, we see that $A_{22} = A_{11}$. Also,

$$\begin{aligned} A_{33} &= \frac{n}{m^2}\, 8\pi^5 C_k \int d\mathbf{v_1}\, \phi_{eq}(v_1) \left(\frac{1}{g} - \frac{g_z^2}{g^3}\right) \\ &= \frac{n}{m^2}\, 8\pi^5 C_k \int_0^{\infty} dv_1\, v_1^2\, \phi_{eq}(v_1) \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, \frac{1}{g} \left(1 - \cos^2\theta'\right) \end{aligned}$$

Furthermore, we may set:

$$g = |\mathbf{g}| \equiv |\mathbf{v} - \mathbf{v_1}| = \left[(\mathbf{v} - \mathbf{v_1}) \cdot (\mathbf{v} - \mathbf{v_1})\right]^{1/2} = \left(v^2 + v_1^2 - 2\mathbf{v} \cdot \mathbf{v_1}\right)^{1/2}$$

and, having chosen the velocity $\mathbf{v}$ to lie along the $z-$axis, we have:

$$g = \left(v^2 + v_1^2 - 2v\, v_1\, \cos\theta\right)^{1/2}$$

so the above expressions become:

$$\begin{aligned} A_{11} = A_{22} &= \frac{n}{m^2}\, 8\pi^5 C_k \int_0^{\infty} dv_1\, v_1^2\, \phi_{eq}(v_1) \int_0^{\pi} d\theta\, \sin\theta\, (2\pi) \left(\frac{1}{g} - \frac{1}{2}\frac{v_1^2}{g^3} \sin^2\theta\right) \\ A_{33} &= \frac{n}{m^2}\, 8\pi^5 C_k \int_0^{\infty} dv_1\, v_1^4\, \phi_{eq}(v_1) \int_0^{\pi} d\theta\, \sin\theta\, (2\pi) \frac{1}{g^3} \sin^2\theta \end{aligned}$$

or, more simply

$$A_{33} = \frac{n}{m^2}\, 16\pi^6 C_k \int_0^{\infty} dv_1\, v_1^4\, \phi_{eq}(v_1) \int_{-1}^{1} d\mu\, \frac{1 - \mu^2}{\left(v^2 + v_1^2 - 2v\, v_1\, \mu\right)^{3/2}}$$

and

$$A_{11} = A_{22} = -\frac{1}{2} A_{33} + \frac{n}{m^2}\, 16\pi^6 C_k \int_0^{\infty} dv_1\, v_1^2\, \phi_{eq}(v_1) \int_{-1}^{1} d\mu\, \frac{1}{\left(v^2 + v_1^2 - 2v\, v_1\, \mu\right)^{1/2}}$$

(we have shifted from θ to $\mu \equiv \cos\theta$).

In order to carry out the integrations in μ, it is convenient to shift from μ to $g = \left(v^2 + v_1^2 - 2v\, v_1\, \mu\right)^{1/2}$, so that:

$$\mu = \frac{v^2 + v_1^2 - g^2}{2v\, v_1} \quad , \qquad d\mu = -\frac{g}{v\, v_1}\, dg$$

The first integral in μ thus finally gives:

$$\int_{-1}^{1} d\mu\, \frac{1 - \mu^2}{\left(v^2 + v_1^2 - 2v\, v_1\, \mu\right)^{3/2}} = \begin{cases} \frac{4}{3\, v^3}\, , & v_1 \leq v \\ \frac{4}{3\, v_1^3}\, , & v_1 > v \end{cases}$$

$(v, v_1 \in \Re_+)$. The remaining velocity integration in A_{33} now reads:

$$
\begin{aligned}
A_{33} &= \frac{n}{m^2}\, 16\pi^6 C_k \left[\int_0^v \frac{4}{3\,v^3} + \int_v^\infty \frac{4}{3\,v_1^3} \right] v_1^4 \, \frac{1}{(\pi\sigma^{\alpha'})^{3/2}}\, e^{-v_1^2/\sigma^{\alpha'}}\, dv_1 \\
&= \cdots \\
&= \frac{n}{m^2}\, 4\sqrt{2}\,\pi^5 C_k \frac{1}{v_{th}}\, \frac{1}{w^3} \Big[\Phi(w) - w\,\Phi'(w) \Big] \qquad \text{(J.10)}
\end{aligned}
$$

where $\Phi(x)$ denotes the *error function*:

$$
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt
$$

(notice that: $\Phi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$) and the (non-dimensional) argument w in the final expression is given by:

$$
w = \frac{v}{\sqrt{\sigma}} = \frac{v}{v_{th}\sqrt{2}} = \left(\frac{m v^2}{2 k_B T} \right)^{1/2}
$$

