Charging and shielding of dust grains in a dusty plasma

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Abstract

The charging of a dust grain in a unmagnetized plasma is examined. A description of the far-field dipole-like potential is presented for the case when the dust grain absorbs electrons and ions from the background plasma. In a Maxwellian plasma the electron and ion currents, which reach the dust grain surface, are calculated. It is shown that the trapped ions produce only an insignificant current and therefore they have little effect on the dust grain charging process. The trapped ion effects on the shielding of a dust grain in a plasma are also examined. It is shown that ions with low angular momentum cannot form a trapped cloud in a potential well. Therefore, it is necessary to examine the clouds of trapped ions with large angular momentum, which can shield the grains. The potential distribution between charged dust grains in a dusty plasma is derived analytically. The profiles of the intergrain potential distribution can have such a shape that the dust grains form a regular periodic structure (dust crystal). The charge of a given grain is not only determined by its own potential, but it also

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strongly depends on the potential of other dust grains that are participating in the dust crystallization. It means that although the short range interaction is provided by the Debye-Hückel potential, the long range interaction contributes to the formation of a dust crystal. For cubic symmetry and small dust grains (when the dust grain sizes are smaller than the inter-grain spacing) the calculation of the potential has been performed analytically. The possibility of both longitudinal (compressional) and transverse oscillations of a cubic dust lattice is discussed.

1 Introduction

When a neutral grain is immersed in a plasma, it will usually attain a negative charge [Shukla 2001; Shukla and Mamun 2002]. The charging and interaction of grains in a plasma is a subject of central importance for the explanation of dust grain attraction in laboratory and space environments [Shukla and Mamun 2002].

There are numerous processes [Mott-Smith and Langmuir 1926; Bernstein and Rabinovitz 1959; Horanyi et al. 1988; Goertz 1989; Northrop 1992; Allen 1992] by which a neutral dust grain in a plasma can be charged. The charging processes include collection of the background plasma electrons and ions by the dust grain, photoionization, secondary electron emission, thermionic emission, etc. Below we consider the first process only - grain charging by electron and ion currents.

The physics of charged dust grains is usually described by the orbit limited motion (OLM) theory, and the charge on the dust grain is determined by balancing the electron and ion currents that reach the dust grain surface. The OLM theory assumes that there are no potential barriers preventing the plasma ions from reaching the positive energy points in phase space.

Recently, there has been a renewed interest in the study of the OLM theory and diverse opinions have emerged [Allen et al. 2000; Lampe et al. 2000, 2001]. While Allen et al. [2000] question the validity of the OLM theory for any finite sized negatively charged dust grain in a Maxwellian plasma, Lampe et al. [2000, 2001] re-examine the limits of validity of the OLM theory by incorporating the potential barriers that exclude some plasma ions. Lampe et al. [2000, 2001] reported that the OLM theory is quite exact in the limit of small grain size, and that it turns out to be quite accurate in calculating the ion current to the dust grain for typical conditions pertinent to dusty plasmas.

Since the dust grain charging involving the OLM theory is subjected to
differing views at present, it is of urgent interest to re-examine the dust particle charging in an unmagnetized plasma [Tskhakaya et al. 2001]. Our objective is also to discuss the role of trapped ions in the shielding of the grain field [Tskhakaya et al. 2001].

About fifteen years ago, Ikezi [1986] theoretically predicted the possibility of a periodic distribution of the potential (dust crystallization) in dusty plasmas. The idea of Ikezi has been experimentally verified by several groups [Chu and I 1994, 1994; Chu et al. 1994; Thomas et al. 1994; Hayashi and Tachibana 1994; Morfill et al. 1997, 1999; Fortov et al. 1996, 1999; Zheng and Earnshaw 1995; Pieper et al. 1996; Hayashi 2001; Molotkov et al. 2000; Zuzic et al. 2000], who reported the fabrication of dust crystals in radio-frequency and glow discharges. However, no satisfactory mechanisms have yet been presented for understanding the formation of periodic potential structures, although several authors have studied dust lattice oscillations [Melandsø 1996; Thomas et al. 1998; Farokhi et al. 1999; Shukla 2000]. A dust crystal is composed of an ensemble of charged macroparticles which are very much affected by the short-range shielded Coulomb potential. However, in reality, the mutual inter-grain interaction is quite complex in that the charge on a specific grain is not only determined by its own surface potential, but also by the potentials of other grains which participate in the formation of the dust crystal.

Due to the repulsive character of the Debye-Hückel potential (also referred to as the Yukawa potential [Morfill et al. 1997, 1999; Fortov et al. 1996, 1999; Zheng and Earnshaw 1995], the symmetry of the dust crystal is determined by the form of the repulsive walls [Chu and I 1994, 1994; Chu et al. 1994] that surround the dust grain gas. The screened Coulomb potential approximation is backed by simulations [Zheng and Earnshaw 1995] and experiments [Morfill et al. 1997, 1999]. For a given symmetry, one can use the methods developed in solid state physics [Kittel 1956] to investigate the properties of a dust crystal. However, contrary to solid state crystals, studies of dust crystals in plasmas are easier to perform because the form of the interaction potentials (viz. Debye-Hückel, dipole-dipole, wake, dust shadowing, etc.) is known. Accordingly, we are able to deduce more complete information on the physics of dust crystals in plasmas [Farokhi et al. 1999; Tskhakaya and Shukla 2001].

We have organized our present chapter in the following fashion. We describe the motion of electrons and ions in the dust grain potential in §2. We derive analytical expressions for the electron and ion current densities in §3 and the ion number density far from the grain in §4, respectively. We then estimate the dust grain potential in §5 and study the effects of
the trapped ions on the shielding of the dust grains in §6. We derive an analytical periodic distribution of the potential in §7, and we study small oscillations of the dust lattice in §8. We finally present a brief discussion in §9.

2 Motion of electrons and ions

Let us consider a multi-component dusty plasma whose constituents are electrons, ions and extremely massive negatively charged dust grains which are discrete particles [Tskhaba et al. 2001]. We assume that the dust grain size $r_d$ is much smaller than the ion Debye radius $\lambda_{Di}$. Since micron-sized dust grains are typically billion times heavier than protons, they are supposed to be immobile on the time scales of interest here. The surfaces of the stationary dust grains are supposed to absorb the electrons and ions during the charging process. Furthermore, we assume that the field and the plasma particle (electrons and ions) distributions have radial symmetry. The finite grain size effect is incorporated by introducing a cut-off scale size $r_d$. The radial motion of the plasma particles is considered as one-dimensional in the field corresponding to the effective potential energy

$$U_{eff}(r; M) = U(r) + \frac{M^2}{2m_ar^2},$$

where the angular momentum $M$ is an integral of motion and $m_\alpha$ is the mass of the species $\alpha$ ($\alpha$ equals $e$ for electrons and $i$ for ions). The dependence of the effective potential on the distance $r$ to the center of force is determined by the potential energy $U(r)$ of the particles in the field that is produced by a dust grain.

