Electron-acoustic solitary waves via vortex electron distribution

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[1] We consider the nonlinear propagation of electron-acoustic waves in a plasma composed of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions. The properties of small but finite amplitude electron-acoustic solitary waves (EASWs) are studied by employing the reductive perturbation technique, whereas those of arbitrary amplitude EASWs are investigated by means of the pseudopotential approach. It is found that the present plasma model supports EASWs having a positive potential, which corresponds to a hole (hump) in the cold (hot) electron number density. It is also shown that as the trapped electron temperature increases, the amplitude of the EASWs increases but their width decreases. The application of our results in interpreting the salient features of the broadband electrostatic noise in the dayside auroral zone is discussed. INDEX TERMS: 0654 Electromagnetics: Plasmas; 7815 Space Plasma Physics: Electrostatic structures; 7839 Space Plasma Physics: Nonlinear phenomena; 7899 Space Plasma Physics: General or miscellaneous; KEYWORDS: Electron acoustic waves, solitary waves, vortex-like electron distribution, two-electron-temperature plasma, broadband electrostatic noise

1. Introduction

[2] The idea of electron-acoustic (EA) mode had been conceived by Fried and Gould [1961] during numerical solutions of the linear electrostatic Vlasov dispersion equation in an unmagnetized, homogenous plasma. Besides the well-known Langmuir and ion-acoustic waves, they noticed the existence of a heavily damped acoustic-like solution of the dispersion equation. It was later shown that in the presence of two distinct groups (cold and hot) of electrons and immobile ions, one indeed obtains a weakly damped EA mode [Watanabe and Taniti, 1977], the properties of which significantly differ from those of the Langmuir waves. Gary and Tokar [1985] performed a parameter survey and found conditions for the existence of the EA waves. The most important condition is \( T_c \ll T_h \), where \( T_c \) (\( T_h \)) is the temperature of cold (hot) electrons. The propagation characteristics of the EA waves have also been studied by Yu and Shukla [1983], Mace and Hellberg [1990, 1993, 2001] and Mace et al. [1991].

[3] Two electron temperature plasmas are known to occur both in laboratory experiments [Derfler and Simonen, 1969; Henry and Treguier, 1972] and in space environments [Dubouloz et al., 1991, 1993; Pottelette et al., 1999; Berthomier et al., 2000; Singh and Lakhina, 2001]. The propagation of the EA waves has received a great deal of renewed interest not only because the two electron temperature plasma is very common in laboratory experiments and in space but also because of the potential importance of the EA waves in interpreting electrostatic component of the broadband electrostatic noise (BEN) observed in the cusp of the terrestrial magnetosphere [Tokar and Gary, 1984; Singh and Lakhina, 2001], in the geomagnetic tail [Schriver and Ashour-Abdalla, 1989], in auroral region [Dubouloz et al., 1991, 1993; Pottelette et al., 1999], etc.

[4] The EA mode has been used to explain various wave emissions in different regions of the Earth’s magnetosphere [Dubouloz et al., 1991, 1993]. It was first applied to interpret the hiss emissions observed in the polar cusp region in association with low-energy (~100 eV) upward moving electron beams [Lin et al., 1984]. The EA mode was also utilized to interpret the generation of the broadband electrostatic noise (BEN) emissions detected in the plasma sheath [Schriver and Ashour-Abdalla, 1989] as well as in the dayside auroral zone [Dubouloz et al., 1991, 1993]. Dubouloz et al. [1991] rigorously studied the BEN observed in the dayside auroral zone and showed that because of the very high electric field amplitudes (100 mV/m) involved, the nonlinear effects must play a significant role in the generation of the BEN in the dayside auroral zone. Dubouloz et al. [1991, 1993] also explained the short-duration (<1 s) burst of the BEN in terms of EA solitary waves: such EA solitary waves passing the satellite would generate electric field spectra. To study the properties of EA solitary structures, Dubouloz et al. [1991] considered a one-dimensional, unmagnetized collisionless plasma consisting of cold electrons, Maxwellian hot electrons, and stationary ions. However, in practice, the hot electrons may not follow a Maxwellian distribution due to the formation of phase space holes caused by the trapping of hot electrons in a wave potential. Accordingly, in most space plasmas the hot electrons follow the trapped/vortex-like distribution [Schamel, 1972, 1973, 2000; Abbasi et al., 1999]. The electron trapping is common not only in space plasmas, but also in laboratory experiments, namely, a