In a similar manner, the calculation for A_{11} yields:

$$
A_{11} = -\frac{1}{2} A_{33} + \frac{n}{m^2}\, 16\pi^6 C_k \int_0^\infty dv_1\, v_1^2\, \phi_{eq}(v_1) \int_{-1}^{1} d\mu\, \frac{1}{\left(v^2 + v_1^2 - 2v\, v_1\, \mu \right)^{1/2}}
$$

The integral in μ now gives:

$$
\int_{-1}^{1} d\mu\, \frac{1}{\left(v^2 + v_1^2 - 2v\, v_1\, \mu \right)^{1/2}} = \begin{cases} \frac{2}{v}\,, & v_1 \le v \\[2mm] \frac{2}{v_1}\,, & v_1 > v \end{cases}
$$

$(v, v_1 \in \Re_+)$ so the remaining integration now reads:

$$
\begin{aligned}
A_{11} &= -\frac{1}{2} A_{33} + \frac{n}{m^2}\, 16\pi^6 C_k \left[\int_0^v \frac{2}{v} + \int_v^\infty \frac{2}{v_1} \right] v_1^2 \, \frac{1}{(\pi\sigma^{\alpha'})^{3/2}}\, e^{-v_1^2/\sigma^{\alpha'}}\, dv_1 \\
&= \cdots \\
&= \frac{n}{m^2}\, 2\sqrt{2}\,\pi^5 C_k \frac{1}{v_{th}}\, \frac{1}{w^3} \Big[(2w^2 - 1)\, \Phi(w) + w\,\Phi'(w) \Big] \qquad \text{(J.11)}
\end{aligned}
$$

The result for A_{22} is exactly the same.

The final result may be conveniently re-arranged in the form:

$$
\begin{aligned}
A_{11} = A_{22} &= A(0) \left\{ \frac{3\sqrt{\pi}}{8}\, \frac{1}{w^3} \Big[(2w^2 - 1)\, \Phi(w) + w\,\Phi'(w) \Big] \right\} \equiv A(0)\, \tilde{H}(w) \\
A_{33} &= A(0) \left\{ \frac{3\sqrt{\pi}}{8}\, \frac{1}{w^3} \Big[\Phi(w) - w\,\Phi'(w) \Big] \right\} \equiv A(0)\, \tilde{G}(w) \qquad \text{(J.12)}
\end{aligned}
$$

where $A(0)$ is defined by:

$$
A(0) = \frac{n_{\alpha'}}{m_\alpha^2}\, \frac{16\sqrt{2}}{3}\, \pi^4 \sqrt{\pi}\, C_k \frac{1}{v_{th}} \qquad \text{(J.13)}
$$

The functions $\tilde{H}(w)$, $\tilde{G}(w)$ are depicted in figure J.2; they are actually associated to a set of functions first derived in [60] in a different context and recovered by different methods in later works (see [3]; also references therein)[4].

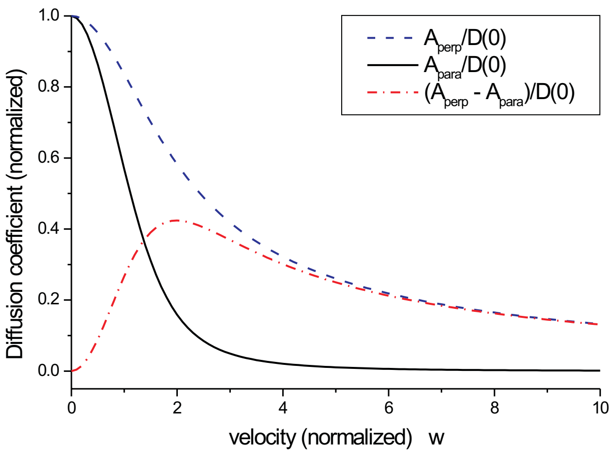

Figure J.2: The diffusion coefficients in the direction parallel (solid line) and perpendicular (dashed line) to the particle velocity $\mathbf{v}$. Their difference $A_\perp - A_\parallel$ is represented in dashed-dot line.

