

Nonlinear Whistlerons*

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Recently, observations from laboratory experiments have revealed amplitude modulation of whistlers by low-frequency perturbations. We here present theoretical and simulation studies of amplitude modulated whistler solitary waves (whistlerons) and their interaction with background low-frequency density perturbations created by the whistler ponderomotive force. We derive a nonlinear Schrödinger equation which governs the evolution of whistlers in the presence of finite-amplitude density perturbations, and a set of equations for arbitrary large amplitude density perturbations in the presence of the whistler ponderomotive force. The governing equations studied analytically in the small amplitude limit, and are solved numerically to show the existence of large scale density perturbations that are self-consistently created by localized whistlerons. Our numerical results are in good agreement with recent experimental results where the formation of modulated whistlers and solitary whistler waves were formed.

I. INTRODUCTION

Almost three decades ago, Stenzel [1] experimentally demonstrated the creation of a magnetic field-aligned density cavities by the ponderomotive force of localized electron whistlers. Observations from a recent laboratory experiment [2] exhibit the creation of modulated whistler wavepackets due to nonlinear effects. Furthermore, instruments on board the CLUSTER spacecraft have been observing broadband intense electromagnetic waves, correlated density fluctuations and solitary waves near the Earth's plasmopause, magnetopause and foreshock [3], revealing signatures of whistler turbulence in the presence of density depletions and enhancements. The Freja satellite [4] also observed the formation of envelope whistler solitary waves correlated with density cavities in the plasma.

A theoretical investigation has in the past predicted the self-channeling of electron whistlers and the creation of a localized density hump [5]. Taking into account the spatio-temporal dependent whistler ponderomotive force [6, 7], investigations of the modulation and filamentation of finite amplitude whistlers interacting with magnetosonic waves [8, 9, 10] and ion-acoustic perturbations [11, 12] have been carried out.

In this article, we investigate nonlinearly interacting electron whistlers and arbitrary large amplitude ion-acoustic perturbations, by using computer simulations, and we find analytical expressions for whistlerons in the low-amplitude limit [13, 14].

II. DERIVATION OF THE GOVERNING EQUATIONS

Let us consider the propagation of nonlinearly coupled whistlers and ion-acoustic perturbations in a fully ionized electron-ion plasma in a uniform external magnetic field $\hat{\mathbf{z}}B_0$, where $\hat{\mathbf{z}}$ is the unit vector along the z direction and B_0 is the magnitude of the magnetic field strength. We consider the propagation of right-hand circularly polarized modulated whistlers of the form

$$\mathbf{E} = \frac{1}{2}E(z, t)(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\exp[i(k_0z - \omega_0t)] + \text{c.c.}, \quad (1)$$

where $E(z, t)$ is the slowly varying envelope of the whistler electric field, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the x and y axes, respectively, and c.c. stands for the complex conjugate. The whistler frequency ω_0 ($\gg \sqrt{\omega_{ce}\omega_{ci}}$), and the wavenumber k_0 are related by the cold plasma dispersion relation

$$\omega_0 = \frac{k_0^2 c^2 \omega_{ce}}{\omega_{pe,0}^2 + k_0^2 c^2}, \quad (2)$$

where c is the speed of light in vacuum, $\omega_{ce} = eB_0/m_e c$ ($\omega_{ci} = eB_0/m_i c$) is the electron (ion) gyrofrequency, $\omega_{pe,0} = (4\pi n_0 e^2/m_e)^{1/2}$ is the electron plasma frequency, e is the magnitude of the electron charge, m_e (m_i) is the electron mass, and n_0 is the unperturbed background electron number density.

The dynamics of modulated whistler wavepacket in the presence of electron density perturbations associated with low-frequency ion-acoustic fluctuations and of the nonlinear frequency-shift caused by the magnetic field-aligned free streaming of electrons (with the flow speed v_{ez}), is governed by the nonlinear Schrödinger equation [12]

$$i(\partial_t + v_g \partial_z)E + \frac{v_g'}{2} \partial_{zz}^2 E + (\omega_0 - \omega)E = 0, \quad (3)$$

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where

$$\omega = \frac{k_0^2 c^2 \omega_{ce}}{\omega_{pe}^2 + k_0^2 c^2} + k_0 v_{ez}, \quad (4)$$

and $\omega_{pe}^2 = \omega_{pe,0}^2 n_e/n_0$ is the local plasma frequency including the electron density n_e of the plasma slow motion. The group velocity and the group dispersion of whistlers are

$$v_g = \frac{\partial \omega_0}{\partial k_0} = 2 \left(1 - \frac{\omega_0}{\omega_{ce}} \right) \frac{\omega_0}{k_0} \quad (5)$$

and

$$v'_g = \frac{\partial^2 \omega_0}{\partial k_0^2} = 2 \left(1 - \frac{\omega_0}{\omega_{ce}} \right) \left(1 - 4 \frac{\omega_0}{\omega_{ce}} \right) \frac{\omega_0}{k_0^2}, \quad (6)$$

respectively.

