Nonperturbative Calculation of Born-Infeld Effects on the Schrödinger Spectrum of the Hydrogen Atom

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We present the first nonperturbative calculations of the nonrelativistic hydrogen spectrum as predicted by first-quantized nonlinear Maxwell-Born-Infeld electrodynamics with point charges. Judged against empirical data our results significantly restrict the range of viable values of the new electromagnetic constant β introduced by Born. We assess Born's own proposal for the value of β.

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In the 20 years since its rediscovery in the 26-dimensional bosonic string theory study by Fradkin and Tseytlin [1], the nonlinear electromagnetic field theory proposed by Born and Infeld [2] has been experiencing an astonishing renaissance. Recent surveys are located in [3,4]. Most investigations since [1] have been conducted from the perspective of the high energy community and involve higher-dimensional versions of the Born-Infeld theory [as in [1,5]] and/or noncommutative analogs of it [as in [6]]. Inevitably this has rekindled the interest in the original four-dimensional theory, the subject of this Letter.

We recall that Born's agenda [7] was to rid (early) QED from its ultraviolet divergencies by quantizing self-regularizing nonlinear classical field equations. It was noted already in [7] that the nonlinear Maxwell-Born-Infeld field equations [8] do not lead to the infinite self-energy problems of a point charge which occur with the linear Maxwell-Lorentz field equations, but the nonlinearity made it difficult to proceed. With the spectacular quantitative successes of renormalized QED since the late 1940s, Born's original motivation became obsolete, or so it would seem. However, as emphasized by Weinberg [9], more than half a century later standard QED is still in need of extrinsic mathematical regularizers to overcome the infinite self-energy problems of a point charge that have been inherited, in a sense, from the classical Maxwell-Lorentz electrodynamics. In view of this, Born's suggestion [7] to pursue some intrinsically self-regularizing nonlinear electromagnetic field theory reads as contemporary as it did in the 1930s; the rediscovery of Born-Infeld type Lagrangians in string theory, which could hardly have been foreseen by its founders, makes Born's suggestion all the more prophetic.

The avoidance of infinite self-energies, as well as some other conceptual items [10], are greatly to the theory's credit but surprisingly little is known about the empirical validity of the Born-Infeld theory. While the theory does not seem to have problems at the classical level [11,12] it remains to be seen whether it will live up to its expectations at the quantum level.

In this vein, a very natural question to ask is the following: what (detectable) effects does a hypothetical Born-Infeld nonlinearity of the electromagnetic fields have on the atomic spectra? This question should have been answered long ago. It was not, presumably because the nonlinearity of the field equations causes "difficulties [...] with the passage to the quantum theory, which appear to be insoluble with present methods of quantization" [13] (p. 32), and by 1969 "[t]he adaption [...] to the principles of quantum theory and the introduction of the spin ha[d] [...] met with no success" [14] (p. 375). As long as this situation prevails, one has to settle for quantum mechanical computations of spectral data in which Born-Infeld effects can be incorporated through the classical fields.

Unfortunately, because the complicated nonlinearity of the field equations has stood in the way of finding relevant solutions with two or more point charges, all previous attempts to compute such quantum mechanical spectra [15–17], have been foiled. In [15] the electron is treated as a test particle in the known [see [7]] Maxwell-Born-Infeld field of a point nucleus to compute hydrogenlike Schrödinger spectra to first order in perturbation theory; however, as we will see in this Letter, test particle theory is misleading for Born-Infeld equations. In [16,17], which have become standard references [see the introductions in [18,19]], Dirac spectra are computed without recourse to test particle theory (albeit with other approximations which are not of concern here), defining the interaction energy as difference of the electrostatic field energy integrals for the bound versus the free configurations. However, the authors of [16,17], who use Coulomb's solution $\boldsymbol{D}_{\mathrm{C}}$ of the displacement field equation $\nabla \cdot \boldsymbol{D} = 4\pi\rho$ with a charge density ρ comprising a single spectral electron and a spherically symmetric nucleus of charge z and a Thomas-Fermi cloud for the remaining $z - 1$ electrons, fail to realize that the nonlinear Born-Infeld law for the electromagnetic vacuum maps this Coulomb field $\boldsymbol{D}_{\mathrm{C}}$ into an electric field $\boldsymbol{E}_{\mathrm{FGRS}} = \mathcal{F}_{\mathrm{BI}}(\boldsymbol{D}_{\mathrm{C}})$ which is not identically curl free [20]; more precisely, $\nabla \times \mathcal{F}_{\mathrm{BI}}(\boldsymbol{D}_{\mathrm{C}}) \neq \boldsymbol{0}$ almost everywhere, invalidating the spectral results of [16,17].