J.3.2 Normalization constant

The functions $\tilde{H}(w)$, $\tilde{G}(w)$ defined above are defined in such a manner, that their limit for vanishing velocity is equal to unity. Therefore, the constant $A(0)$ actually represents $A_\perp(v = 0) = A_\parallel(v = 0)$; it is given by:

$$\begin{aligned} A_{\alpha,\alpha'}(0) &= \frac{n_{\alpha'}}{m_\alpha^2} C_k \frac{16\pi^4\sqrt{2\pi}}{3v_{th}^{\alpha'}} \approx \frac{n_{\alpha'}}{m_\alpha^2} \frac{16\pi^4\sqrt{2\pi}}{3v_{th}^{\alpha'}} \tilde{V}_0^2 \ln \Lambda_{Coulomb} \\ &= e_\alpha^2\, e_{\alpha'}^2\, \frac{n_{\alpha'}}{m_\alpha^2} \frac{4\sqrt{2\pi}\, m_{\alpha'}^{1/2}}{3\, T_{\alpha'}^{1/2}} \ln \frac{k_{max}}{k_{Debye}} = \ldots \end{aligned}$$

It is related to the constant $D(0)$ defined in Chapter 8.

[4] More precisely, $\tilde{H}(w)$, $\tilde{G}(w)$ are related to $H(w)$, $G(w)$ appearing in [3] (see §37-38 therein) by: $\tilde{G}(w) = \frac{3\sqrt{\pi}}{4}\frac{1}{w}G(w)$, $\tilde{H}(w) = \frac{3\sqrt{\pi}}{4}\frac{1}{w}H(w)$.

J.3.3 Final result for A_{ij}

The above results correspond to a reference frame with the $z-$axis along the particle velocity $\mathbf{v}$. The velocity diffusion matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} A_\perp & 0 & 0 \\ 0 & A_\perp & 0 \\ 0 & 0 & A_\parallel \end{pmatrix} \tag{J.14}$$

[5]where $A_\perp$, $A_\parallel$ are given by (J.12a, b) respectively.

The diffusion matrix presented above is of the form:

$$\mathbf{A} = A_\perp \mathbf{I} + (A_\parallel - A_\perp)\hat{e}_3\,\hat{e}_3$$

The diffusion coefficients are depicted in figure J.2 against velocity v.

J.3.4 Expression in an arbitrary frame

In an arbitrary frame, $\hat{e}_3$ should be replaced by the unit vector $\hat{v} = \frac{1}{v}\mathbf{v}$. The above expressions finally become:

$$D_{rs} = \frac{A_\perp\, v^2\, \delta_{rs} + (A_\parallel - A_\perp) v_r v_s}{v^2} \tag{J.15}$$

For instance, the diagonal elements are of the form e.g.

$$D_{xx} = \frac{A_\perp\, v^2\; + (A_\parallel - A_\perp) v_x^2}{v^2} = ... = \frac{1}{v^2}[A_\parallel v_x^2 + A_\perp (v_y^2 + v_z^2)]$$

(the analogous expressions for A_{yy}, A_{zz} can be obtained by inspection); the off-diagonal elements are e.g.

$$A_{xy} = (A_\parallel - A_\perp)\frac{v_x v_y}{v^2}$$

(plus similar expressions for the remaining elements); notice that $A_\parallel - A_\perp \leq 0$.

Note that the diffusion matrix is *symmetric*, i.e. $A_{ij} = A_{ji}$.

J.3.5 Dynamical friction vector $\mathcal{F}_i$

We have seen that the drift coefficient $\mathcal{F}$ is related to a velocity derivative of the velocity diffusion coefficients. In the particular frame taken above (taking $\mathbf{v}$ along $\hat{z}$), the vector $\mathcal{F}$ is given by:

$$\mathcal{F}\,\hat{v} = (1+\mu)\,\frac{1}{v}\frac{\partial A_\parallel}{\partial v}\,\mathbf{v} \;\equiv\; -\eta(v)\,\mathbf{v}$$

[5]A similar calculation results in the same form for the matrix $\mathbf{G}$; however, details are of no importance here and will be omitted.