The equations for the ion motion involved in the low-frequency (in comparison with the whistler wave frequency) ion-acoustic perturbations are

$$\partial_t n_i + \partial_z (n_i v_{iz}) = 0 \quad (7)$$

and

$$\partial_t v_{iz} + \frac{1}{2} \partial_z v_{iz}^2 = -\frac{e}{m_i} \partial_z \phi - \frac{1}{m_i n_i} \partial_z p_i, \quad (8)$$

where, for an adiabatic compression in one space dimension, the ion pressure is given by $p_i = p_{i,0} (n_i/n_0)^3$. Here, the unperturbed ion pressure is denoted by $p_{i,0} = n_0 T_i$, where T_i is the ion temperature.

The electron dynamics in the plasma slow motion is governed by the continuity and momentum equations, *viz.*

$$\partial_t n_e + \partial_z (n_e v_{ez}) = 0 \quad (9)$$

and

$$0 = \frac{e}{T_e} \partial_z \phi - \partial_z \ln \left(\frac{n_e}{n_0} \right) + F, \quad (10)$$

where T_e is the electron temperature, ϕ is the ambipolar potential, and the low-frequency ponderomotive force of electron whistlers is

$$F = \frac{\omega_{pe,0}^2}{\omega_0 (\omega_{ce} - \omega_0)} \left(\partial_z + \frac{2}{v_g} \partial_t \right) \frac{|E|^2}{4\pi n_0 T_e}. \quad (11)$$

The system of equations is closed by means of quasi-neutrality

$$n_i = n_e \equiv n, \quad (12)$$

which is justified if $\omega_0 < \omega_{ce}$ is fulfilled with some margin. The continuity equations for the electrons and ions give $v_{iz} = v_{ez} \equiv v_z$, so that

$$\partial_t n + \partial_z (n v_z) = 0. \quad (13)$$

Eliminating $\partial_z \phi$ from the governing equations for low-frequency density perturbations, we have

$$\partial_t v_z + \frac{1}{2} \partial_z v_z^2 = -\frac{T_e}{m_i} \left[\partial_z \ln \left(\frac{n}{n_0} \right) - F \right] - \frac{1}{m_i n} \partial_z p_i. \quad (14)$$

The nonlinear Schrödinger equation for the whistler electric field together with the low-frequency equations form a closed set for our purposes.

A. Dimensionless variables

In order to investigate numerically the interaction between whistlers and large amplitude ion-acoustic perturbations, it is convenient to normalize the governing equations into dimensionless units, so that relevant parameters can be chosen. We introduce the dimensionless variables $\xi = \omega_{pi,0} z / C_s$, where the sound speed is $C_s = [(T_e + 3T_i)/m_i]^{1/2}$, $\tau = \omega_{pi,0} t$, $N = n/n_0$, $u = v_z / C_s$ and $\mathcal{E} = E / \sqrt{4\pi n_0 (T_e + 3T_i)}$; the only free dimensionless parameters of the system are $\Omega_c = \omega_{ce} / \omega_{pi,0}$, $\kappa = k_0 c / \omega_{pe,0}$, $\eta = T_i / T_e$ and $V_g = v_g / C_s$. The normalized system of equations are of the form

$$\partial_\tau N = -\partial_\xi (N u), \quad (15)$$

$$\begin{aligned} \partial_\tau \left(u - \frac{2\alpha}{V_g} |\mathcal{E}|^2 \right) \\ = \partial_\xi \left[-\frac{u^2}{2} - \frac{\ln N + 1.5\eta N^2}{1 + 3\eta} + \alpha |\mathcal{E}|^2 \right], \end{aligned} \quad (16)$$