It is important to determine the regions for finite and infinite particle motions, as the electron and ion distributions in those regions have different forms. Therefore, we introduce certain assumptions for the dependence of the potential energy $U(r)$ on $r$. At short distances (viz. $r$ is smaller than the ion Debye radius $\lambda_{Di}$) from the grain, the Debye screening of the grain charge by the plasma particles is not significant and the potential energy $U(r)$ decreases as $1/r$, i.e. slower than $1/r^2$. When $r \sim \lambda_{Di}$, $U(r)$ decreases exponentially due to screening, i.e. faster than $1/r^2$. The behavior of $U(r)$ at large distances (viz. $r \gg \lambda_{Di}$) significantly depends on the conditions at the dust grain surface. If the dust grain surface absorbs electrons and ions, the potential energy $U(r)$ at $r \to \infty$ decays as $1/r^2$. Although this has been noted by many authors [Lampe et al. 2000, 2001], the limits of its validity
and its physical origin have not been explained. In the following, we present a brief derivation [Alpert et al. 1964] of the dipole field (for a negatively charged dust grain surrounded by ion clouds).

In steady state, the kinetic equation for the ions is of the form

\[ \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{r}} - \frac{1}{m_i} \frac{\partial U(r)}{\partial \mathbf{r}} \cdot \frac{\partial f_i}{\partial \mathbf{v}} = A(r, \mathbf{v}) \delta(S(r)), \]  

(2)

where \( f_i \) is the ion distribution function and the value \( U(r) \), defined as a singly charged ion potential energy, is negative. The right-hand side in (2) describes the absorption of particles by the grains, \( A(r, \mathbf{v}) \) represents the rate of change of ions per unit time, and \( S(r) = 0 \) determines the dust grain surface. Inserting \( f_i = f_{i0} + \delta f_i \), where \( f_{i0} \) is a Maxwellian distribution function with temperature \( T_i \), in (2) and Fourier decomposing in space, we obtain

\[ i \mathbf{k} \cdot \mathbf{v} \delta f_k - i \frac{U_k}{m_i} \mathbf{k} \cdot \frac{\partial f_{i0}}{\partial \mathbf{v}} = \frac{i}{m_i(2\pi)^3} \int d\mathbf{k}' U_k' k' \cdot \frac{\partial \delta f_{k'} - k'}{\partial \mathbf{v}} + \int d\mathbf{r} A(r, \mathbf{v}) \delta(S(r)) \exp(-i \mathbf{k} \cdot \mathbf{r}), \]  

(3)

where \( \delta f_k \) is the Fourier transform of \( \delta f_i \), which is not assumed to be small. We are interested in finding the behavior of the perturbations at large distances \( (r \to \infty) \), which means that in the right-hand side of (3) we have to consider the limit \( k \to 0 \). The convergence of the integral on the right-hand side is ensured since we expect \( U_k \) and \( \delta f_k \) to decay as \( 1/k \) at \( k \to 0 \). Hence, we obtain

\[ i \mathbf{k} \cdot \mathbf{v} \left( \delta f_k + \frac{U_k}{T_i} f_{i0} \right) = I(\mathbf{v}), \]  

(4)

where

\[ I(\mathbf{v}) = \int d\mathbf{r} A(r, \mathbf{v}) \delta(S(r)) + \frac{1}{m_i} \int d\mathbf{r} \frac{\partial U(r)}{\partial \mathbf{r}} \cdot \frac{\partial \delta f_i}{\partial \mathbf{r}}. \]  

(5)

The function \( I(\mathbf{v}) \) is determined by the properties of the field in the nearest zone of the dust grain.

Let us assume that the weakly correlated electrons follow a Boltzmann distribution, so that the corresponding density is

\[ n_e(\phi) = n_{e0} \exp[U(r)/T_e], \]  

(6)
where \( n_{e0} \) is the number density of electrons when \( U = 0 \) and \( T_e \) is the electron temperature. Substituting (5) and (6) into Poisson’s equation we readily obtain

\[
\nabla^2 U = 4\pi e^2 \left( n_{e0} \exp[U(r)/T_e] - n_{\theta} - \int \delta f_i d\nu \right),
\]

(7)

where \( e \) is the magnitude of the electron charge and \( n_{\theta} (n_{e0}) \) is the unperturbed ion (electron) number density. Expanding the exponential function in powers of \( U(r)/T_e \) and Fourier decomposing, we obtain from (7)

\[
(k^2 + k_D^2) U_k + \frac{2\pi n_{e0} e^2}{T_e^2} \frac{1}{(2\pi)^3} \int d\nu' U_{\nu'} U_{k-\nu'} + ... = 4\pi e^2 \int \delta f_k d\nu,
\]

(8)

where \( k_D = (4\pi n_{e0} e^2/T_e)^{1/2} \) is the inverse electron Debye radius. The linear term (with respect to \( U_k \)) is of order \( 1/k \), while the nonlinear terms tend to a constant value at \( k \to 0 \). Thus, neglecting the nonlinear term and ignoring the departure from the quasi-neutrality [the \( k^2 \) term in the left-hand side of (8)], we obtain the Fourier component of the ion number density

\[
\delta n_k = \int \delta f_k d\nu = \frac{U_k}{T_e} n_{e0}.
\]

(9)

We note that (9) remains valid even in the nonlinear regime [Alpert et al. 1964] when \( k \to 0 \). Combining (9) with (4) we obtain

\[
\delta n_k \simeq - \frac{i}{1 + T_e/T_i} \int d\nu \frac{I(\nu)}{k \cdot \nu - \nu \epsilon} \quad \text{at} \quad \epsilon \to +0
\]

(10)

from which it emerges that

\[
\delta n_k \sim \frac{1}{k} \quad \text{and} \quad U_k \simeq \frac{1}{k} \quad \text{for} \quad k \to 0.
\]

(11)

Consequently, in real space, we would have

\[
U(r) = U_\infty \frac{r_d^2}{r^2},
\]

(12)

where \( U_\infty \) is the potential at the dust grain surface if the \( 1/r^2 \) dependence would have been extrapolated over all the distances up to the grain surface. The quantity \( |U_\infty| \) is always smaller than the real potential on the dust grain surface \( |U(r_d)| \) [Alpert et al. 1964], viz.

\[
|U_\infty| < |U(r_d)|,
\]

(13)
as will also be shown later [Tskhakaya et al. 2001]. The expression (12) can be physically understood as follows [Alpert et al. 1964]. When the perturbation at large distances is caused by the absorption of plasma particles, then the quasi-neutral density perturbation (and consequently the field potential and the potential energy) is proportional to the solid angle under which the dust grain is seen from the given point \( r \). Since the solid angle is proportional to \( r_d^2/r^2 \), we retrieve (12) on a physical ground as well. We note that (12) remains valid for distances that are smaller than the collisional mean free path \( \lambda_m \gg r \). The inter-particle collisions destroy the dependence law (12).