Recently, a consistent first quantization of the nonlinear Maxwell-Born-Infeld field equations with point charges was achieved using the electromagnetic potentials [21]. Moreover, an explicit integral formula for the electron's

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electrostatic potential in certain proton-electron configurations (treated as point charges) was derived; this integral formula is readily extended to nuclear charges $z > 1$ (see below). Thus the stage has been set for a systematic investigation of the simplest atomic and ionic spectra, the hydrogenlike ones.

In order to keep technical matters as simple as possible, here we only address the nonrelativistic Schrödinger equation of a spinless electron bound to an infinitely massive point nucleus. We plan to deal with the fine details contributed by relativity, spin, and the finite mass and size of the nucleus elsewhere. Furthermore, detailed evaluations of the interactions and the eigenvalues are carried out only for the hydrogen atom ($z = 1$); the details of hydrogenlike interactions and ionic spectra for nuclear charges $z > 1$ are beyond the scope of this Letter.

In units of $\hbar$ for both action and magnitude of angular momentum, elementary charge e for charge, electron rest mass m_e for mass, speed of light c for velocity, and Compton wavelength of the electron $\lambda_C = \hbar/m_e c$ for both length and time, a hydrogenlike spectrum is determined by the following dimensionless stationary Schrödinger equation on the electron's configuration space [21],

$$-\tfrac{1}{2}\nabla_e^2\psi(\boldsymbol{s}_e) - \alpha\phi_\beta(\boldsymbol{s}_e)\psi(\boldsymbol{s}_e) = E\psi(\boldsymbol{s}_e), \tag{1}$$

where $\boldsymbol{s}_e$ is the electron's generic configuration space coordinate and the subscript $_e$ on ∇_e^2 indicates differentiation with respect to $\boldsymbol{s}_e$. The fine structure constant $\alpha \equiv e^2/\hbar c \approx 1/137.036$ is the dimensionless electromagnetic coupling constant for the dimensionless total electrostatic potential ϕ_β defined below. The positive parameter β is Born's electromagnetic vacuum constant ("aether constant" for short) [22], which enters through the Born-Infeld aether law, relating $\boldsymbol{E}$ (and $\boldsymbol{H}$) with $\boldsymbol{D}$ (and $\boldsymbol{B}$). Born [7] argued that $\beta = \beta_B$ with

$$\beta_B \approx 1.2361\alpha. \tag{2}$$

Our spectral results allow us to assess the viability of (2) and Born's reasoning for it.

The total electrostatic potential $\phi_\beta(\boldsymbol{s})$ at the actual space point $\boldsymbol{s}$ is determined by the electrostatic Maxwell-Born-Infeld equation $\nabla \cdot \mathcal{F}_{BI}^{-1}(-\nabla\phi_\beta) = 4\pi\rho$ with ρ consisting of one positive and one negative point charge with values z and -1 at generic positions $\boldsymbol{s}_n$ and $\boldsymbol{s}_e$, respectively; explicitly,

$$-\nabla \cdot \frac{\nabla\phi_\beta(\boldsymbol{s})}{\sqrt{1 - \beta^4|\nabla\phi_\beta(\boldsymbol{s})|^2}} = 4\pi(z\delta_{\boldsymbol{s}_n}(\boldsymbol{s}) - \delta_{\boldsymbol{s}_e}(\boldsymbol{s})), \tag{3}$$