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The manuscript is organized as follows. The basic equations governing our plasma model are given in section 2. The small and arbitrary amplitude EASWs are studied in section 3. A brief discussion as well an application are presented in section 3.3.

2. Governing Equations

[6] We consider a one-dimensional, collisionless plasma consisting of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions. The dynamics of the nonlinear EA waves (with phase speed much larger than the hot electron thermal speed) is governed by

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z}(n_e u_e) = 0, \tag{1}
\]

\[
\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial z} = \alpha \frac{\partial \phi}{\partial z}, \tag{2}
\]

\[
\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha} \frac{n_e}{n_h} - (1 + \frac{1}{\alpha}), \tag{3}
\]

where \( n_e (n_h) \) is the cold (hot) electron number density normalized by its equilibrium value \( n_{e0} (n_{h0}) \), \( u_e \) is the cold electron fluid speed normalized by \( C_c = (k_B T_e/\alpha m)^{1/2} \), and \( \phi \) is the electrostatic wave potential normalized by \( k_B T_e/\epsilon \). Furthermore, \( \alpha = n_{e0}/n_{h0} \), \( m \) is the electron mass, \( e \) is the magnitude of the electron charge, and \( k_B \) is the Boltzmann constant. The time and space variables are in units of the cold electron plasma period \( \omega_{pe}^{-1} = (m/4\pi n_{e0} e^2)^{1/2} \) and the hot electron Debye length \( \lambda_{De} = (k_B T_e/4\pi n_{h0} e^2)^{1/2} \), respectively.

[7] To model the hot electron distribution in presence of trapped particles, we employ a vortex-like electron distribution of Schamel [1972] which solves the electron Vlasov equation. Thus we have \( f_{hi} = f_{hi}^0 + f_{hi}^1 \) where

\[
f_{hi}^0 = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{z^2}{2\phi} \right) \right] \quad |v| > \sqrt{2\phi}, \tag{4}
\]

\[
f_{hi}^1 = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{z^2}{2\phi} \right) \right] \quad |v| \leq \sqrt{2\phi}. \tag{5}
\]

Here \( v \) is the hot electron speed normalized by the hot electron thermal speed \( v_{Th} = (k_B T_e/m)^{1/2} \) and \( |\beta| = (T_h/T_{Th}) \), which is the ratio of the free hot electron temperature \( T_{Th} \) to the hot trapped electron temperature \( T_{Th} \), is a parameter determining the number of trapped electrons. It has been assumed that the speed of the nonlinear EA waves is much smaller than the hot electron thermal speed. We note that the electron distribution function, as prescribed above, is continuous in velocity space and satisfies the regularity requirements for an admissible Bernstein-Greene-Kruskal (BGK) solution [Schamel, 1972].

[8] It is obvious from equations (4) and (5) that \( \beta = 1 (\beta = 0) \) represents a Maxwellian (flat-topped) distribution, whereas \( \beta < 0 \) represents a vortex-like excited trapped electron distribution corresponding to an underpopulation of trapped electrons. The situation \( \beta > 0 \) is of our present interest. Thus, integrating the electron distributions over the velocity space, the hot electron number density \( n_h \) for \( |\beta| < 1 \) can be expressed as [Schamel, 1972]

\[
n_h = l(\phi) + \frac{2}{\sqrt{\pi} |\beta|} W_p \left( \sqrt{-|\beta|} \right), \tag{6a}
\]

where \( I(x) = [1 - \text{erf}(\sqrt{x})] \exp(x) \) and \( W_p(x) = \exp(-x^2) \exp(x^2) \mathrm{d}y \). For \( \phi < 1 \), equation (6a) gives

\[
n_h = 1 + \phi - \frac{4(1 - \beta)}{3\sqrt{\pi}} \phi^{3/2} + \frac{1}{2} \phi^2 + \ldots. \tag{6b}
\]