For electrons that are repelled by a negatively charged dust grain, \( U_{\text{eff}} \) decreases monotonically because the potential energy of electrons is positive, viz. \( U_e(r) = -U(r) > 0 \). Thus, the electrons are reflected from a monotonically decreasing potential well. For ions \( U(r) < 0 \) and the character of the dependence of their effective potential energy on \( r \) significantly depends on the angular momentum \( M \). It then follows from (1) that the extreme values of \( U_{\text{eff}} \) are determined from the condition

\[
\frac{dU_{\text{eff}}}{dr} = -\frac{1}{r^3} \left[ \frac{M^2}{m_i} - r^3 \frac{dU(r)}{dr} \right] = 0. \tag{14}
\]

According to the above discussions of the behavior of the potential energy \( U(r) \) for absorbing grains, the qualitative dependence of the term \( r^3 dU/dr \) on \( r \) must have the form depicted in figure 1. The maximum value of \( r^3 dU/dr \)

![Figure 1](image)

**Figure 1:** Qualitative dependence of the auxiliary function \( r^3 dU/dr \) on \( r \).

is reached at a certain point \( r_k \geq \lambda_{D_i} \). By using the Debye-Hückel shielded
potential, we obtain \( r_k \approx 1.61 \lambda_D \). From equation (12) and figure 1 we determine

\[
M_0^2 = m_i \left( r^3 \frac{dU}{dr} \right)_{r \to \infty} = 2m_i r^2 U_\infty \tag{15}
\]

and

\[
M_k^2 = m_i \left( r^3 \frac{dU}{dr} \right)_{r=r_k} \tag{16}
\]

At \( M \leq M_0 \), the effective potential energy \( U_{eff} \) has only one extreme value that is minimum. If the angular momentum is in the range

\[
M_0 < M < M_k, \tag{17}
\]

then \( U_{eff}(r) \) will have both maximum and minimum at points \( r_m \) and \( r_n \), respectively. The values of these extreme points depend on the angular momentum \( M \), but the condition \( r_n \leq r_k \leq r_m \) is always satisfied. At \( M = M_k \) the extreme points coincide, viz. \( r_n = r_m \), and at this point the function \( U_{eff}(r) \) has an inflection point. If \( M > M_k \), the effective potential energy \( U_{eff}(r) \) decreases monotonously with the increase of \( r \).

Let us determine the critical value of the angular momentum \( M_p \) from the condition of equality of the maximum value \( U_{eff}(r_m; M) \) and an effective potential energy on the surface of the grain, which is

\[
U_{eff}(r_d; M_p) = U_{eff}(r_m; M_p). \tag{18}
\]

Equation (18) yields

\[
M_p^2 = 2m_i r_d^2 \left| U(r_d) \right| - \left| U(r_m) \right| \left( 1 - \frac{r_d^2}{r_m^2} \right)^{-1}. \tag{19}
\]

Since \( r_m \geq r_k > r_d \) and \( \left| U(r_d) \right| \gg \left| U(r_m) \right| \), we obtain the approximate value of \( M_p \), namely

\[
M_p^2 \approx 2m_i r_d^2 \left| U(r_d) \right|. \tag{20}
\]

It is obvious from (13), (15) and (20) that \( M_p > M_0 \). By means of (1) and (18) we can make sure that \( U_{eff}(r_d; M) > U_{eff}(r_m; M) \) at \( M > M_p \), while \( U_{eff}(r_d; M) < U_{eff}(r_m; M) \) at \( M < M_p \). The value of the zero point \( r_s \) of the effective potential energy \( U_{eff}(r_s; M) \) is determined from the condition

\[
U_{eff}(r_s; M) = 0 \quad \text{at} \quad M \leq M_0. \tag{21}
\]
According to (1) and (13) \( r_s \) is always less that the radius of the grain, viz. \( r_s < r_d \). Accounting for the above mentioned properties of the function \( U_{eff}(r; M) \), we can determine the surface that separates regions of infinite and finite particle motions in the velocity space \((v_r, v_\theta)\), where \( v_r \) and \( v_\theta \) are the components of the velocity along and across the radial direction. We are interested in calculating the current associated with those particles which reach the dust grain surface. According to figure 2, they must have the energy

\[
E(v_r, v_\theta, r) = \frac{m_i v_r^2}{2} + U_{eff}(r; M) \geq U_{eff}(r_d; M). \tag{22}
\]

The usual definition of the angular momentum reads

\[
M = m_i v_\theta r. \tag{23}
\]

Three cases are apparent. First, if \( M \leq M_0 \), the particles with the positive energy

\[
E(v_r, v_\theta, r) \geq 0 \tag{24}
\]
will move infinitely and all such particles will reach the grain surface since \( r_s < r_d \).

Second, for \( M_0 < M \leq M_p \) all the free ions attach onto the grain surface since for such values of angular momentum, as is mentioned above, the condition \( U_{\text{eff}}(r_d;M) < U_{\text{eff}}(r_m;M) \) is fulfilled. The region of infinite ion motion is determined by the condition

\[
E(v_r,v_{\theta},r) \geq U_{\text{eff}}(r_m;M).
\]

(25)

Third, if \( M > M_p \), then \( U_{\text{eff}}(r_m;M) < U_{\text{eff}}(r_d;M) \), and only those particles whose energy satisfies (22) are able to reach the dust grain surface.

3 Current densities

In plasmas, the electron thermal speed is much larger than the ion thermal speed. Accordingly, the electrons will reach the dust grain surface faster than the ions. The grain surface will thus acquire a huge negative charge. When it has a critical capacity of absorbing electrons, it will start repelling electrons and attracting ions which form a sheath around the grain.

In the following, we discuss the form of the electron and ion currents that reach a dust grain surface in a plasma with Maxwellian particle distributions

\[
f_\alpha(v_r,v_{\theta},r) = n_\alpha \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \exp \left[ -\frac{m_\alpha (v_r^2 + v_{\theta}^2)}{2T_\alpha} \mp \frac{|U(r)|}{T_\alpha} \right],
\]

(26)

where \( \alpha = e, i \) and the \((-\uparrow)\) sign in the argument of the exponential function corresponds to electrons (ions). The current of the plasma particles per unit area with external unit normal vector \( \mathbf{n} \) located at the point \( r \) will be determined by the expression

\[
J_\alpha = -2\pi q_\alpha \int (\mathbf{n} \cdot \mathbf{r}) f_\alpha(v_r,v_{\theta},r)v_{\theta} dv_{\theta} dv_r,
\]

(27)

where \( q_e = -e \) and \( q_i = e \) and the integration in (27) has to be performed over velocities for which \( \mathbf{n} \cdot \mathbf{v} < 0 \), i.e. \( v_r < 0 \). In (26) we have denoted \( |U(r)| = e|\phi(r)| \), where \( \phi(r) < 0 \) is the electric potential at the point \( r \).