where the definition of the *dynamical friction coefficient* $\eta(v)$ is obvious. Substituting with the definition of $A_{\parallel}$ from above, the algebraic value $\mathcal{F}$ of the vector is found to be:

$$\begin{aligned}\mathcal{F} &= (1+\mu)\frac{\partial A_{\parallel}(v)}{\partial v} = (1+\mu)\frac{1}{v_{th}\sqrt{2}}\frac{\partial A_{\parallel}(w)}{\partial w} \\ &= (1+\mu)\,\frac{D(0)}{v_{th}\sqrt{2}}\frac{\partial \tilde{G}(w)}{\partial w} \\ &= -(1+\mu)\,\frac{D(0)}{v_{th}\sqrt{2}}\,\frac{3\sqrt{\pi}}{4\,w^4}\left[3\,\Phi(w) - w\,(2w^2+3)\,\Phi'(w)\right] \qquad \text{(J.16)}\end{aligned}$$

We see that the vector $\mathcal{F}(v)$ is always directed opposite to the particle velocity $\mathbf{v}$, and therefore represents the dynamical friction force felt by the particle; notice that:

$$\mathcal{F}\cdot\mathbf{v} < 0 \qquad \forall v \in \Re_+$$

The absolute value of this quantity represents the *energy loss rate* due to friction; it is depicted, together with the vector norm $\mathcal{F}$, in Figure J.3a. The friction coefficient $\eta(v)$ plotted against the (normalized) velocity w in Figure J.3b. We see that *the faster* the particle *the less* friction it suffers.

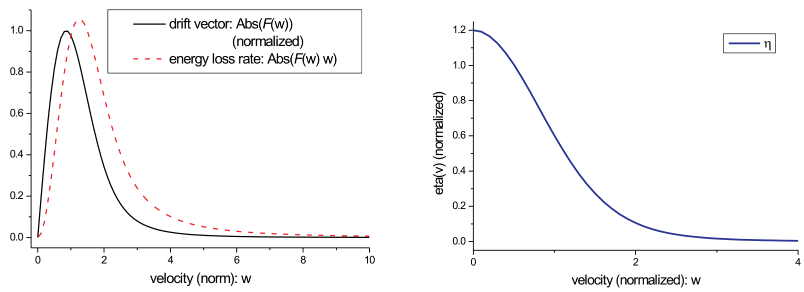

Figure J.3:
(a) The norm of the drift coefficient (solid line) and the (absolute value of the) energy loss rate $\mathcal{F}\cdot\mathbf{v}$ (dashed line) versus particle velocity (normalized) w.
(b) The friction coefficient $\eta(v)$ as a function of velocity w.

J.4 An alternative form for the diffusion coefficients

Remember the original expression (J.2) for the coefficients. The evaluation presented in the preceding paragraphs was the result of carrying out the time integration first (from $\tau = 0$ to infinity) and then evaluating the Fourier integrals. What follows is an alternative evaluation method, which will be convenient for comparison with the results for the magnetized case, presented elsewhere in this text. Following the method outlined in Chapter 8, we may start by evaluating the velocity ($\mathbf{v_1}-$) integral, then carry on with the Fourier ($\mathbf{k}-$) integration and thus leave the time ($\tau-$) integral for the end.

Eliminating the velocity integral in (J.2) (see in §8.3.1 for details) we obtain:

$$D_{ij} = \frac{n}{m^2}(2\pi)^3 \int_0^t d\tau \int d\mathbf{k}\, e^{-\sigma\, k_\perp^2 \tau^2/4}\, e^{-\sigma\, k_\parallel^2 \tau^2/4}\, k_\perp^2\, e^{i\, k_\perp v_\perp \tau}\, e^{i\, k_\parallel v_\parallel \tau}\, \tilde{V}_k^2\, k_i\, k_j \tag{J.17}$$

Choosing a frame with the $z-$ axis lying along particle velocity $\mathbf{v}$ (see figure above), we may express the integral in cylindrical coordinates:

$$\mathbf{D} = \frac{n}{m^2}(2\pi)^3 \int_0^t d\tau \int_{-\infty}^{\infty} dk_\parallel \int_0^{\infty} dk_\perp \int_0^{2\pi} dk_\alpha \; e^{-\sigma\, k_\perp^2 \tau^2/4}\, e^{-\sigma\, k_\parallel^2 \tau^2/4}\, k_\perp^2\, e^{i\, k_\parallel v_\parallel \tau}\, \tilde{V}_k^2 \begin{pmatrix} k_\perp\, \cos\theta \\ k_\perp\, \sin\theta \\ k_\parallel \end{pmatrix} \begin{pmatrix} k_\perp\, \cos\theta \\ k_\perp\, \sin\theta \\ k_\parallel \end{pmatrix} \tag{J.18}$$