with the asymptotic condition that $\phi_\beta(\boldsymbol{s}) \to 0$ for $|\boldsymbol{s}| \to \infty$. The solution of (3) depends on $\boldsymbol{s}$ as variable and on $\boldsymbol{s}_n$ and $\boldsymbol{s}_e$ as parameters; we sometimes emphasize this by writing $\phi_\beta(\boldsymbol{s}|\boldsymbol{s}_n, \boldsymbol{s}_e)$. While no explicit formula for $\phi_\beta(\boldsymbol{s}|\boldsymbol{s}_n, \boldsymbol{s}_e)$ is known, (1) reveals that we need only $\phi_\beta(\boldsymbol{s}_e|\boldsymbol{s}_n, \boldsymbol{s}_e)$. Fortunately, although $\nabla \times \mathcal{F}_{BI}(\boldsymbol{D}_C(\boldsymbol{s})) \neq \boldsymbol{0}$ for almost every $\boldsymbol{s}$ in space, we do have $\nabla \times \mathcal{F}_{BI}(\boldsymbol{D}_C(\boldsymbol{s})) = \boldsymbol{0}$ for all $\boldsymbol{s}$ on the straight line through the point charges [this result generalizes to the vanishing of $\nabla \times \mathcal{F}_{BI}(\boldsymbol{D}_C(\boldsymbol{s}))$ on the straight line through the respective centers of any two spherically symmetric charge distributions]. Hence, an electrostatic potential function $\phi(\boldsymbol{s})$ solving (3) for space points $\boldsymbol{s}$ on that line can be computed through the line integral $\phi(\boldsymbol{s}) = \int_{\boldsymbol{s}}^\infty \mathcal{F}_{BI}(\boldsymbol{D}_C(\boldsymbol{s}')) \cdot d\boldsymbol{s}'$. Assuming $\boldsymbol{D} = \boldsymbol{D}_C$ in leading order in β, on this line we can approximately set $\phi(\boldsymbol{s}) = \phi_\beta(\boldsymbol{s})$. For $z > 1$ the integral is formidable, but when the nucleus is a proton ($z = 1$ and $\boldsymbol{s}_n = \boldsymbol{s}_p$), the integral can be recast into the more manageable form [21]

$$\phi_\beta(\boldsymbol{s}_e|\boldsymbol{s}_p, \boldsymbol{s}_e) = \frac{1}{\beta}\int_{2\sqrt{2}\beta/r}^\infty \frac{f'(y)}{\sqrt{1 + x^4}}dx, \tag{4}$$

where $r = |\boldsymbol{s}_p - \boldsymbol{s}_e|$, $xy = \beta/r$, and f' is the derivative of

$$f(y) = \sqrt{\tfrac{1}{4} + y^2 - y\sqrt{1 + y^2}}. \tag{5}$$

For the remainder of this Letter, $z = 1$.

A look at the integral (4) makes it plain that $\beta\phi_\beta(\boldsymbol{s}_e|\boldsymbol{s}_p, \boldsymbol{s}_e)$ depends on $\boldsymbol{s}_p$, $\boldsymbol{s}_e$, and β only through the combination $|\boldsymbol{s}_p - \boldsymbol{s}_e|/\beta$; hence, $\beta\phi_\beta(\boldsymbol{s}_e|\boldsymbol{s}_p, \boldsymbol{s}_e) =: W(r/\beta)$ is a function of r/β. And while W does not seem to be expressible in terms of known functions, (4) lends itself readily to an analysis when the electron is far from, respectively, near the proton. Note that "far" and "near" are relative to β.

If the electron is far from the nucleus, i.e., if $r \geq 2\sqrt{2}\beta$, then $W(r/\beta)$ can be expanded in an asymptotic series in powers of β/r so that asymptotically exact to four orders as $r \to \infty$

$$W(r/\beta) = \sum_{k=0}^{3} b_k(\beta/r)^k + o((\beta/r)^3), \tag{6}$$

with $b_0 = -\frac{1}{4}B(\frac{1}{4}, \frac{1}{4})$, $b_1 = 1$, $b_2 = \frac{3}{4}B(\frac{3}{4}, \frac{3}{4})$, and $b_3 = 2$, where $B(.,.)$ is Euler's Beta function. Equation (6) reveals three important results. First, when $r \to \infty$ the electric potential at the location of the electron, $\phi_\beta(\boldsymbol{s}_e)$, converges to the finite electron self-potential in Born-Infeld theory, defined by setting $\boldsymbol{s} = \boldsymbol{s}_e$ in Born's solution

$$\phi_B^{(-)}(\boldsymbol{s}|\boldsymbol{s}_e) = -\frac{1}{\beta}\int_{|\boldsymbol{s}-\boldsymbol{s}_e|/\beta}^\infty \frac{dx}{\sqrt{1 + x^4}} \tag{7}$$