3. Solitary Waves

[9] To study small but finite amplitude EASWs, we use the reductive perturbation method [Waschimi and Taniuti, 1966; Schamel, 1973] and to study arbitrary amplitude EASWs [Mace et al., 1991], we use the pseudopotential approach [Bernstein et al., 1957]. The properties of such small and arbitrary amplitude EASWs may be described as follows.

3.1. Small-Amplitude Solitary Waves

[10] To study small but finite amplitude EASWs, we follow the reductive perturbation technique of Schamel [1973] and construct a weakly nonlinear theory for the EASWs by introducing the stretched coordinates \( \xi = \sqrt{\epsilon} (z - v_0 t) \) and \( \tau = \epsilon^{3/4} t \), where \( \epsilon \) is a smallness parameter measuring the weakness of the nonlinearity and \( v_0 \) is the wave speed normalized by \( C_c \). We also expand \( n_e, u_e, \phi \) in power series of \( \epsilon \):

\[
n_e = 1 + \epsilon n_e^{(1)} + \epsilon^{3/2} n_e^{(2)} + \ldots, \tag{7}
\]

\[
u_e = \epsilon u_e^{(1)} + \epsilon^{3/2} u_e^{(2)} + \ldots, \tag{8}
\]

\[
\phi = \epsilon \phi^{(1)} + \epsilon^{3/2} \phi^{(2)} + \ldots. \tag{9}
\]

and develop equations in various powers of \( \epsilon \). To lowest order in \( \epsilon \), equations (1)–(3) and (6b) give \( n_e^{(1)} = u_e^{(1)} v_0 \), \( u_e^{(1)} = -\phi^{(1)} / v_0 \), and \( v_0 = 1 \). To next higher order in \( \epsilon \), we obtain a set of equations:

\[
\frac{\partial n_e^{(1)}}{\partial t} - v_0 \frac{\partial n_e^{(2)}}{\partial \xi} + \frac{\partial u_e^{(2)}}{\partial \xi} = 0, \tag{10}
\]

\[
\frac{\partial u_e^{(1)}}{\partial t} - v_0 \frac{\partial u_e^{(2)}}{\partial \xi} - \frac{\partial \phi^{(2)}}{\partial \xi} = 0, \tag{11}
\]

\[
\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - \frac{1}{\alpha} \frac{n_e^{(2)} - \phi^{(2)}}{\phi^{(1)}} + \frac{4(1 - \beta)}{3\sqrt{\pi}} \left[ \phi^{(1)} \right]^{3/2} = 0. \tag{12}
\]
Figure 1. (top) Variation of the Mach number $M$ with the amplitude $\phi_m$ for $\alpha = 5$, $\beta = -0.5$ (solid curve), $\beta = -0.6$ (dotted curve), and $\beta = -0.7$ (dashed curve). (bottom) Variation of $\Delta$ with $\phi_m$ for $\alpha = 5$, $\beta = -0.5$ (solid curve), $\beta = -0.6$ (dotted curve), and $\beta = -0.7$ (dashed curve).