The calculation is straightforward. First, the off-diagonal elements are seen to cancel once the angle integration is carried out. We thus obtain:

$$D_{ij} = \left[A_\perp(\delta_{i1} + \delta_{i2}) \,+\, A_\parallel \delta_{i3}\right]$$

where:

$$D_{\{\perp,\parallel\}} = \frac{4\, n\, e^4}{m_\alpha^2}\, e^{-v^2/\sigma} \int_0^t d\tau \int_0^{\infty} dk_\perp\, k_\perp^{\{3,1\}}\, e^{-\sigma\, k_\perp^2 \tau^2/4} \left[\int_{-\infty}^{\infty} dk_\parallel\; e^{-\sigma\, k_\parallel^2\, \tau^2/4}\, \frac{k_\parallel^{\{0,2\}} e^{-\sigma\, k_\parallel\, v_\parallel\, \tau}}{(k^2 + k_D^2)^2}\right] \tag{J.19}$$

The $k_\parallel-$ integration may now be carried out, just as it appears in Chapter 9 (this integral has been evaluated in the Appendix); we will not give any further details here, since they will be provided in the following chapter.

J.5 Comparison with the magnetized case

Let us point out that these expressions are *exactly* the same as the result obtained from the field-dependent expressions for $D_{\{\perp,\parallel\}}(v_\perp, v_\parallel;\, \Omega)$ derived in Chapter 9, by considering the vanishing field limit $\Omega \to 0$ and taking $v_\perp = 0$ (since $\mathbf{v} = v\,\hat{e}_3$, i.e. $v = v_\parallel$ in this frame) i.e.

$$A_{\{\perp,\parallel\}}(v) = \lim_{\Omega\to 0} D_{\{\perp,\parallel\}}(v_\perp = 0, v_\parallel = v;\, \Omega)$$

where $D_{\{\perp,\parallel\}}(v_\perp, v_\parallel;\, \Omega)$, $A_{\{\perp,\parallel\}}(v)$ are given by (8.21), (J.19) respectively.

In conclusion, in the vanishing-field limit of our general expressions, derived previously, we recover the well-known previous result for (*unmagnetized*) electrostatic plasma.

Abbreviations / Notation

Particle σ (or σ_α): test-particle (species: α);
Particle 1 (or $1_R^{\alpha'}$): one reservoir particle (species: α') chosen at random;
BLGE: Balescu-Lennard-Guernsey Equation;
df : distribution function;
em, EM : electromagnetic;
FP, FPE : Fokker - Planck, Fokker - Planck Equation;
FT : Fourier Transform;
GME : Generalized Master Equation;
LE: Landau Equation;
lhs : left-hand-side;
ME: Master Equation;
mf : mean-field;
MFPE : Markovian Fokker - Planck Equation;
MME : Markovian Master Equation;
MSD : mean-square-displacement;
ODE : ordinary differential equation;
PDE : partial differential equation;
pdf : probability distribution function;
QMFPE : Quasi - Markovian Fokker - Planck Equation;
QMME : Quasi - Markovian Master Equation;
R : Reservoir (heat-bath, thermostat);
rdf : reduced distribution function;
rhs : right-hand-side;
species α, α', β etc.: particle type (e.g. e (electron), i (ion) etc.;
each specific definition of species e.g. α implies
a value of particle mass m_α and charge e_α,
i.e. a specific value of $Z = |e_\alpha/e_e|$ and $A = m_\alpha/m_p$;
t.p. : test-particle Σ;
VE: Vlasov Equation;
w.c.a. : weak-coupling approximation.

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De ces baisers puissants comme un dictame,
De ces transports plus vifs que des rayons,
Que reste-t-il? C'est affreux, ô mon âme!
Rien qu'un dessin fort pâle, aux trois crayons…

Charles Baudelaire, *Les fleurs du mal*

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C.P. 231 - Physique Statistique et Plasmas

Kinetic Theory for a Test Particle Weakly-Coupled to a Heat Bath. Application au plasma magnétisé

Synopsis of the thesis

Relying on first principles of Non-Equilibrium Statistical Mechanics, we have undertaken the derivation of a kinetic equation for a test-particle weakly interacting with a large heat-bath in equilibrium. Both sub-systems are subject to an external force field.

