Solitary potentials in dusty plasmas

A. A. Mamun and R. A. Cairns
School of Mathematical and Computational Sciences, University of St. Andrews, St. Andrews, Fife KY16 9SS, United Kingdom

P. K. Shukla
Institut für Theoretische Physik IV, Ruhr-Universität Bochum, D-44780 Bochum, Germany

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It is found that a dusty plasma with inertial dust fluid and Boltzmann distributed ions admits only negative solitary potentials associated with nonlinear dust-acoustic waves. The dynamics of small-amplitude disturbances is governed by the Korteweg–de Vries (KdV) equation, the stationary solution of which assumes the inverted bell-shaped secant hyperbolic squared profile. The associated dust and ion density perturbations are, on the other hand, positive. The solitary potentials can be identified as nonlinear structures in low-temperature dusty plasmas such as those in laboratory and astrophysical environments. © 1996 American Institute of Physics.

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About five years ago, Rao et al.1 presented a novel low phase velocity (in comparison with the electron and ion thermal velocities) dust-acoustic wave in unmagnetized dusty plasmas whose constituents are inertially charged dust fluids and Boltzmann distributed electrons and ions. Thus, in dust-acoustic waves the dust particle mass provides the inertia, whereas the tension comes from the pressures of inertialess electrons and ions. A recent laboratory experiment2 has conclusively verified the theoretical prediction of Rao et al.,1 and have also reported the nonlinear features of the dust-acoustic waves.

The laboratory observations2 of low phase velocity dust-acoustic waves is associated with significant depletion of the electron number density, suggesting that the wave dynamics is governed by the inertia of the dust fluid and the pressure of inertialess ions only. Our objective here is to present a two-fluid model of the dusty plasma in which \( n_e = Z_d n_d \), where \( n_e(n_d) \) is the electron (dust) number density and \( Z_d m_d m_i \) is number of the charge residing on the dust grains, and report the existence of only compressive dust-acoustic solitons, associated with negative potentials. Here, \( m_i/m_d \) is the mass of the ions (dust particles).

Let us consider a two-component dusty plasma with extremely massive, micron-sized, negatively charged inertial dust grains and Boltzmann distributed ions. Thus, at equilibrium, we have \( n_i = Z_d n_d \), where \( n_i \) and \( n_d \) are the unperturbed ion and dust number densities, respectively. The dynamics of low phase velocity (lying between the ion and dust thermal velocities, viz. \( v_{td} < v_p < v_{di} \)) dust-acoustic oscillations is governed by

\[
\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x} (n_d u_d) = 0, \tag{1}
\]

where \( n_d \) is the dust particle density normalized by \( n_{d0} \), \( u_d \) is the fluid velocity normalized by the dust-acoustic speed \( c_d = (T_d/m_d)^{1/2} \), and \( \Phi \) is the electrostatic wave potential normalized by \( T_i/e \), where \( T_i \) is the ion temperature and \( e \) is the magnitude of the electron charge. The time and space variables are in the units of the dust plasma period \( T_d = (m_d/4\pi e^2 Z_d^2 n_{d0}^2)^{1/2} \) and the Debye length \( \lambda_{Dd} = (T_i/4\pi Z_d n_{d0} e^2)^{1/2} \), respectively. We note that in (3), we have replaced the ion number density \( n_i \) by \( Z_d n_{d0} \exp(-\Phi) \), and have assumed that \( Z_d n_{d0} \approx n_e \), which are valid, provided that the wave phase velocity is much smaller than the ion thermal velocity \( v_{di} = (\sqrt{T_i/m_i}) \), and the electron number density is sufficiently depleted during the charging of the dust grain, on account of the attachment of the background plasma electrons on the surface of the dust grains. This scenario is relevant to the recent laboratory experiment2 and also to the Saturn’s F ring.