For the electron current, we have

\[
J_e = -e n_0 V_{Te} \exp \left[ -\frac{e|\phi(r)|}{T_e} \right],
\]

(28)

where \( V_{Te} = (T_e/2\pi m_e)^{1/2} \).
The boundaries confining the region in velocity space from which the ions can get onto the dust grain surface significantly depend on the value of $M$. For this reason, the determination of the current of infinite ions should be made separately for each of the three characteristic cases of the angular momentum regimes, as described in the previous section. We obtain the following expressions for the currents of free ions. First, for the region $M \leq M_0$, we have

$$J_{i1} = e n_{i0} V_{Ti} \frac{r_d^2 |U_{\infty}|}{r_i^2},$$  \hspace{1cm} (29)

where $V_{Ti} = (T_i/2\pi m_i)^{1/2}$. Second, in the region $M_0 < M \leq M_p$, we have

$$J_{i2} = e n_{i0} V_{Ti} \frac{r_m^2}{r_d^2} \left[ \exp \left( -\frac{r_d^2}{r_m^2} \frac{|U_{\infty}|}{T_i} \right) - \exp \left( -\frac{r_d^2}{r_m^2} \frac{e |\phi(r_d)|}{T_i} \right) \right] \times \exp \left( \frac{e |\phi(r_m)|}{T_i} \right).$$  \hspace{1cm} (30)

Third, in the region $M > M_p$ we have

$$J_{i3} = e n_{i0} V_{Ti} \exp \left[ \frac{e |\phi(r_d)|}{T_i} - \frac{M_p^2}{2m_i r_d^2 T_i} \right].$$  \hspace{1cm} (31)

The total current of ions with an infinite motion onto the grain surface is the sum of (29)-(31), which differs from the expression deduced by Alpert et al. [1964]. The reason for the discrepancy is that Alpert et al. [1964] did not introduce the critical value for the angular momentum $M_p$ and that the ions were not classified by the values of the ratio $M/M_p$. If the potential energy of the ions at a distance $r_m$ is much less than the thermal energy, viz. $|U(r_m)| \ll T_i$ and also if $r_d \ll r_m$, then by means of (20) and (31) we obtain

$$J_i = e n_{i0} V_{Ti} \left( 1 + \frac{e |\phi(r_d)|}{T_i} \right),$$  \hspace{1cm} (32)

which agrees with Mott-Smith and Langmuir [1926]. This is expected since near the grain the field decreases according to Coulomb's law. It is important to note that in (32) the smallness of the ratio $e |\phi(r_d)|/T_i$ is not assumed.

The determination of the current of finite ions (viz those trapped by the potential well; see figure 2) is difficult, since it is necessary to determine their distribution function. If the ions are not absorbed by the grain, then it is appropriate to use the Boltzmann distribution for the ions in those
regions where the motion is finite. However, when the grain absorbs ions, the
determination of the distribution function of trapped ions is a complicated
independent problem [Alpert et al. 1964; Gurevich 1968]. But in the case
of a rarefied plasma when the mean free path of the ions exceeds the ion
Debye radius and the grain absorbs the ions, we can estimate the current of
the trapped ions.

In steady state, the decrease of the number of ions in the region of finite
motion (due to absorption on the dust grain surface) is compensated by
the flow of ions from the region of infinite motion into the region of finite
motion. This flow is caused by collisions of ions with each other and with
the neutral particles. The particles which are trapped this way may then be
absorbed by the dust grain. Consequently, in the stationary case, the flow of
trapped ions on the surface of the grain cannot exceed the flow from the
region of infinite motion into the region of finite motion.

The integral of ion collisions with each other and with neutrals can be
represented in the form of a divergence in phase space [Silin 1971], and the
kinetic equation can be written in the form

\[ \mathbf{v} \cdot \frac{\partial f_i}{\partial r} - \frac{\partial U(r)}{\partial r} \cdot \frac{\partial f_i}{\partial \mathbf{p}} = - \frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{S}(f_i, f_\beta), \quad (33) \]

where \( \mathbf{p} = m_i v \), \( \mathbf{S}(f_i, f_\beta) \) is the flow density in the momentum space and
\( \beta = i, n \). The expression for \( \mathbf{S} \) can be derived for ion-ion collisions from the
Landau collision integral and for ion-neutral collisions from the Boltzmann
collision integral. However, in the case of rare collisions, one can use, to a
first approximation, the Maxwell-Boltzmann distribution for both ions and
neutrals. We have also to consider the inhomogeneous character of those
distribution functions, otherwise the collision integrals would vanish. Let us
determine the current density of the trapped ions in the form

\[ J_\parallel^i(r) = e \int v_r f_i \, dv, \quad (34) \]

where the integration is carried out over velocities from the region of finite
motion. Then from (33) we obtain

\[ \frac{\partial}{\partial r} J_\parallel^i(r) = - \frac{e}{m_i} \oint_\sigma \mathbf{S} \cdot d\sigma, \quad (35) \]

where \( \sigma \) is the surface (in the velocity space) enclosing the region of finite
motion. The right-hand side of (35) is caused by the ion flow through this
surface. The current on the surface of the grain, \( r \approx r_d \), is of interest to
us. Since \( r_m \gg r_d \) the equation for the surface in the first two cases of the previous section has the form

\[
v_r^2 + v_\theta^2 \simeq \frac{2}{m_i} |U(r_d)|. \tag{36}
\]

As a result, for the current of trapped ions on the dust grain surface, we obtain the following estimates. For ion-ion collisions, we have

\[
J_i^i \approx e n_i V_{T_i} \frac{\lambda_{D_i} \lambda_{i \Lambda}}{\lambda_{i i}} \frac{|U(r_d)|}{T_i}, \tag{37}
\]

where \( \lambda_{i i} \) is the mean free path for ion-ion collisions and \( \Lambda \) is the usual Coulomb logarithm. For ion-neutral collisions we have

\[
J_i^i \approx e n_i V_{T_i} \frac{d}{\lambda_{i n}} \frac{|U(r_d)|}{T_i}, \tag{38}
\]

where \( d \) is the radius of interaction of ions and neutrals and \( \lambda_{i n} \) is the mean free path for these collisions. Since in our case \( \lambda_{i i} \gg \lambda_{D_i} \) and \( \lambda_{i n} \gg d \), a comparison of (37) and (38) with (32) reveals that

\[
J_i^i \ll J_i. \tag{39}
\]

Hence, in a rarefied plasma the ion current on the dust grain surface is mainly determined by the ions which execute an infinite motion.

4 Particle number densities

Let us now determine the electron and ion number densities far from the grain (viz. \( r \gg \lambda_{D_i} \)). The general definition of the particle number density is

\[
n_\alpha = 2\pi \int f_{\alpha}(v_r, v_\theta, r) v_\theta dv_\theta dv_r, \tag{40}
\]

where the function \( f_{\alpha}(v_r, v_\theta, r) \) is defined by (26). The limits of integration must be determined according to the character of the particle motion. For example, for the ions one has to follow the conclusions about the ion motion that are indicated in figure 2. The grain absorbs only those electrons which have the energy

\[
E_e = \frac{m_e}{2} (v_r^2 + v_\theta^2) + |U(r)| \geq \frac{M_e^2}{2m_e r_d^2} + |U(r_d)| \tag{41}
\]
for \( n_r < 0 \). Here, we have introduced \( M_e = m_e v_{0r} r \). Far from the grain (viz. \( r \gg \lambda_D \)), when \( |U(r)|/T_e \ll 1 \), after integrating (40) outside the area (41), we obtain

\[
\frac{n_e}{n_{e0}} = 1 - \frac{|U(r)|}{T_e} + \frac{r_d^2}{2 r^2} \left( \frac{|U(r_d)|}{T_e} - \frac{1}{2} \right) \left( 1 - \text{erf} \left( \sqrt{\frac{|U(r_d)|}{T_e}} \right) \right)
- \frac{r_d^2}{2 \sqrt{\pi} r^2} \sqrt{\frac{|U(r_d)|}{T_e}} \exp\left(-\frac{|U(r_d)|}{T_e}\right),
\]

(42)

where

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp(-t^2)
\]

(43)

is the error function. For the ions we have to consider different ranges of their angular momentum, as described in §2.