We begin the thesis by presenting a general formalism which is appropriate for a kinetic description of the system. Perturbation theory based on the Liouville equation leads to a Generalized Master Equation. A Fokker-Planck-type equation follows in the usual *"Markovian"* approximation. Nevertheless, it is shown that this collision operator, denoted by "Θ", is ill-defined; namely, the positivity of the phase-space distribution function (*df*) $f(\mathbf{x}, \mathbf{v}; t)$ is not preserved. Relying on a method introduced in the theory of open quantum systems, we have obtained a *correct* kinetic operator (the "Φ"-operator), presenting the expected mathematical properties. This methodology is valid in any particular test-particle problem, provided that an explicit solution of the free (collisionless) motion problem is known.

The formalism has been applied in the particular case of an electrostatic plasma embedded in a magnetic field, assumed to be uniform and stationary. We have constructed the Φ− operator for this problem, obtained a Fokker-Planck-type kinetic equation and confirmed its mathematical properties. A set of exact expressions for the diffusion and drift coefficients have been obtained, taking into account the (long-range) inter-particle interaction law and the magnitude of the external field. Considering Debye-type interactions and a Maxwellian reservoir state, we have computed the coefficients and force correlation functions and studied their numerical behaviour in terms of the physical parameters involved. We have mainly focused on their dependence on the magnitude of the magnetic field, in various regimes.

Adopting certain assumptions for the coefficients (leading to an Ornstein-Uhlenbeck-type problem), the kinetic equation has been solved analytically; an exact expression has been obtained for the *df* as a function of time, describing thermal relaxation towards equilibrium. Particle diffusion in space, as a result of collisions, arises naturally from the modified Φ-collision operator. This result would be impossible to obtain via the old (Θ-) kinetic operator.

Finally, a set of evolution equations for velocity moments have been derived, pointing out the modification of the collisional terms therein, and comparing to the Landau collision term (for *unmagnetized* plasma) which is often used, for simplicity.

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Théorie cinétique pour une particule test faiblement couplée à un reservoir thermique. Application aux plasmas magnétisés

Résumé de la thèse

A partir des premiers principes de la Mécanique Statistique de Non-Equilibre, nous avons entrepris la dérivation d'une équation cinétique pour une particule témoin *(« particule - test »)* en interaction faible avec un grand reservoir thermique à l'équilibre. Les deux sous-systèmes sont soumis à un champ de forces extérieur.

La thèse commence par une présentation du formalisme général permettant une description cinétique du système. La théorie des perturbation sur base de l'équation de Liouville fournit une équation maîtresse généralisée *(« Generalized Master Equation »)*. Dès lors, une équation du type Fokker-Planck suit dans une approximation «quasi - Markovienne ». Néanmoins, on montre que cet opérateur cinétique ("Θ") ne possède pas de caractère mathématique bien défini; à savoir, la positivité de la fonction de distribution $f(\mathbf{x},\ \mathbf{v};\ t)$ d'état n'est pas assurée. En appuyant sur une méthode introduite dans la théorie des systèmes quantiques ouverts, nous avons obtenu un opérateur cinétique modifié (désormais appelé "Φ") caractérisé par les propriétés mathématiques correct. Cette méthodologie est valable pour tout problème de ce type, pour lequel une solution exacte du problème dynamique à l'ordre zero (sans collisions) existe.

Le formalisme est alors appliqué au cas particulier d'un plasma électrostatique soumis à un champ magnétique extérieur, supposé uniforme et stationnaire. On a donc obtenu une nouvelle équation cinétique du type Fokker-Planck, ainsi que des expressions analytiques exactes pour les coefficients de diffusion et de dérive. Celles-ci tiennent compte explicitement du champ magnétique et du potentiel d'interaction électrostatique (à longue portée). En considérant des interactions du type Debye et un reservoir Maxwellien, on a établi des expressions analytiques pour les fonctions de correlation et les coefficients de transport. On a donc étudié leur comportement numérique en fonction des paramètres physiques, en mettant l'accent sur l'effet du champ magnétique.

Le traitement analytique de l'équation cinétique dans un cas particulier (description du type Ornstein-Uhlenbeck) nous a ensuite permis d'obtenir une solution exacte pour la fonction de distribution *f* en fonction du temps, celle-ci décrit un processus de relaxation thermique vers l'état d'équilibre. Le comportement diffusif des particules, dû aux collisions, apparaît donc naturellement avec l'opérateur de collisions modifié ("Φ"). Ce résultat serait impossible à obtenir à partir de l'ancien opérateur Θ.

Enfin, on a établi des nouvelles équations d'évolution pour les valeurs moyennes (moments de vitesses), en mettant en évidence la modification du terme de collisions présent dans ces équations et comparant celui-ci au terme de Landau (sans champ) habituellement utilisé.