for the electrostatic potential at $\boldsymbol{s}$ generated by a single (negative) unit point charge at $\boldsymbol{s}_e$ [7] [NB: (7) solves (3) when $z = 0$]. We recall [7]: there is no short distance Coulomb singularity of the single particle potential in Born-Infeld theory. Second, to leading order for large separation of electron and proton, the potential $\phi_\beta(\boldsymbol{s}_e)$ varies with r reciprocally; i.e., we recover Coulomb's law for the pair potential from the nonlinear field equation (3). Third, there are higher order corrections to Coulomb's law. Indeed, when the electron is near the nucleus, deviations from Coulomb's law become significant. More pre-

cisely, for $r < 2\sqrt{2}\beta$, the function $W(r/\beta)$ can be expanded into a Taylor series in powers of r/β,

$$W(r/\beta) = \sum_{k=0}^{\infty} a_k (r/\beta)^{4k+1} \tag{8}$$

with explicitly computable expansion coefficients a_k. The first four of them read as follows: $a_0 = -1/2$, $a_1 = 3/40 - 3\pi/138$, $a_2 = -29/672 + 225\pi/16384$, and $a_3 = 1667/54912 - 20265\pi/2\,097\,152$. Note, in particular, that $W(0) = 0$: there is no short distance Coulomb singularity of the pair potential in Born-Infeld theory [21].

Our discussion of ϕ_β supplies all the information we need to solve the Schrödinger equation (1). To facilitate the comparison with the familiar Schrödinger equation for the Coulomb interaction, we write the eigenvalues as $E = \frac{\alpha}{\beta} \times \frac{1}{4}B(\frac{1}{4}, \frac{1}{4}) + \varepsilon$ and the total potential as $\phi_\beta(\boldsymbol{s}_e) = -\frac{1}{\beta} \times \frac{1}{4}\boldsymbol{B}(\frac{1}{4}, \frac{1}{4}) + \frac{Z(r/\beta)}{r}$, where $Z(r/\beta)$ is the effective Coulomb charge of the proton "seen" from a distance r. The self-potential terms on left- and right-hand side of (1) then cancel out, leaving us to solve

$$-\frac{1}{2}\nabla_e^2 \psi(\boldsymbol{s}_e) - \alpha \frac{Z(r/\beta)}{r} \psi(\boldsymbol{s}_e) = \varepsilon \psi(\boldsymbol{s}_e). \tag{9}$$

The Born-Infeld Schrödinger and the Coulomb Schrödinger potentials are compared in Fig. 1.

Our first important spectral result states that the Coulomb limit $\beta \to 0$ of Eq. (9) exists. In this case, $Z(r/\beta) \to 1$ for all $r > 0$, which follows from (4) [21]. Hence, in the limit $\beta \to 0$ the spectrum of (9) converges to the familiar Rydberg law, i.e.,

$$\varepsilon_{n,\ell,m}^{(0)} = -\frac{1}{2n^2}\alpha^2, \qquad n = 1, 2, \ldots, \tag{10}$$

where $n = 1, 2, 3, \ldots$ and $\ell = 0, 1, \ldots, n-1$ and $m = -\ell, \ldots, 0, \ldots, \ell$ are the usual main, secondary, and magnetic quantum numbers. As is well known, ℓ and m do not contribute to the energy eigenvalues $\varepsilon_{n,\ell,m}^{(0)}$, so that n^2 of them coincide; we recall that this high degeneracy is due to the $O(4)$ invariance of (9) when $\beta = 0$.

For all $0 < \beta < \infty$, the $O(4)$ invariance is broken and the energy eigenvalues $\varepsilon_{n,\ell,m}^{(\beta)}$ in general display only the $2\ell + 1$-fold degeneracy corresponding to the manifest $O(3)$ invariance of (9); i.e., $\varepsilon_{n,\ell,m}^{(\beta)}$ does not depend on the quantum number m. The $O(3)$ symmetry allows us to treat (9) by the usual separation of variables. Shifting the origin of space to $\boldsymbol{s}_p$, the electron-proton distance r becomes the radial variable of standard spherical coordinates r, ϑ, φ. In these coordinates the eigen-wave functions take the form $\psi_{n,\ell,m}^{(\beta)}(\boldsymbol{s}_e) = R_{n,\ell}^{(\beta)}(r) Y_\ell^m(\vartheta, \varphi)$, where the $Y_\ell^m(\vartheta, \varphi)$ are spherical harmonics, and the $R_{n,\ell}^{(\beta)}(r)$ satisfy the Sturm-Liouville problem

$$(r^2 R')' - [\ell(\ell+1) - 2\alpha r Z(r/\beta) - 2\varepsilon r^2]R = 0 \tag{11}$$

for $\int_0^\infty r^2 R^2(r) dr < \infty$. We solved this radial problem by standard shooting technique, using MAPLE's Runge-Kutta-Fehlberg45 method.