For \( M \leq M_0 \) all the free ions impact the grain surface and are absorbed. According to the results given in the previous section the number of the trapped ions is negligibly small. Then the necessary area of integration in the velocity space is determined by the conditions

\[
v_r^2 + v_\theta^2 \geq \frac{2}{m_i} |U(r)|,
\]

(44)

\[
0 \leq v_\theta \leq \frac{M_0}{m_i r} \quad \text{and} \quad v_r < 0.
\]

(45)

Far from the grain, when \( |U(r)|/T_i \ll 1 \), we obtain

\[
n_i = \frac{n_{i0} |U(r)|}{2 T_i}.
\]

(46)

For the angular momentum in the region \( M_0 < M \leq M_p \) the area in the velocity space for ions impinging on the grains is determined by the relations

\[
v_r^2 + v_\theta^2 \geq v_\theta^2 \frac{r^2}{r_m^2} + \frac{2}{m_i} [U(r) - |U(r_m)|],
\]

(47)

\[
\frac{M_0}{m_i r} < v_\theta \leq \frac{M_p}{m_i r}, \quad \text{and} \quad v_r \leq 0.
\]

(48)
The quantities $M_0$ and $M_p$ are defined by (15) and (20). Performing the integration outside of this area we find the ion number density far from the grain (viz. $r \gg \lambda_{Di}$ and $r_m \geq \lambda_{Di}$)

$$n_i \simeq n_{i0} \frac{2}{\sqrt{\pi}} \frac{r_d^3}{r^3} \left( \frac{|U(r_d)|}{T_i} \right)^{3/2},$$

which is negligible because $n_i \propto 1/r^3$. We note that the number density of free ions which are reflected by the potential hump, $|U(r_m)|$ is also small (cf. curve 2 in figure 2). For the angular momentum $M > M_p$ (cf. curve 1 in figure 2) only the ions from the area

$$v_r^2 + v_\theta^2 \geq \frac{v_\theta^2}{r_d^2} \frac{r_r^2}{r_d^2} \left( \frac{|U(r_d)|}{|U(r)|} \right),$$

and $v_r < 0$ are absorbed by the grain. Those free ions which have velocity satisfying the opposite inequality in (50) can move both in the positive and negative radial directions, $v_r < 0$ and $v_r > 0$. Far from the grain ($r \gg \lambda_{Di}$) the area of integration in velocity space is presented in figure 3 (the shaded regions). The curve in the region $v_r > 0$ in figure 3 is determined by (50)
when the left- and right-hand sides are equal. Performing the corresponding integration for the ion number density we obtain

\[ n_i = n_{i0} \left[ 1 + \frac{|U(r)|}{T_i} - \frac{r_d^2}{r^2} \left( \frac{|U(r_d)|}{T_i} + \frac{1}{4} \right) \right]. \quad (51) \]

By adding (46) and (51) we obtain the total ion number density far from the grain

\[ n_i = n_{i0} \left[ 1 + \frac{3}{2} \frac{|U(r)|}{T_i} - \frac{r_d^2}{r^2} \left( \frac{|U(r_d)|}{T_i} + \frac{1}{4} \right) \right]. \quad (52) \]

The quasi-neutrality condition \( n_i = n_e \) must hold far from the grain. Using the relation (12) and equating (42) and (52) we obtain

\[
3 \frac{|U_\infty| - |U(r_d)|}{T_i} = - \frac{|U(r_d)|}{T_i} - \text{erf} \left( \sqrt{\frac{|U(r_d)|}{T_e}} \left( \frac{|U(r_d)|}{T_e} - \frac{1}{2} \right) \right) \\
- \frac{1}{\sqrt{\pi}} \sqrt{\frac{|U(r_d)|}{T_e}} \exp \left( -\frac{|U(r_d)|}{T_e} \right).
\]

In the next section [e.g. Eq. (56)] it will be shown that \( |U(r_d)|/T_e > 1 \). Hence it follows from (53) that \( |U_\infty| < |U(r_d)| \), which was used in §2 [cf. Eq. (13)].

5 Dust grain potential

In steady state the potential on the dust grain surface is determined by putting to zero the sum of the electron and ion currents at \( r = r_d \). It means that

\[ J_i(r_d) + J_e(r_d) = 0. \quad (54) \]

For \( n_{d0} = n_{i0} \) we find that

\[
\left[ 1 + \frac{e|\phi(r_d)|}{T_i} \right] \exp \left[ \frac{e|\phi(r_d)|}{T_e} \right] = \frac{V_T e}{V_{T_i}}.
\]

Equation (55) has a solution

\[
\frac{e|\phi(r_d)|}{T_e} = \frac{|U(r_d)|}{T_e} > \frac{1}{2}, \quad (56)
\]
if $1 \leq \frac{T_e}{T_i} < \frac{m_i}{m_e}$. The relation (56) has been used in the preceding section for the analysis of (53). Taking $V_{T_e}/V_{T_i} \simeq 200$ for an isothermal plasma with $T_e \approx T_i \approx T$, we obtain from (55)

$$|\phi(r_d)| \simeq 3.75T/e.$$  

(57)

The electric field strength near the grain is

$$E_0 = \frac{|\phi(r_d)|}{r_d}.$$  

(58)

For $r_d \approx 10^{-2}\mu m$ and $T \sim 0.01$ eV, the dust grain potential and the electric field strength turn out to be $6.5 \times 10^{-2}$ V and $6.5 \times 10^4$ V/cm, respectively. Strong electric fields can cause dust grain disruption and produce additional ionization of neutral grains.

6 Trapped ion effects on shielding of dust grains

Recently, Lampe et al. [2001a, 2001b] investigated the effects of trapped ions on the shielding of a dust grain in a plasma. Specifically, they employed the OLM theory for the ion motion in the electrostatic field around a single dust grain, and showed that ion-neutral charge exchange can give rise to a buildup of negative energy trapped ions which may dominate the shielding cloud of the dust grain field.

Our objective here is to examine the validity of the trapped particle effects on the shielding of a dust grain in an unmagnetized plasma [Tskhakaya et al. 2001]. We will conclude that most of the trapped ions in the potential well with the negative effective potential energy, as considered by Lampe et al. [2001b], have in general only a rather minor effect on the shielding process.