and

$$\begin{aligned} \partial_\tau \mathcal{E} = -V_g \partial_\xi \mathcal{E} + i \left[P \partial_{\xi\xi}^2 \mathcal{E} \right. \\ \left. + \left(\frac{1}{1 + \kappa^2} - \frac{1}{N + \kappa^2} - \frac{2}{(1 + \kappa^2)^2} \frac{u}{V_g} \right) \Omega_c \kappa^2 \mathcal{E} \right], \end{aligned} \quad (17)$$

where the constants are $\alpha = (1 + \kappa^2)^2 \omega_{pe,0}^2 / \omega_{ce}^2 \kappa^2$ and $P = (1 + \kappa^2)(1 - 3\kappa^2) V_g^2 / 4\kappa^2 \Omega_c$. The sign of the coefficient P , multiplying the dispersive term in Eq. (3), depends on κ : When $\kappa < 1/\sqrt{3}$, P is positive and for $\kappa < 1/\sqrt{3}$ we see that P is negative.

III. SMALL-AMPLITUDE SOLITARY WAVES

In the small-amplitude limit, *viz.* $N = 1 + N_1$, $u = u_1$, where N_1 , $u_1 \ll 1$, Eqs. (1)–(3) yield

$$\partial_\tau N_1 = -\partial_\xi u_1, \quad (18)$$

$$\partial_\tau \left(u_1 - \frac{2\alpha}{V_g} |\mathcal{E}|^2 \right) = \partial_\xi (-N_1 + \alpha |\mathcal{E}|^2), \quad (19)$$

and

$$\begin{aligned} \partial_\tau \mathcal{E} = -V_g \partial_\xi \mathcal{E} \\ + i \left[P \partial_{\xi\xi}^2 \mathcal{E} + \left(N_1 - \frac{u}{V_g} \right) \frac{\Omega_c \kappa^2 \mathcal{E}}{(1 + \kappa^2)^2} \right], \end{aligned} \quad (20)$$

where the only nonlinearity kept is the ponderomotive force terms involving $|\mathcal{E}|^2$. It is important to remember that our nonlinear Schrödinger equation for the whistler field is based on a Taylor expansion of the dispersion relation for whistler waves around a wavenumber k_0 . Thus, this model is only accurate for wave envelopes moving with speeds close to the group speed V_g , and other speeds of the wave envelopes may give unphysical results. Here, we look for whistler envelope solitary waves moving with the group speed V_g , so that N_1 and u_1 depends only on $\chi = \xi - V_g\tau$, while the electric field envelope is assumed to be of the form $\mathcal{E} = W(\chi) \exp(i\Omega\tau - ik\xi)$, where W is a real-valued function of one argument. Using the boundary conditions $N_1 = 0$, $u_1 = 0$ and $W = 0$ at $|\xi| = \infty$, we have $k = 0$, $N_1 = -W^2\alpha/(1 - V_g^2)$ and $u_1 = V_g N_1$. We here note that subsonic ($V_g < 1$) solitary waves are characterized by a density cavity while supersonic ($V_g > 1$) envelope solitary waves are characterized by a density hump. The system of equations (4) to (6) is then reduced to the cubic Schrödinger equation

$$P\partial_{\chi\chi}^2 W + QW^3 - \Omega W = 0, \quad (21)$$

where $Q = \alpha\Omega_c\kappa^2/(1+\kappa^2)(1-V_g^2)$. Localized solutions of Eq. (7) only exist if the product PQ is positive. We note that $P > 0$ ($P < 0$) when the whistler frequency $\omega_0 < \omega_{ce}/4$ ($\omega_0 > \omega_{ce}/4$), and that $Q < 0$ ($Q > 0$) when $|V_g| < 1$ ($|V_g| > 1$), so in the frequency band where $\omega_0 < \omega_{ce}/4$, only subsonic solitary waves, characterized by a localized density cavity can exist, while in the frequency band $\omega_0 > \omega_{ce}/4$, only supersonic solitary waves characterized by a localized density hump exist. Equation (7) has exact solitary wave solutions of the form

$$W = \left(\frac{2\Omega}{Q}\right)^{1/2} \text{sech} \left[\left(\frac{\Omega}{P}\right)^{1/2} (\xi - V_g\tau - \xi_0) \right], \quad (22)$$

where V_g and Ω and the displacement ξ_0 are the three free parameters for a given set of physical plasma parameters. Finally, we recall that the dispersion relation for the electron whistlers used here is valid if $\omega_0 > \sqrt{\omega_{ce}\omega_{ci}}$. For subsonic whistlers having the group speed $v_g = C_s V_g$ (where $V_g < 1$), where $v_g \approx 2\omega_0/k_0$ and $\omega_0 \approx k_0^2 c^2 \omega_{ce}/\omega_{pe,0}^2$, we have $ck_0/\omega_{pe,0} = (C_s/c)(\omega_{pe,0}/\omega_{ce})V_g/2 > (m_e/m_i)^{1/4}$.