We note that in the linear limit, (1)–(3) gives the dust-acoustic waves, whose phase velocity is \( \omega/k \approx c_d/(1 + k^2\lambda_{Dd}^2)^{1/2} \). For long-wavelength disturbances, the latter reduces to \( \omega/k \approx (Z_d T_i/m_d)^{1/2} \), indicating that the wave is supported by the pressure of the ions and inertia of the dust fluid.

In order to investigate the properties of large-amplitude stationary dust-acoustic solitons, we assume that all the dependent variables in (1)–(3) dependent only on a single variable \( \xi = x - Mt \), where again \( \xi \) is normalized by \( \lambda_{Dd} \) and \( M \) is the Mach number (soliton velocity/c_d). Thus, in the stationary frame, we obtain, from (1) and (2),

\[
\frac{\partial n_d}{\partial \xi} = -\frac{M}{\sqrt{M^2 + 2\Phi}}, \tag{4}
\]

where we have imposed the appropriate boundary conditions for localized disturbances, viz. \( \Phi \rightarrow 0 \), \( u_d \rightarrow 0 \), and \( n \rightarrow 1 \) at \( \xi \rightarrow \pm \infty \).

Substituting for \( n_d \) from (4) into Poisson’s equation (3), and multiplying both sides of the resulting equation by \( d\Phi/d\xi \), integrating once, and imposing the appropriate boundary
conditions for localized solutions, namely, \( \Phi \to 0 \) and \( d\Phi/d\xi \to 0 \) at \( \xi \to \pm \infty \), we obtain
\[
\frac{1}{2} \left( \frac{d\Phi}{d\xi} \right)^2 + V(\Phi) = 0,
\]
where the Sagdeev potential\(^3\) for our purposes reads as
\[
V(\Phi) = 1 + M^2 - M^2(1 + 2\Phi/M^2)^{1/2} - \exp(-\Phi).
\]
Equation (5) can be regarded as an “energy integral” of an oscillating particle of unit mass, with a velocity \( d\Phi/d\xi \) and position \( \Phi \) in a potential \( V(\Phi) \). It is clear from (5) that \( V(\Phi) = 0 \) and \( dV(\Phi)/d\Phi = 0 \) at \( \Phi = 0 \). Solitary wave solutions of (5) exist if (i) \( (d^2V/d\Phi^2)_{\Phi=0} < 0 \), so that the fixed point at the origin is unstable; (ii) there exists a nonzero \( \Phi_m \), the maximum (or minimum) value of \( \Phi \), at which \( V(\Phi_m) = 0 \); and (iii) \( V(\Phi) < 0 \) when \( \Phi \) lies between 0 and \( \Phi_m \).

We have numerically analyzed (6) for different values of \( M \) and the results are displayed in Figs. 1 and 2. It is seen that the Sagdeev potential is never zero for any positive value of \( \Phi \) and \( M > 1 \), thereby ruling out the possibility of compressive (\( \Phi > 0 \)) dust-acoustic solitons.

It is, however, clear that solitons with \( \Phi < 0 \) exist. It is of interest to find out whether or not there exists an upper limit of \( M \) for which solitons exist. This upper limit of \( M \) can be found by the condition \( V(\Phi_e) \to 0 \), where \( \Phi_e = -M^2/2 \) is the minimum value of \( \Phi \) for which the dust density \( n_d \) is real. Thus, we have \( 1 + M^2 - \exp(M^2/2) > 0 \); the latter holds for \( M < 1.58 \). Clearly, finite-amplitude dust-acoustic solitons exist for \( 1 < M < 1.58 \), as indicated in Figs. 1 and 2. Figure 3

\[\text{FIG. 1. Here } V(\Phi) \text{ vs } \Phi \text{ for different values of } M \text{ at an interval of } 0.01. \text{ The bottom (top) curve is for } M=1.05 (M=1.00).\]

\[\text{FIG. 2. Here } V(\Phi) \text{ vs } \Phi, \text{ for different values of } M \text{ at an interval of } 0=0.02. \text{ The bottom (top) curve is for } M=1.58 (M=1.5).\]