Lampe et al. [2001b] have accounted for the first interval for the ion angular momentum, $M \leq M_0$ (cf. §2), and used a corresponding restriction on the total energy of the ions, $E \leq 0$ (cf. Eq. (2a) of Lampe et al. [2001b]). This case corresponds to the curve 1 in figure 2. According to (13) and the remark made just after (53),

$$\frac{|U_\infty|}{|U(r_d)|} < 1.$$  

(59)

The dependence of $|U_\infty|$ on the grain surface potential is depicted on page 317 in Alpert et al. [1964] for the case when the electron and ion temperatures are equal, (viz. $T_e \approx T_i$) and it is shown that always $|U_\infty|/|U(r_d)| <$
1/3. With increasing $|U(r_d)|$ this ratio decreases. Obviously (12) is valid only for large $r$. However, if one extrapolates that equation to $r \to r_d$, one finds, in view of (59), that the dependence in (12) on $r$ does not compensate a sharp decrease of the potential energy $U(r)$ at the distances $r > \lambda_D$. From (59) it also follows that the zero point $r_s$ of the effective potential energy $U_{\text{eff}}(r; M)$ at $M \leq M_0$ is always close to the center ($r = 0$) in comparison with the grain surface. That means $r_s < r_d$. Actually, from (1) one finds that $U(r; M)$ at $r = r_d$ for $M \leq M_0$ is negative (see the curve 1 in figure 2), i.e.

$$U_{\text{eff}}(r_d; M) \leq |U_{\infty}| - |U(r_d)| < 0, \quad \text{for } M \leq M_0. \quad (60)$$

Moreover, it turns out that at $M \leq M_0$ also the distance for the minimum point of $U_{\text{eff}}(r; M)$ from the center ($r = 0$) can be less than the grain radius (see the curve 1 in figure 2). For the distances $r \leq \lambda_D$, we can choose the test function

$$U(r) \simeq -\frac{r_d}{r} |U(r_d)| \exp(-r/\lambda_D) \quad (61)$$

for the potential energy. Then from (14) we obtain for $M \leq M_0$

$$r_{\min} \equiv r_{\min} \simeq 2r_d \frac{|U_{\infty}|}{|U(r_d)|}. \quad (62)$$

For $|U_{\infty}| < (1/2)|U(r_d)|$ we have $r_{\min} < r_d$. This result is physically expected, as the ions with small angular momentum impact the dust grain surface and are absorbed.

At this point, it is worth stressing that the free ions fall into the potential well within a time $\tau = \lambda_{mf}/V_i$, where $\lambda_{mf}$ is the ion mean free path due to ion-neutral and ion-ion collisions and $V_i$ is the relative speed between the colliding particles; this speed can be larger than the ion thermal speed $V_Ti$ (see the end of this section). The time scale $\tau$ is much larger than the characteristic time $\tau_s(= \lambda_D/V_i)$ for ions to traverse the potential well, because usually $\lambda_{mf} \gg \lambda_D$ [Shukla 2001; Shukla and Mamun 2002; Lampe et al. 2001b]. It turns out that ions, falling into the potential well very quickly stick onto the grain surface and are lost due to absorption. It means that the ions do not have enough time to be accumulated in the well and to form a stationary trapped cloud, contrary to what was assumed in [Lampe et al. 2001b].

So we can conclude that for ions, with the angular momentum $M$ smaller than a certain critical value $M_0$, the dust grain surface is within the
well and the ions falling into the potential well are immediately lost due to absorption onto the dust grain surface. Hence, the formation of ion clouds, that can shield the grain, is possible only for ions with \( M > M_0 \). For the latter case, we have to properly calculate the distribution function and the corresponding density for the ions which are trapped in a potential well. This leads to a lengthy analysis because a correct calculation requires that the trapped ions satisfy the boundary condition on the surface in the velocity space, which separates the region of infinite and finite motion [Lifshitz and Pitaevskii 1981].

Dusty plasmas typically have \( T_i \ll T_e \) and \( e |\phi(r_d)| \sim T_e \). Accordingly, we have

\[
\frac{|U(r_d)|}{T_i} = \frac{e |\phi(r_d)|}{T_i} \gg 1. \tag{63}
\]

Lampe et al. [2001b] assumed that the ion temperature is equal to the temperature of the neutrals and that the ion-neutral charge exchange collision frequency \( \nu \) does not depend on energy. However, from (63) it follows that the velocity of the ions in the electric field exceeds their thermal speed. Consequently, the relative velocity of the colliding species in ion-neutral charge-exchange collisions differs essentially from their thermal speed. In this case, charge exchange collisions can be described to a good approximation by a constant cross section [McDaniel and Mason 1973]. It means that for problems connected with the ion motion in the grain field at \( T_i \ll T_e \), the constancy of the mean free path at charge-exchange collisions \( \lambda_{mfp} = \text{const} \) is more applicable than the constancy of the collision frequency \( \nu = \text{const} \).

7 Periodic potential distribution of potential

In this section, we construct a theory for the three-dimensional (3D) periodic distribution of the electric potential in a dusty plasma. Specifically, we show that charged macroparticles (dust grains) can organize themselves in the form of regular periodic structures if the external boundary conditions are appropriately considered [Tsakhakaya and Shukla 2001]. For this purpose, we calculate the spatial distribution of electrostatic potentials for 3D dust lattices accounting for interactions between all the dust grains that form the dust crystal. Furthermore, we have also investigated oscillations of a dust lattice and found that transverse lattice oscillations have anisotropic behavior relative to the direction of the lattice wave propagation. We note
that our investigation of 3D dusty crystals and associated oscillations is superior to those existing in solid state physics where there does not exist a self-consistent theory for 3D crystals including the interaction forces.

Let us consider again a multi-component dusty plasma whose constituents are electrons, ions and extremely massive negatively charged dust grains which are discrete particles. The plasma potential \( \phi(r) \) in such a system is determined from Poisson’s equation

\[
\nabla^2 \phi(r) - 4\pi e (n_e - n_i) = -4\pi \sum_n q_n(r),
\]

where \( n_e(\phi) \) and \( n_i(\phi) \) are the electron and ion number densities and the summation on the right-hand side of (64) is taken over all grains and \( n \equiv (n_1, n_2, n_3) \). Here, \( n_1, n_2 \) and \( n_3 \) enumerate the grain position along the \( x, y \) and \( z \) axes, respectively. The charge of the grain \( q_n(r) \) can be represented in the form

\[
q_n = \frac{Q_n}{S_n} \int_{S_n} dS_n(r') \delta \left( r - (r_n + r') \right)
\]

by assuming that the grains are conductors. Here, \( \delta(r) \) is the Dirac delta function. The integral in (65) has been taken over the dust grain surface \( S_n \). The center of mass of the grain has been denoted by \( r_n \equiv (x_{n1}, y_{n2}, z_{n3}) \equiv (x_n^{(1)}, x_n^{(2)}, x_n^{(3)}) \). In the following, the superscripts \( (1), (2), (3) \) shall denote the \( x, y \) and \( z \) components, respectively, i.e. \( A^{(1)} = A_x, A^{(2)} = A_y, A^{(3)} = A_z \) and \( x^{(1)} = x, x^{(2)} = y, x^{(3)} = z \). We stress that the charge of the \( n \)-th grain, \( Q_n \), is not only determined by its own potential, but it also depends on the potential of other grains [Landau and Lifshitz 1960]. Thus, we have

\[
Q_n = \sum_{n'} C_{nn'}(r_n, r_{n'}) \phi_{n'}(r_n),
\]

where \( C_{nn'} \) and \( C_{nn'} \) \( (n \neq n') \) are the capacity and electrostatic induction coefficients, respectively. These coefficients depend on the shape and the relative position of conducting grains.