IV. NUMERICAL RESULTS

We have investigated the properties of modulated whistler wave packets by solving numerically Eqs. (1)–(3). We have here chosen parameters from a recent experiment, where the formation of localized whistler envelopes have been observed [2]. In the experiment, one has $n_0 = 1.2 \times 10^{12} \text{ cm}^{-3}$ and $B_0 = 100 \text{ G}$, so that $\omega_{pe,0} = 6.7 \times 10^{10} \text{ s}^{-1}$ and $\omega_{ce} = 1.76 \times 10^9 \text{ s}^{-1}$, respectively. Hence, $\omega_{ce}/\omega_{pe,0} = 0.026$. The frequency of the whistler wave is $\omega_0 = 2\pi \times 160 \times 10^6 \text{ s}^{-1} = 1.0 \times 10^9 \text{ s}^{-1}$, so that $\omega_0/\omega_{ce} \approx 0.57 > 0.25$. Thus, the whistlers have

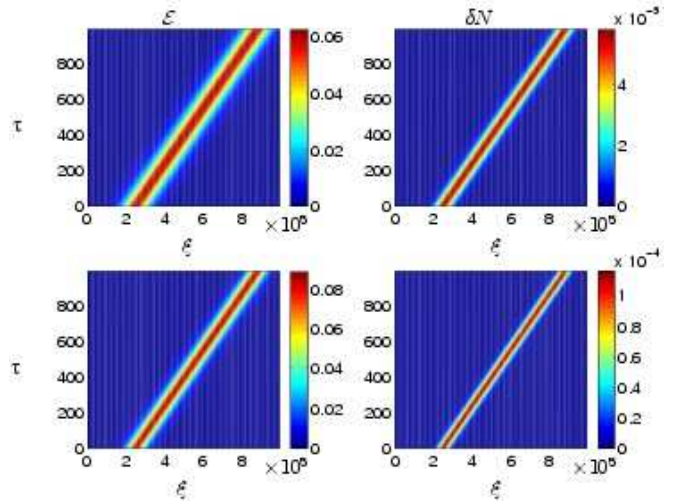


FIG. 1: Simulations of solitary whistler waves, with its associated electric field envelope $|\mathcal{E}|$ (left panels) and density perturbation $\delta N = N - 1$ (right panels). Parameters are: $V_g = 640$, $\kappa = 1.15$, $\Omega_c = \omega_{ce}/\omega_{pi,0} = 7.05$, $m_i/m_e = 73400$ (argon ions) and $\eta = 0.05$. For the initial condition, we used the small-amplitude solitary wave solution $\mathcal{E} = W = (2\Omega/Q)^{1/2} \text{sech}[(\Omega/P)^{1/2}(\xi - 2.5 \times 10^5)]$, with $Q = -0.025$ and $P = -7.57 \times 10^4$ for the given parameters, and $\Omega = -0.5 \times 10^{-4}$ (upper panels) and $\Omega = -1 \times 10^{-4}$ (lower panels). For the density and velocity, we used $N = 1 + N_1$ and $u = u_1$, where $N_1 = -W^2\alpha/(1 - V_g^2)$ and $u_1 = V_g N_1$.

negative group dispersion. From the dispersion relation of whistlers, we have $\kappa \approx 1.15$, which gives $k_0 \approx 257 \text{ m}^{-1}$. The latter corresponds to whistlers with a wavelength of 2.4 cm. Furthermore, the whistler group velocity is $v_g = 3.36 \times 10^6 \text{ m/s}$. The argon ion-electron plasma ($m_i/m_e = 73400$) had the temperatures of $T_e = 10 \text{ eV}$ and $T_i = 0.5 \text{ eV}$, giving the sound speed $5.25 \times 10^3 \text{ m/s}$, and the normalized group velocity $V_g = v_g/C_s = 640$. In Fig. 1, we have illustrated localized whistler envelope solitons, in which the electric field envelope (left panels) is accompanied with a density hump (right panels). We notice that the density hump is relatively small, due to the large (in comparison with the acoustic speed) group velocity of the whistler waves. In Fig. 2, we can see the development of a large-amplitude whistler pulse, which has been launched in a plasma perturbed by ion-acoustic waves, with a density modulation of one percent (see the caption of Fig. 2). This simulates, to some extent, the experiment by Kostrov *et al.*, where the density and magnetic field were perturbed by a low-frequency conical refraction wave, giving rise to a modulation of the electron whistlers. Here, as in the experiment, we observe that a modulated electron whistler pulse (middle panel of Fig. 2) develops into isolated solitary electron whistler waves (lower panel). We note that the wavelength of the whistlers is $\approx 2.5 \text{ cm}$, while the typical