Combining equations (10)–(12), we deduce a modified KdV equation

$$\frac{\partial \phi^{(1)}}{\partial \tau} + \left(1 - \beta \right) \sqrt{\phi^{(1)}} \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = 0. \quad (13)$$

The stationary solitary wave solution of the modified KdV equation (13) can be obtained by transforming the space variable $\zeta$ to $\eta = \zeta - u_0 \tau$ and by imposing the appropriate boundary conditions, namely, $\phi^{(1)} \rightarrow 0$, $d\phi^{(1)}/d\eta \rightarrow 0$, $d^2\phi^{(1)}/d\eta^2 \rightarrow 0$ at $\eta \rightarrow \pm \infty$. Thus the steady state solution of equation (13) can be expressed as [Schamel, 1973]

$$\phi^{(1)} = \phi^{(1)}_{\text{sech}^4[(\zeta - u_0 \tau)/\delta]}, \quad \text{(14)}$$

where the amplitude $\phi^{(1)}_{\text{sech}^4}$ and the width $\delta$ are given by $\phi^{(1)}_{\text{sech}^4} = [15u_0\sqrt{\pi/8(1 - \beta)}]^2 > 0$ and $\delta = \sqrt{8/\alpha_0}$, respectively. Equation (14) reveals the existence of small but finite amplitude EASWs with a positive potential. This corresponds to the rarefaction of the cold electron density, i.e., the cold electron density holes since $n_c^{(1)} = -\alpha \phi^{(1)}$. It also reveals that as $u_0$ increases, the amplitude increases but the width decreases and that the amplitude of these density holes decreases with $|\beta|$ but increases with $\alpha$.

### 3.2. Arbitrary Amplitude Solitary Waves

To study stationary arbitrary amplitude EASWs, we suppose that all the dependent variables depend only on a single variable $\xi = \zeta - Mt$ (where $\xi$ is normalized by $\lambda_{\text{DB}}$ and $M$ is the Mach number = solitary wave speed/$C_0$), use the steady state condition, impose the appropriate boundary conditions for localized perturbations (namely, $n_c \rightarrow 1$, $u_c \rightarrow 0$, $\varphi \rightarrow 0$, and $d\varphi/dz \rightarrow 0$ at $\xi \rightarrow \pm \infty$), and reduce equations (1)–(3) and (6a) to the form of an energy integral [Bernstein et al., 1957]

$$\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + V(\phi) = 0, \quad \text{(15)}$$

where the pseudopotential $V(\phi)$ for our purposes reads

$$V(\phi) = 1 + \left(1 + \frac{M^2}{\alpha \phi} \right) \left(1 - \sqrt{1 + \frac{2\alpha \phi}{M^2}} - I(\phi) \right)$$

$$- \frac{2}{\sqrt{\pi}} \left(1 - \frac{1}{\beta}\right) \sqrt{\phi} - \frac{2}{\beta \sqrt{\pi |\beta|}} W\left(\sqrt{-\beta \phi}\right). \quad \text{(16)}$$

It is obvious from equation (16) that $V(\phi) = dV(\phi)/d\phi = 0$ at $\varphi = 0$. Therefore solitary wave solutions of equation (15) exist if (1) $V(\phi) < 0$ for $0 < \phi < \phi_m$ and (2) $V(\phi) = 0$ for $\phi = \phi_m$ where $\phi_m$ is the amplitude of the solitary waves. The first condition can be satisfied only if $(d^2V/d\phi^2)|_{\phi=0} > 0$ so that the fixed point at the origin is unstable. The first condition can be obtained by expanding $V(\phi)$ around $\phi = 0$ and by setting the quadratic term (coefficient of $\phi^2$) equal to zero. Thus the first condition, which provides the critical Mach number $M_c$ for which solitary wave solution of equation (15) exists, becomes $M > M_c = 1$. The second condition $V(\phi_m) = 0$, which provides a relation between the amplitude $\phi_m$ and the solitary wave speed $M$ (known as the nonlinear dispersion relation), can be expressed as

$$1 + \left(1 + \frac{M^2}{\alpha \phi_m} \right) \left(1 - \sqrt{1 + \frac{2\alpha \phi_m}{M^2}} - I(\phi_m) \right)$$

$$- \frac{2}{\sqrt{\pi}} \left(1 - \frac{1}{\beta}\right) \sqrt{\phi_m} - \frac{2}{\beta \sqrt{\pi |\beta|}} W\left(\sqrt{-\beta \phi_m}\right) = 0. \quad \text{(17)}$$