Equation (64) has to be solved with appropriate boundary conditions on the grain surfaces which read: (i) the grain surface must be equipotential. It means that regardless of the direction of approach to any point on the grain surface, the solution of (64) must satisfy the condition

\[
\phi|_{S_n} = \phi_n = \text{constant}.
\]
(ii) the derivative of the potential (viz. the electric field) on the grain surface has a jump. Thus, substituting (65) and (66) into (64) and integrating the resultant equation over the grain volume we obtain the second boundary condition

$$\int_{S_n} \nabla \phi dS_n = -4\pi \sum_{n'} C_{nn'} \phi_{n'}(r_n) = -4\pi Q_n. \quad (68)$$

Let us assume that weakly correlated electrons and ions follow Boltzmann distributions, so that the corresponding densities are, respectively,

$$n_e(\phi) = n_{e0} \exp(\epsilon \phi / T_e) \quad \text{and} \quad n_i = n_{i0} \exp(-\epsilon \phi / T_i), \quad (69)$$

where $n_{e0}$ and $n_{i0}$ are as usual the number densities of the electrons and ions when $\phi = 0$, and $T_e > T_i$. Substituting (65), (66) and (69) into (64) for small potentials we readily obtain

$$\nabla^2 \varphi - 2k_D^2 \varphi = -\sum_{nn'} \frac{C_{nn'}}{S_n} (\varphi_{n'} + \hat{\phi}) \int_{S_n} dS_n(r') \delta(r - (r_n + r')). \quad (70)$$

Here, $\varphi = -e\phi / T_i (\ll 1)$ and $k_D^2 = 2\pi e^2 (n_{e0}/T_e + n_{i0}/T_i)$. Equation (70) exhibits that non-equal electron and ion number densities (at $\phi = 0$) simply shift the potential by $\hat{\phi}$, where $\hat{\phi} = (n_{e0} - n_{i0}) / T_i (n_{e0}/T_e + n_{i0}/T_i)$ is a constant. Thus, in the following, without loss of generality, we shall restrict ourselves to the case when the electron and ion densities at $\phi = 0$ are equal, i.e. $n_{e0} = n_{i0}$.

The general method for describing the dust grain system in the plasma is as follows: The solution of (70) has to be found by solving it in a volume outside the grains [this means we have to neglect the right-hand side of equation (64) and solve the homogeneous equation] and match the solutions on the surfaces of each dust grain by using the boundary conditions (67) and (68). We consider the dust grain size to be much smaller than the inter-grain spacing, viz. $a_n \ll d$. (here for convenience the radius of the $n$th grain is denoted by $a_n$, different from §1, where the grains radius is denoted by $r_d$). Then, we have $C_{nn} \simeq a_n$ and $C_{nn'} \ll C_{nn}$, where $n \neq n'$. Hence, equation (70) reduces to the Schrödinger equation

$$\nabla^2 \varphi(r) + [-2k_D^2 \varphi(r) - V(r)] \varphi(r) = 0, \quad (71)$$

where the potential energy is

$$V(r) = -\sum_{n} a_n \delta(r - r_n). \quad (72)$$
As an application of the above mentioned method, we shall consider a simple case. Keeping in mind the cubic symmetry, we present the solution of the homogeneous equation (71) by the method of separation of variables. That is, we introduce \( \varphi(r) = \prod_{i=1}^{3} \varphi_i(x^{(i)}) \) and obtain

\[
\varphi \left( x^{(1)}, x^{(2)}, x^{(3)} \right) = \varphi_n \prod_{i=1}^{3} \left\{ \alpha_n^{(i)} \sinh \left[ s_i \left( x^{(i)} - x^{(i)}_{n_i} \right) \right] \right. \\
+ \cosh \left[ s_i \left( x^{(i)} - x^{(i)}_{n_i} \right) \right] \right\},
\]

which is defined in the rectangular volume \( x^{(i)}_{n_i} < x^{(i)} < x^{(i)}_{n_i+1} \). Here, \( \varphi_n \) is the potential of the grain with coordinates \( x_{n_1}, y_{n_2}, z_{n_3} \) and \( s_1^2 + s_2^2 + s_3^2 = 2k_0^2 \). Obviously the quantities \( 1/s_1, 1/s_2 \) and \( 1/s_3 \) represent the characteristic scales of potential screening along the \( x, y \) and \( z \) axes, respectively. The boundary condition (67) is then reduced to the requirement of the potential continuity on the three planes \( x^{(i)} = x^{(i)}_{n_i} \), where \( i = 1, 2, 3 \). This allows us to deduce

\[
\alpha_n^{(i)} = \left\{ \frac{\varphi_{n+1}^{(i)}}{\varphi_n} \sinh^{-1} \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right] - \coth \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right] \right\} \\
\sinh^{-1} \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right],
\]

where \( I_i = \delta_{ik} \) is the Kronecker symbol and the notation \( \varphi_{n+I_i} \) means \( \varphi_{n_1+\delta_{i1}, n_2+\delta_{i2}, n_3+\delta_{i3}} \). Substituting (74) into (73), we obtain the distribution of the potential

\[
\varphi \left( x^{(1)}, x^{(2)}, x^{(3)} \right) = \frac{1}{\varphi_n} \prod_{i=1}^{3} \sinh^{-1} \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right] \\
\left\{ \varphi_{n+I_i} \sinh \left[ s_i \left( x^{(i)} - x^{(i)}_{n_i} \right) \right] + \varphi_n \sinh \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right] \right\}.
\]

The second boundary condition, given by (68), yields

\[
\varphi_n \left\{ \sum_{i=1}^{3} s_i \left[ \tanh \left( \frac{s_i}{2} \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right) + \tanh \left( \frac{s_i}{2} \left( x^{(i)}_{n_i} - x^{(i)}_{n_i-1} \right) \right) \right] - \frac{1}{\alpha_n} \right\} \\
\approx \sum_{i=1}^{3} \left\{ \frac{\varphi_{n+I_i} - \varphi_n}{\sinh \left[ s_i \left( x^{(i)}_{n_i+1} - x^{(i)}_{n_i} \right) \right]} + \frac{\varphi_{n-I_i} - \varphi_n}{\sinh \left[ s_i \left( x^{(i)}_{n_i} - x^{(i)}_{n_i-1} \right) \right]} \right\}.
\]
In obtaining (76), without loosing the essential physics, we have supposed that in every cage the grains have the form of a small cube with the site length \( 2a_n \) and the surface integral has been performed over such cubic surfaces. Also used is the condition \( s_i a_n < 1 \).