It is instructive to discuss first the dependence of the ground state energy $\varepsilon_0(\beta) \equiv \varepsilon_{1,0,0}^{(\beta)}$ on β. In Fig. 2 we display our numerically computed values of $\varepsilon_0(\beta)$ for a selection of β values versus β, together with semiexplicit upper and lower bounds on $\varepsilon_0(\beta)$, computed analytically except for numerical quadratures.

There are several remarkable features visible in Fig. 2. First of all, there are two values of β at which the ground state energy $\varepsilon_0(\beta)$ coincides with the familiar Coulomb value $-\alpha^2/2$. Thus there are two regimes where the Born-Infeld theory yields a binding (or ionization) energy $[-\varepsilon_0(\beta)]$ compatible with the empirical data: a perturbative one near $\beta = 0$ (the obvious one, as already noted) and a highly nonobvious and nonperturbative regime near $\beta \approx 1.83297/\alpha$. Too far away from these two values, the binding energy would be either unrealistically large or small. Between these two β values, the binding energy is enhanced compared to $\alpha^2/2$, reaching a maximum at about $\beta \approx 0.24774/\alpha$, while to the right of $\beta \approx 1.83297/\alpha$, the binding energy is diminished, converging to zero as $\beta \to \infty$. We remark that our semiexplicit upper and lower bounds allow us to rigorously prove this nonmonotonic behavior of the binding energy.

We emphasize that the nonmonotonic behavior of $\varepsilon_0(\beta)$ is a quite nontrivial result; in particular, there is no hint of it

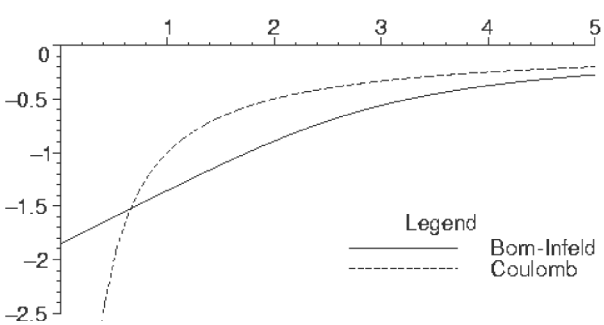

FIG. 1. $-(\beta/r)Z(r/\beta)$ and $-\beta/r$ vs r/β.

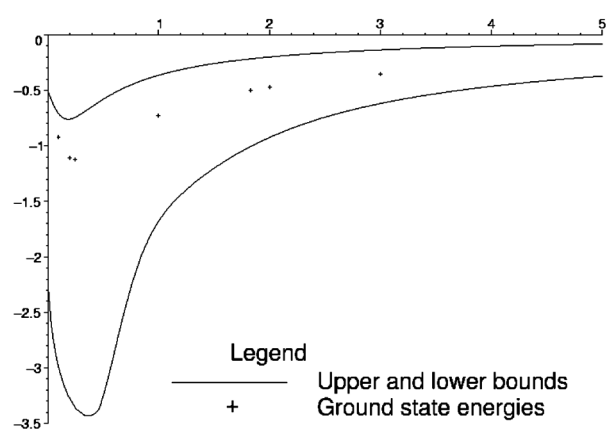

FIG. 2. ε_0/α^2 vs $\alpha\beta$.

when the electron is treated (in "first approximation") as a test particle which "feels" only the proton's electrostatic potential computed from the Born-Infeld equations with a single point source, neglecting the electron's own feedback [15]; i.e., $\phi_\beta(\boldsymbol{s}_e) \equiv \phi_\beta(\boldsymbol{s}_e|\boldsymbol{s}_p, \boldsymbol{s}_e)$ in (9) is replaced by Born's [7] solution for a positive point charge, $\phi_{\mathrm{B}}^{(+)}(\boldsymbol{s}_e|\boldsymbol{s}_p) = -\phi_{\mathrm{B}}^{(-)}(\boldsymbol{s}_p|\boldsymbol{s}_e)$. For r large, $\phi_{\mathrm{B}}^{(+)}(\boldsymbol{s}_e|\boldsymbol{s}_p) \sim 1/r$, too. But since $\sqrt{1+x^4} > x^2$, it follows from (7) that $\phi_{\mathrm{B}}^{(+)}(\boldsymbol{s}_e|\boldsymbol{s}_p) < 1/r$, so test particle theory predicts a diminished binding energy for all β.