\[\text{FIG. 3. Here } \Phi \text{ vs } \xi = M_t \text{ for } M=1.1 \text{ (solid curve), } M=1.2 \text{ (dotted curve), and } M=1.3 \text{ (dashed curve).}\]
ploying the reductive perturbation technique\(^4\) and the stretched coordinates \(\xi = \epsilon^{1/2}(x - v_0 t)\) and \(\tau = \epsilon^{1/2} t\), where \(\epsilon\) is a smallness parameter measuring the weakness of the amplitude or dispersion, and \(v_0\) is the unknown soliton velocity (normalized by \(c_d\)), to be determined later. We can then expand the variables \(n_d\), \(u_d\), and \(\Phi\) about the unperturbed states in power series of \(\epsilon\), as done in Ref. 4, and develop equations in various powers of \(\epsilon\). To lowest order in \(\epsilon\), (1) and (2) give \(n_d^{(1)} = -\Phi^{(1)}/v_0^2, u_d^{(1)} = -\Phi^{(1)}/v_0,\) and \(v_0 = 1\). To next higher order in \(\epsilon\), we have a set of equations, which read as

\[
\frac{\partial n_d^{(1)}}{\partial \tau} - v_0 \frac{\partial n_d^{(2)}}{\partial \xi} + \frac{\partial u_d^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi} (n_d^{(1)} u_d^{(1)}) = 0, \tag{7}
\]

\[
\frac{\partial u_d^{(1)}}{\partial \tau} - v_0 \frac{\partial u_d^{(2)}}{\partial \xi} - \frac{\partial \Phi^{(2)}}{\partial \xi} + u_d^{(1)} \frac{\partial u_d^{(1)}}{\partial \xi} = 0, \tag{8}
\]

and

\[
\frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} - \Phi^{(2)} - n^{(2)} + \frac{1}{2} (\Phi^{(1)})^2 = 0. \tag{9}
\]

From (7), (8), and (9), we readily obtain

\[
\frac{\partial \Phi^{(1)}}{\partial \tau} + a \Phi^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} + b \Phi^{(1)} \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} = 0, \tag{10}
\]

which is the KdV equation with \(a = -1\) and \(b = \frac{1}{2}\). The stationary localized solution of (10) is

\[
\Phi = \Phi_m \mathrm{sech}^2\left(\left(\xi - u_d t\right)/\delta\right), \tag{11}
\]

where the amplitude \(\Phi_m\) and the width \(\delta\) are given by \(\Phi_m = -3u_0\) and \(\delta = \sqrt{2/|u_0|}\), respectively. As the soliton velocity \(u_0 > 0\), Eq. (11) clearly indicates that there exists only solitons with negative potential of amplitude \(3u_0\) and width \(\sqrt{2/|u_0|}\). It is found that as \(u_0\) increases the amplitude \(\Phi_m\) increases while the width \(\delta\) decreases.

In this Brief Communication, we have investigated the nonlinear properties of dust-acoustic waves in dusty plasmas whose constituents are cold dust particulates and thermal ions having the Boltzmann distribution. The plasma model is appropriate when the electron number density is sufficiently depleted, so that \(Z_d n_d >> n_e\). Our results show that in such a plasma we can have only negative potential structures associated with the nonlinear dust-acoustic waves. The latter travel at a speed larger than the dust-acoustic speed \(c_d\). Furthermore, it is found the dynamics of weakly nonlinear and weakly dispersive dust-acoustic waves is governed by the standard KdV equation, in which the coefficient of the nonlinear and the wave dispersion are opposite to each other.

The stationary solution of the KdV equation can be represented in the form of inverted secant hyperbolic squared profile. Thus, the potential polarity of the dust-acoustic solitons in our dusty plasma is different from the usual ion-acoustic solitons in an electron–ion plasma without the dust component.\(^3\) However, the ion density and dust density profiles associated with nonlinear dust-acoustic waves are compressive. In conclusion, we stress that the results of our investigation should be useful in understanding the nonlinear features of localized electrostatic disturbances in laboratory and space plasmas, in which negatively charged dust particulates and ions are the major plasma species.

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