When the grains have identical potentials, viz. \( \varphi_n = \varphi_0 = \text{constant}, \) we obtain \( x^{(i)}_{n_{i+1}} - x^{(i)}_{n_i} = d^{(i)} \), which implies that along every axis the intergrain distances are equal, and the following relation is satisfied

\[
\sum_{i=1}^{3} s_i \tanh \left( \frac{s_i d^{(i)}}{2} \right) \simeq \frac{1}{2a}, \quad (77)
\]

where \( a_n \approx a \). On the other hand, if the dust grains are separated with equal distances and the relation (77) is satisfied the grain surface potentials turn out to be equal as well. It follows that for the existence of a regular periodic dust crystal the plasma parameters have to satisfy some definite condition, given by (77).

In the general case when the grains potentials are not equal, the relation (76) connects the potential of different grains to each other, i.e. \( \varphi_n = \sum_{m \neq 0} D_{n,m} \varphi_{n_1+m_1,n_2+m_2,n_3+m_3} \), where \( D_{n,m} \) can be expressed as a combination of the coefficients of \( \varphi_n \) in (76). Consequently, the potential (75) can be seen to arise from the effect of the whole grain system in the dust crystal.

8 Small oscillations of a dust lattice

Let us now consider small oscillations of the grains around their equilibrium position. For simplicity we consider cubic symmetry and take uniform charge and mass distributions for the dust grains. Thus, \( Q_n = Q_0, \ m_n = m_0, \ s_1 = s_2 = s_3 = s \) and \( d^{(1)} = d^{(2)} = d^{(3)} = d \). Then, for the total electrostatic interaction energy for the grains we have

\[
U = Q_0 \varphi_0 \sum_{n} \prod_{i=1}^{3} \cosh \left( \frac{s_i}{2} \left( x^{(i)}_{n_{i+1}} + x^{(i)}_{n_{i-1}} - 2x^{(i)}_{n_i} \right) \right), \quad (78)
\]

where from the sum in (78) we have excluded the interaction of the charged grain with its own field. By using (78) we can construct the equation of motion for the dust grain displacement \( \xi^{(i)}_{n_i} \). The dust grains execute small oscillations around their equilibrium position \( x^{(i)}_{0n_i} \) with the displacement
\[ \xi^{(i)}_{n_i} = x^{(i)}_{n_i} - x^{(i)}_{0_{n_i}}. \] The equation for the \( n \)-th grain displacement is

\[ \frac{\partial^2 \xi^{(i)}_{n_i}}{\partial t^2} = -\frac{s^2 Q_0 \varphi_0}{4 m_0} \cosh^{-3}(sd) \left( \xi^{(i)}_{n_i+1} + \xi^{(i)}_{n_i-1} - 2 \xi^{(i)}_{n_i} \right), \quad (79) \]

 Seeking a wave-train solution of the form \( \xi^{(i)}_{n_n} = A^{(i)} \exp(i k \cdot n d - i \omega t) \), where \( k \) is the wave vector and \( \omega \) is the frequency, we obtain from (79)

\[ \omega^2 = \frac{s^2 Q_0 \varphi_0}{4 m_0} \frac{k^2 d^2}{\cosh^3(s d)} \left[ \frac{\sin(k_d/2)}{k_d/2} \right]^2. \quad (80) \]

We emphasize that (80) determines the frequencies of both longitudinal (compressional) and transverse oscillations. If the wave vector is aligned along one of the main lattice axes, there appear only longitudinal waves as \( k \times A = 0 \). On the other hand, if the wave vector forms an angle with the main axis of a cubic cell, then there also develop transversal components of the oscillations as \( k \times A \neq 0 \). Physically, the longitudinal and transverse particle dynamics can be understood as follows. The minimum of the potential energy is located along the main axis of a cubic crystal lattice, which provides channels for guiding the dust grains. When the wave propagates along one of these axes, it pushes the particles along the channels and the direction of the particle displacement coincides with the wave vector, and only longitudinal (compressional) waves appear. On the other hand, if the wave vector forms an angle with the crystal axis, then the displacement of the dust grains, which move along the channels, would have both longitudinal and transverse components with respect to the wave vector.

We have considered a physically realistic situation in which the charge of a specific dust grain is not only determined by its own surface potential, but it also incorporates the effects of other dust grain potentials. Thus, we have analytically deduced an expression for the electrostatic potential when electrons and ions are Boltzmann distributed, and when dust grains are strongly interacting not only with their neighbours but also with other grains that are participating in the formation of the periodic structure. The results, given above, show that a regular crystal structure appears only when the plasma parameters satisfy a specific condition that is expressed by equation (77). It should be mentioned that the formalism developed here can be applied to any symmetry (depending on the symmetry of the bounding walls confining the dusty plasma).
9 Discussion

We have considered a realistic dusty plasma system, and have, particularly, focused on various basic features of dusty plasmas, namely the motion of the electrons and the ions in the dust grain potential, the electron and ion current densities, the ion number density far from the dust grain, the dust grain potential, effects of the trapped ions on the shielding of the dust grains, the distribution of the periodic potential, the oscillations of dust lattices, etc.

We have first studied the dust grain charging in a plasma accounting for the absorption of electrons and ions by a dust grain. It is found that besides the shielded Debye-Hückel potential there also appears a dipole-like potential at large distances (in comparison with the ion Debye radius). The effective potential of the dust grain is then drastically modified. We have analyzed the dynamics of the plasma particles in the effective potential, and discussed the charging currents that are essential for the determination of the potential of a dust grain. We have found that the current due to the trapped ions on the dust grain surface is insignificant, and that the dust grain charging is solely determined by the currents of the free electrons and ions that reach the dust grain surface.

We have then examined the effects of the trapped ions on the shielding of a dust grain and clarified the underlying physics. We have shown that the shapes of the effective potential and the potential well strongly depend on the ion angular momentum $M$. Therefore, the ion trapping in the potential well is controlled by $M$. When $M$ is smaller than a certain critical value $M_0$, the dust grain surface is within the well and the ions falling into the potential well are thus immediately lost due to absorption onto the dust grain surface. Hence, the formation of ion clouds that shield the grain is possible only for $M > M_0$. This will be addressed in more detail in chapter 2.

We have also presented a theory for a dust crystal and its oscillations. We have considered a physically realistic situation in which the charge of a specific dust grain is not only determined by its own surface potential, but also incorporates the effects of other dust grain potentials. We have thus been able to deduce an analytic expression for the electrostatic potential when the electrons and ions are Boltzmann distributed, and when the dust grains are strongly interacting not only with their neighbours, but also with other grains that are participating in the dust crystal assembly. Clearly, short and long range interactions between dust grains are self-consistently included in our theory for dust crystal. Our results show that a regular crystal structure appears only when the plasma parameters satisfy a specific condition that is
given by equation (77). We have finally presented a transparent treatment for longitudinal and transverse oscillations of our dust crystal. In particular, we have derived analytical expressions for the frequencies of the dust lattice oscillations and explained the underlying physics of those modes.

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**References**