For a judicious selection of β values we computed several higher eigenvalues. Of particular interest are Born's value (2) and the value $\beta \approx 1.83297/\alpha$ where the binding energy coincides with the Coulomb value $\alpha^2/2$ at $\beta = 0$. From Table I and Fig. 2 we are able to delineate the physically viable range of β values. By inspecting the excited energies, we can immediately rule out $\beta \approx 1.83297/\alpha$, and by continuity also its neighborhood. That leaves only the perturbative regime of sufficiently small β. But how small is "sufficiently small"? In particular, is β_{B} "small enough"?

The second row in Table I lists spectral data for $\beta = \beta_{\mathrm{B}}$. Note that $-\varepsilon_0(\beta_{\mathrm{B}})/\alpha^2$ deviates from $-\varepsilon_0(0)/\alpha^2 = 1/2$ (first row) by 1.6×10^{-4}, and from the empirical data even by 4.3×10^{-4}. Of course, $-\varepsilon_0(0)/\alpha^2$ differs itself from the empirical data by 2.7×10^{-4}, but as is well known, after correcting $-\varepsilon_0(0)/\alpha^2$ for the finite mass of the proton, the difference to the empirical data reduces to only -3.14×10^{-6}, and the agreement improves even more with relativistic corrections. It is therefore to be expected that even after correcting $-\varepsilon_0(\beta_{\mathrm{B}})/\alpha^2$ for the finite proton mass and relativistic effects, the difference to empirical data will remain at about 10^{-4}.

Even more dramatic is the splitting of $0.066\alpha^2$ between the $2s$ and $2p$ energies computed with (2), which is a factor 10^4 bigger than the $2p_{3/2} - 2p_{1/2}$ "fine structure" splitting (which is not even visible at the level of precision in our table). Hence, even from the spectrum of the simplest atom we conclude that Born's value (2) is not physically viable.

Pending verification of our results through a more refined treatment, viable values of β, as far as spectral results go, must be much smaller than α. We plan to study just how small β must be using a relativistic theory with spin. But would the elimination of (2) not be a bearer of bad tidings for the Born-Infeld theory? Not yet. Born did not use detectable energy differences to compute (2) but equated the static field energy of a point charge to the electron's empirical rest energy, which yields

$$\frac{1}{4\pi}\frac{\alpha}{\beta^4}\int\left(\sqrt{1-\beta^4|\nabla\phi_{\mathrm{B}}|^2}^{\,-1} - 1\right)d^3\boldsymbol{s} = 1 \qquad (12)$$

in our units of $m_{\mathrm{e}}c^2 = 1$. The integral equals $\frac{\alpha}{\beta}\frac{1}{6}\mathrm{B}(\frac{1}{4},\frac{1}{4})$, giving (2). Unless (12) can be tied to a dynamical concept, such as scattering of an electron, the elimination of (2) by our spectral results is not bad news for the Born-Infeld theory. But this clearly calls for a deeper inquiry.

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TABLE 1. Listed in the table below are the energies of the ground, the first excited s, and the first p states for these β values, as well as the corresponding empirical data [23]. We display $\alpha\beta$ rather than β, and $-\varepsilon/\alpha^2$ rather than ε; also, we suppress the magnetic quantum number m.

$\alpha\beta$	$-\varepsilon_{1,0}^{(\beta)}/\alpha^2$	$-\varepsilon_{2,0}^{(\beta)}/\alpha^2$	$-\varepsilon_{2,1}^{(\beta)}/\alpha^2$
0.0000	0.50000	0.12500	0.12500
6.6×10^{-5}	0.50016	0.12502	0.19101
1.83297	0.50000	0.19766	0.36737
Empirical	0.49973	0.12493	0.12493

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