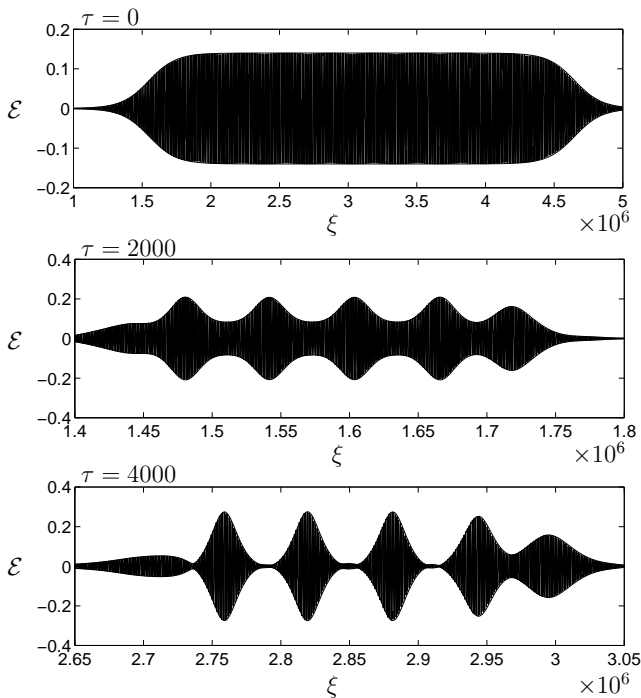


FIG. 2: The electric field as a function of normalized space ξ at the times $\tau = 0$ (upper panel), $\tau = 2000$ (middle panel) and $\tau = 4000$ (lower panel). The initial condition was a pulse on the form $\mathcal{E} = 0.07\{1 - \tanh[5 \cos(\pi\xi/3.1 \times 10^5)]\}$, while the density was perturbed as $N = 1 + 0.01 \cos(20\pi\xi/6.2 \times 10^5)$. Parameters are: $V_g = 640$, $\kappa = 1.15$, $\Omega_c = \omega_{ce}/\omega_{pi,0} = 7.05$, $m_i/m_e = 73400$, $\eta = 0.05$. The (scaled by $C_s/\omega_{pi,0}$) wavelength of the high-frequency wave is $\lambda_0 = 1.16 \times 10^3$, corresponding to ~ 2.5 cm in dimensional units. The envelope $\pm|\mathcal{E}|$ was plotted together with the real part of the wave $\mathcal{E} \exp(iK_0\xi)$, where $K_0 = k_0 C_s/\omega_{pi} = 2\kappa^2 \Omega_c/(1 + \kappa^2)^2 V_g$. We see the formation of separate wavepackets of the modulated high-frequency wave.

width of a solitary pulse is $\Delta\xi \approx 3 \times 10^4$ in the scaled length units, corresponding to ≈ 64 cm, so that each solitary wave train contains 25 wavelengths of the high-frequency whistlers. In one experiment, illustrated in the lower panel of Fig 4 in Ref. [2], one finds that the width of the solitary whistler pulse in time is $0.2 \mu\text{s}$, which with the group speed $v_g = 3.36 \times 10^6$ m/s gives the width ~ 60 cm in space of the solitary wave packets, in good agreement with our numerical results. From the relation $N_1 = -W^2\alpha/(1 - V_g^2)$ valid for solitary whistlers in the small-amplitude limit, and with the amplitude of $W = |\mathcal{E}|$ approximately 0.3 seen in the lower panel of Fig. 2, we can estimate the relative amplitude of the density hump associated with the solitary waves to be of the order 10^{-3} , i.e. much smaller than the modulation $\sim 10^{-2}$ due to the ion-acoustic waves excited in the initial condition.