It is clear that for an arbitrary value of the amplitude $\phi_m$ a simple analytical relation between $\phi_m$, $M$, and $\beta$ is not possible to obtain. We have numerically solved equation (17) and have shown how the solitary wave speed (Mach number) $M$ changes with the amplitude $\phi_m$ and the parameter $\beta$. The results are displayed in the upper part of Figure 1. It is evident that the Mach number $M$ increases with the amplitude $\phi_m$ but decreases with increasing the value of the parameter $\beta$.

[12] The width $\Delta$ of the EASWs is related with the depth of the classical potential $|V_{\text{min}}|$ by [Bujarbarua and Schamel, 1981]

$$\Delta = \frac{\phi_m}{\sqrt{|V_{\text{min}}|}}, \quad \text{(18)}$$

which is obtained by renormalizing equation (15). We have numerically obtained $\Delta$ as a function of $\phi_m$ by finding the corresponding $M$ from equation (17), $\phi$ (corresponding to $\partial V(\phi)/\partial \phi = 0$), and $|V_{\text{min}}|$. The results are displayed in Figure 1(bottom), which exhibits that the width $\Delta$ increases
with the amplitude $\phi_m$ as well as with the value of the parameter $\beta$.

We have also numerically analyzed the pseudopotential $V(\phi)$. The results are depicted in Figure 2. It shows that the amplitude $\phi_m$ increases with increasing the Mach number $M$ but decreases with increasing the value of the parameter $\beta$.

4. Discussion and Application

We have considered a plasma model consisting of a cold electron fluid, hot electrons obeying a trapped/vortex-like distribution, and stationary ions and have theoretically investigated the properties of small as well as arbitrary amplitude EASWs. We have found that the present plasma model supports the existence of small as well as arbitrary amplitude EASWs with a positive potential. This corresponds to a hole (hump) in the cold (hot) electron number density. It is found that as the Mach number $M$ increases, both the amplitude $\phi_m$ and the width $\Delta$ of the EASWs increase. We also observe that as the value of the trapping parameter $\beta$ is increased, the amplitude $\phi_m$ decreases, but the width $\Delta$ increases.

To have some numerical appreciations of our results, we choose the values of the parameters corresponding to the dayside auroral zone [Dubouloz et al., 1993]: $T_e \approx 5$ eV, $T_h \approx 250$ eV, $n_{oh} \approx 0.5$ cm$^{-3}$, and $n_{60} \approx 2.5$ cm$^{-3}$. These parameters correspond to $\alpha \approx 5$, $\lambda_{Dh} \approx 7430$ cm, the electric field amplitude $E_0 = \phi_m (k_g T_e / e \lambda_{Dh}) \approx \phi_m (250/74.3)$ V/m $\approx 100$ mV/m for $\phi_m \approx 0.03$ which we obtained for $M \approx 1.08$. When we consider a lower value of $M$ (namely, $M \approx 1.08$), we consider small-amplitude EASWs, our results agree with the observation of the Viking satellite in the dayside auroral zone where the electric field amplitude of $\approx 100$ mV/m is observed [Dubouloz et al., 1991, 1993]. However, our analysis is valid for any higher value of $M$, i.e., for large-amplitude EASWs, and thus it may be useful in understanding the salient features of large-amplitude EASWs in space and laboratory plasmas.

It is important to mention that our dependent variables $n_e$, $u_e$, and $\phi$ have been assumed to vary with only one spatial coordinate ($z$) but not with the other two ($x$ and $y$). To generalize the stationary nonlinear BGK states [Bernstein et al., 1957] to more than one dimension is a very complicated problem and is beyond the scope of our present investigation.

References


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