Next, we study the properties of subsonic whistler

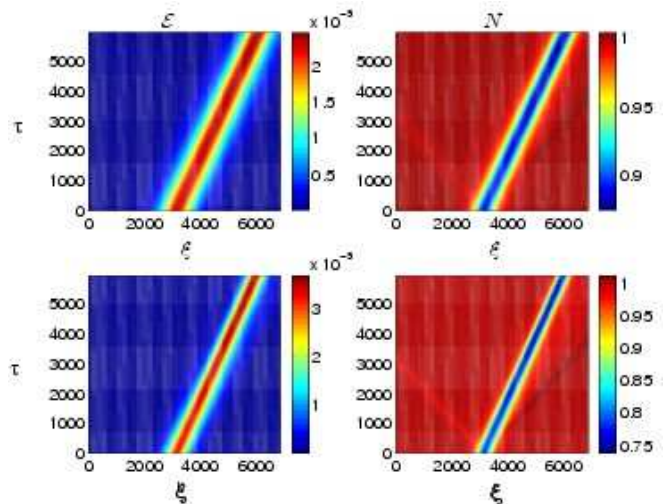


FIG. 3: Simulations of solitary whistler waves, with its associated electric field envelope $|\mathcal{E}|$ (left panels) and density N (right panels). Parameters are: $V_g = 0.5$, $\kappa = 0.2$, $\Omega_c = \omega_{ce}/\omega_{pi,0} = 0.072$, $m_i/m_e = 30000$, $\eta = 0.1$. For the initial condition, we used the small-amplitude solitary wave solution $\mathcal{E} = W = (2\Omega/Q)^{1/2} \text{sech}[(\Omega/P)^{1/2}(\xi - 3000)]$, with $Q = 5.5 \times 10^5$ and $P = 10.8$ for the given parameters, and $\Omega = 1.5 \times 10^{-4}$ (upper panels) and $\Omega = 3 \times 10^{-4}$ (lower panels). For the density and velocity, we used $N = 1 + N_1$ and $u = u_1$, where $N_1 = -W^2\alpha/(1 - V_g^2)$ and $u_1 = V_g N_1$.

envelope solitary pulses which have the normalized group speed $V_g = 0.5$. Here, the restrictive condition $ck_0/\omega_{pe,0} = (C_s/c)(\omega_{pe,0}/\omega_{ce})V_g/2 > (m_e/m_i)^{1/4}$ requires somewhat higher values of the plasma temperature and $\omega_{pe,0}/\omega_{ce}$ for their existence. With $m_i/m_e = 30000$, we have $(m_e/m_i)^{1/4} \approx 0.1$. We take $\kappa = ck_0/\omega_{pe,0} = 0.2$, $C_s = 10^5$ m/s (corresponding to $T_e \sim 1400$ eV) $\eta = 0.1$, and $\omega_{pe,0}/\omega_{ce} = 2400$. Thus, $\Omega_c = 0.072$ and $\omega_0/\omega_{ce} \approx 0.039$. For these values of the parameters, there exist solitary whistler pulse solutions, which we have displayed in Fig. 3. We have used the exact solution in the small-amplitude limit as an initial condition for the simulation of the full system of equations (1)–(3). The bell-shaped whistler electric field envelope is accompanied with a large-amplitude plasma density cavity.

V. DISCUSSION

We have presented theoretical and simulation studies of nonlinearly interacting electron whistlers and arbitrary large amplitude ion-acoustic perturbations in a magnetized plasma. For this purpose, we have derived a set of equations which describe the spatio-temporal evolution of a modulated whistler packet in the presence of slowly varying plasma density perturbations. The ponderomotive force of the latter, in turn, modifies the lo-

cal plasma density in a self-consistent manner. Numerical solutions of the governing nonlinear equations reveal that subsonic envelope whistler solitons are characterized by a bell-shaped whistler electric fields that are trapped in self-created density cavity. This happens when the whistler wave frequency is smaller than $\omega_{ce}/4$, where the waves have positive group dispersion. When the whistler wave frequency is larger than $\omega_{ce}/4$, one encounters negative group dispersive whistlers and the supersonic whistler envelope solitons are characterized by a bell-shaped whistler electric fields which create a density hump. Modulated whistler wavepackets have indeed been observed in a laboratory experiment [2] as well as near the plasmopause [3] and in the auroral zone [4]. Our results are in excellent agreement with the experimental

results [2], while we think that a multi-dimensional study, including channelling of whistler waves in density ducts, is required to interpret the observations by Cluster and Freja satellites.

Acknowledgments

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